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A

MANUAL

OF

SPHERICAL AND PRACTICAL ASTRONOMY:

EMBRACING

THE GENERAL PROBLEMS OF SPHERICAL ASTRONOMY, THE SPECIAL APPLICATIONS TO NAUTICAL ASTRONOMY, AND THE THEORY AND USE OF FIXED AND PORTABLE ASTRONOMICAL INSTRUMENTS.

WITH AN APPENDIX ON THE METHOD OF LEAST SQUARES.

BY

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THEORY AND USE
OF
ASTRONOMICAL INSTRUMENTS.

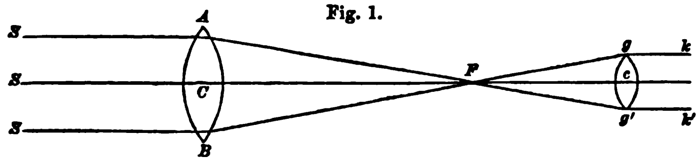
CHAPTER I.

THE TELESCOPE.

1. The complete theory of the telescope considered simply as an optical instrument is too extensive a subject to be condensed into a chapter of the present work: it must be sought for in the larger works on optics.* I shall, therefore, confine myself to such points as appear to be immediately needed by the observer for the intelligent use of his instruments. The following explanations, at once elementary and practical, some of which are not to be found in optical works, are chiefly derived from Sawitsch.†

2. *The simple astronomical telescope.*—The astronomical telescope, in its simplest form, consists of two bi-convex lenses; the larger, *AB* (Fig. 1), which is turned towards the object, is called the

Fig. 1.



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* See Herschel's *Treatise on Light;* Prechtel's *Practische Dioptrik;* Biot's *Astronomie Physique,* Vols. I. and II.; Potter's *Optics;* Coddington's *Optics;* Lloyd's *Treatise on Light and Vision;* Littrow's *Analytische Dioptrik;* Pearson's *Practical Astronomy.*

† *Abriss der practischen Astronomie, von* Dr. A. Sawitsch, *aus dem Russischen übersetzt von* Dr. W. C. Gœtze. Hamburg, 1850.

objective, or, more commonly, the *object glass;* and the smaller, *gg'*, through which the observer looks, is called the *ocular*, or, more commonly, the *eye glass* or *eye piece.* The two surfaces of both these lenses are segments of spherical surfaces of different radii. The *optical axis* of a lens is the straight line which passes through the centres of the two spherical surfaces which bound the lens. The optical axis of the telescope is coincident with that of the object glass. When the telescope is well constructed, the optical axis of the ocular should always be *parallel* to that of the objective, even when (as is usual in the larger instruments) the ocular is movable, this motion being in a plane at right angles to the axis of the telescope. Where the ocular has no motion, its axis should coincide with that of the objective, and, consequently, with that of the telescope.

3. Let us now suppose that our telescope, or rather its optical axis, is directed towards a star *S.* Then, on account of the great distance of the star, we can assume that all the rays from it to various points of the object glass, as *SA*, *SC*, *SB*, are parallel to each other. The ray *SC*, which passes along the optical axis itself, suffers no deviation from the refractive power of the lens, since it enters and leaves the lens at right angles to the refracting surfaces; but all other rays, as *SA* and *SB*, are refracted both when entering the lens and when leaving it, and, when the lens is small in proportion to the radii of curvature of its surfaces, these rays will all converge to a common point *F* in the axis of the telescope. This common point in which a system of parallel rays meet is the *principal focus*, usually called simply *the focus*, of the lens, and the distance *FC* from the centre *C* of the lens is called the *focal length* of the lens. If the radiant point *S* is so near to the telescope that the lines *SA*, *SB* are sensibly divergent, the lens will not bring them together at the principal focus, but at a point more remote; that is, the actual focus will be farther from the lens than *F.* If the radiant point is at a distance from the lens equal to the principal focal distance, the divergent rays from this point will simply be rendered parallel by the lens, or the actual focus will be removed to an infinite distance. For all astronomical purposes we need consider only the principal focus, regarding the rays, even from the nearest celestial body, the moon, as sensibly parallel. The telescopes used in surveying instruments (where the terrestrial objects observed are at various

distances from the lens, and these distances all small) are provided with a ready means of adjusting the position of the objective, by sliding the part of the telescope tube containing it out and in: so that the actual focus may always occupy the same absolute position in the optical axis, and, consequently, always be at the same distance from the ocular. The same result is also obtained by giving the portion of the tube containing the ocular a sliding motion.

4. All the parallel rays from a distant radiant point, as a star *S*, which are converged to the focus *F*, form an image of the star in that focus. Conversely, if the radiant point be placed at *F*, all the divergent rays *SA*, *SB*, &c. will emerge from the lens in parallel lines *AS*, *BS*, &c. We shall hereafter have occasion to make several important applications of this property of a lens: here we shall apply it at once to show how a distinct view of the image of a star at *F* is obtained. The eye lens *gg'*, being placed in the line *CF* produced, at a distance *Fc* equal to its own principal focal distance, it follows, from the property of a lens just stated, that the divergent rays *Fg*, *Fg'* will emerge in parallel lines *gk*, *g'k'*, and will, consequently, enter the eye of the observer in parallel lines, thus giving a distinct view of the star; for the eye, in persons who are neither far-sighted nor near-sighted, is naturally adapted for distinct vision when the rays entering it are parallel. Without the telescope we should see only those rays from the star which fall upon the pupil of the eye; but when we look at the image of the star at the focus of a telescope, we see it with greater distinctness, because we then receive into the eye all the rays which have entered the object glass and have been united at the focus. In this consists the *first* great advantage in the use of the telescope.

5. Let a very fine thread be stretched in the focus *F* of the telescope at right angles to the optical axis. This thread will be visible through the ocular when the latter is so placed that its focus coincides with *F*: consequently, when the telescope is directed towards a star, we shall have distinct vision of both the star and this thread at the same time. If two threads are placed at the focus at right angles to each other, their intersection will determine a fixed point in the field of view, which by moving the telescope may be brought upon the object to be

observed. By bringing this point successively upon different celestial objects, their relative positions can be measured with the greatest precision; and in this consists the *second* great advantage in the use of the telescope. Since the apparent thickness of these threads is increased by the magnifying power of the ocular it is necessary to use a very fine material: the spider's web is that which is almost universally used.

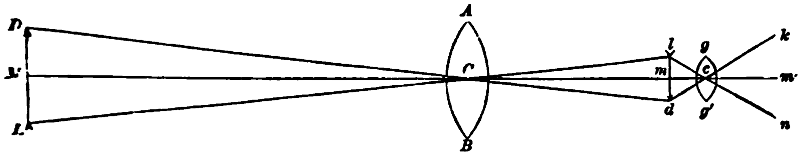
The *line of sight* is the straight line drawn from the thread through the optical centre of the objective; for this line represents the direction of a distant point (as a star), when the telescope is so directed that an image of the point is formed at the thread. This line is also called the *line of collimation;* but we shall hereafter, for the sake of brevity, call it the *sight-line.*

6. The spider lines, or threads, are usually stretched across a ring, or diaphragm, which is placed in a tube which slides in the principal tube of the telescope. The ocular also slides without affecting the threads: so that by means of these two motions we can bring the threads exactly into the common focus of the objective and ocular. It is to be observed that the motion of the ocular is necessary merely for adaptation to the eyes of different observers. The threads, being once accurately placed in the focus of the objective, must not be disturbed; but the ocular may be drawn out or pushed in by each observer until he obtains a distinct view of the threads. To ascertain whether the threads are accurately placed in the focus of the objective, first adjust the ocular for distinct vision of the threads, then, bringing a thread upon a very distinct point, as a slow moving star, observe whether a motion of the eye in any direction towards the edge of the eye lens causes the star to leave the thread; for, if the image of the star is exactly *on* the thread, it ought to be seen on it even from a side view; but, if it is before or behind the thread, it will be seen on it only from a direct front view.

7. *Magnifying power.*—Let us suppose the telescope to be directed towards a *very distant* object *DL* (Fig. 2). From its upper extremity *D* a multitude of rays proceed which fall upon all parts of the objective *AB*, and which (in consequence of the great distance of the object) may all be regarded as parallel to the line *DCd* which passes through the middle point of the lens. All these rays are brought to a focus in this line *DCd* at a point *d* whose

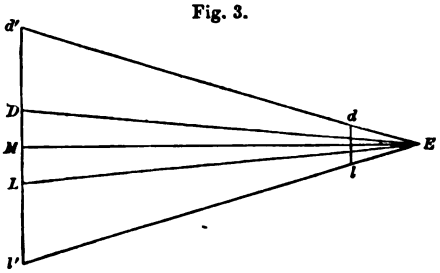
distance from the lens is equal to the focal length of the lens. There exists then at the point *d* a distinct image of the point *D*. In a

Fig. 2.



similar manner an image of every point of the object is found at the same distance behind the object glass: so that there will exist at the focus of the lens a complete, though very small, image of the object. This image will be inverted; for, while the image of the upper point *D* is formed at *d*, that of the lowest point *L* is formed at *l*, the axes of the systems of rays from the several points of the object crossing at the middle point *C* of the lens. If the focus of the ocular is coincident with that of the objective, and, consequently, also with the image *dl*, the rays which diverge from a point *d* of the image and fall upon the ocular *gg'* will emerge from the latter in lines parallel to each other and to the line *dck* which is drawn from *d* through the centre of the ocular; and, the same being true of rays from every point of the image, those from the extreme point *l* emerge in lines parallel to the line *lcn*. Hence the rays from the two extreme points *d* and *l* of the image enter the eye of the observer at an angle with each other equal to *nck* or *lcd;* and this angle is the *apparent angular magnitude* of the image to the eye. But without the telescope the apparent angular magnitude of the object, the eye being at *C*, would be $DCL = dCl$; which angle may be assumed to be the same as that under which the object is seen from the actual position of the eye behind the ocular, the length of the telescope being inconsiderable in relation to the distance of the object. Now, the apparent *linear* magnitudes of the object and its image seen thus under different angles can be compared by referring them to the same absolute distance. Thus, referring the image *dl* (Fig. 3) to the actual distance of the

Fig. 3.



object DL, by the lines Edd', Ell' drawn from the eye at E, we have

$$d'l' : DL = d'M : DM = \tan \tfrac{1}{2}\, dEl : \tan \tfrac{1}{2}\, DEL$$

Hence, denoting the magnifying power by G, we have

$$G = \frac{d'l'}{DL} = \frac{\tan \frac{1}{2}\, dEl}{\tan \frac{1}{2}\, DEL} \qquad (1)$$

whence the proposition, (A), *The magnifying power of the telescope is equal to the tangent of half the apparent angular magnitude of the image seen through the ocular, divided by the tangent of half the apparent angular magnitude of the object seen without the telescope.*

Referring again to Fig. 2, we have the apparent magnitude of the image as seen through the ocular $= lcd$, and that of the object as seen by the naked eye $= lCd$, and

$$\tan \tfrac{1}{2}\, lcd : \tan \tfrac{1}{2}\, lCd = \frac{lm}{mc} : \frac{lm}{mC} = mC : mc$$

or

$$G = \frac{\tan \frac{1}{2}\, lcd}{\tan \frac{1}{2}\, lCd} = \frac{mC}{mc} \qquad (2)$$

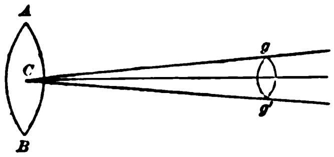
whence the proposition, (B), *The magnifying power of the telescope is equal to the quotient of the focal length of the objective divided by the focal length of the ocular.*

This principle serves for the calculation of the magnifying power when the focal lengths of the glasses are known, at least for the simple astronomical telescope here considered. A mode of obtaining the magnifying power of any telescope by direct observation will be given below.

We see then that with the same objective we can have various magnifying powers by simply varying the ocular; and the less the focal length of the ocular, the greater will be the magnifying power. The more the telescope magnifies, the *nearer* will the object appear to us, and, consequently, the more distinctly will its several parts be seen. Herein consists the *third* essential advantage in the employment of the telescope.

8. *The field of view.*—By the field of view is meant the space which can be viewed with the telescope at one and the same time. The magnitude of the field depends upon the angle gCg' (Fig. 4), which is contained by two rays from the centre of the objective to the extremities

Fig. 4.



of a diameter gg' of the ocular; and consequently it depends upon the magnitude of the ocular and its distance from the objective. Most telescopes have *diaphragms*, or opaque rings, placed within the tube to cut off rays from the extreme edges of the objective, as well as stray light falling down the tube. If the inner edge of any diaphragm trenches upon the lines Cg, Cg', the magnitude of the field will be diminished, and will then depend upon the *free aperture* of the diaphragm, or upon that portion of the ocular upon which rays from the centre of the objective can fall.

As it is difficult to construct large eye pieces which shall give as perfect images near their edges as in the centre, it is usual to obtain a large field with a small eye piece by giving the latter a sliding motion at right angles to the axis of the telescope. In this case the whole available field depends also upon the quantity of motion possessed by the eye piece. Usually this motion can be given only in one direction, in which case the whole available field is oblong, its breadth being limited by the dimensions of the eye piece, and its length by the quantity of motion. Sometimes, however, two motions are provided, at right angles to each other, and then the whole of the free circular aperture of the diaphragm becomes available for the field.

9. *Brightness of images produced by the telescope, and the intensity of their light.* The image which the telescope gives of an object must possess a sufficient degree of brightness to make an impression upon our eye. Let us suppose two telescopes, the object glasses of which are of different diameters, to have the same magnifying power. Then the brightness of the two images formed will be proportional to the quantity of light which falls on the surface of the two objectives respectively; but these surfaces are proportional to the squares of the diameters of the objectives, and hence the brightness of the images is proportional to the square of these diameters. On the other hand, let us suppose two telescopes, with object glasses of equal diameters, to have different magnifying powers; then one and the same quantity of light is distributed over the larger and over the smaller image, and, consequently, in this case the brightness of the image is inversely proportional to the square of the magnifying powers.

It is to be observed, however, that not all the rays which fall upon the object glass reach the eye, partly on account of the want of absolute transparency of the glass, and still more on account of the reflection of a number of rays from the surfaces of the lens. Some light is also lost occasionally, when the breadth of the eye glass is not sufficient to embrace all the rays which proceed in a cone from the image of a radiant point formed at the focus, or when the pupil of the eye is not large enough to receive the whole cylinder which these rays form after passing through the eye glass. Thus, in Fig. 1, let $SABS$ be the cylinder of rays from a very distant point, falling upon the free opening of the object glass; $g'k'kg$, the cylinder of light which emerges from the eye glass; F the common focus of the two glasses. On account of the similarity of the triangles ABF and $g'gF$, we have

$$AB : g'g = CF : Fc$$

But the magnifying power G is (Art. 7) equal to $\frac{CF}{Fc}$; consequently, also,

$$G = \frac{AB}{g'g}$$

Now, *all* the rays which fall upon the object glass will enter the pupil of our eye only when $g'g$ is either equal to the diameter d of the pupil, or is less than d. In the first case we shall have $G = \frac{AB}{d}$; in the second, $G > \frac{AB}{d}$. But if $G < \frac{AB}{d}$, we must have $gg' > d$, or the diameter of the cylinder of light emerging from the eye glass greater than the diameter of the pupil: in that case, therefore, some of the light must be lost to the eye.

Since every point of an object seen through a telescope must appear as a point, whatever may be the magnifying power of the telescope, it follows that the *intensity* of the illumination of the several points of the image in the telescope depends upon the quantity of light which proceeds from each point of the object and reaches our eye. We must, therefore, not confound *intensity* with the *brightness* which results from the impression of the whole image upon the eye. The intensity of the light is independent of the magnifying power, while the brightness is, as we have seen, inversely proportional to the square of the magnifying power. According to these principles, the following explanation of the working of the telescope, given by the distinguished Olbers, will be readily understood:

"Let B be the brightness, I the intensity of light of an object seen through the telescope; both being supposed to be, for the naked eye, equal to unity. Let D be the diameter of the object glass, d that of the pupil of the eye, G the magnifying power of the telescope, and $1:m$ the ratio in which the light is diminished by its passage through all the glasses of the telescope; then we have

$$B = m \cdot \frac{D^2}{d^2 G^2} \qquad I = m \cdot \frac{D^2}{d^2} \tag{3}$$

Now, so long as $G < \frac{D}{d}$, which, however, occurs only in telescopes of large objective apertures and low magnifying power, the quantity B must remain constant and $= m$; for, if G is less than $\frac{D}{d}$, the diameter of the cylinder of emergent rays from the ocular will be greater than can be received by the pupil; the eye then receives no more of the light than it would if the objective had the diameter Gd. Hence, the greatest value of B is m, and can never be greater in the telescope. Since in the best achromatic telescopes $m = 0.85$, we see that the brightness of an object is always greatest with the naked eye. As soon as G is greater than $\frac{D}{d}$, the brightness rapidly diminishes as the square of G.

"On the other hand, I, or the intensity of the light, is constant as soon as $G =$ or $> \frac{D}{d}$, provided that the field of view always includes the whole of the magnified object. I can therefore become very great when D is great; and this is the reason why exceedingly faint stars can be seen through a telescope with a large objective. The diameter d of the pupil (which may be assumed to be about 0.2 of an inch) is not only different in different observers, but also varies with the absolute intensity of the light of the object viewed,—*e.g.* it is less when we view the moon, greater when we view Saturn; less when we view the moon through a telescope of five inches aperture than through one of two inches aperture.

"The sky, or 'ground of the heavens,' has a certain degree of brightness, not only in daytime, in twilight and moonlight, but even at night in the absence of the moon. This brightness of the sky also diminishes in the telescope as $m \cdot \frac{D^2}{d^2 G^2}$, and therefore the ratio of the brightness of an observed object to the brightness of the sky remains constant for all magnifying powers. This is the reason why for considerable magnifying powers we do not observe a correspondingly great decrease of brightness. But, if we call this brightness of the sky b, although the ratio $B:b$ remains constant, our eye can, nevertheless, no longer distinguish the *difference* $B - b$ of the brightness of the object and the sky when this difference is very small. Hence, faint nebulæ, tails of comets, &c. become invisible under high magnifying powers. The intensity of the light of the portion of the sky which we see in the telescope varies inversely as G^2, nearly.* This intensity of the light of the field may be so great as wholly to prevent our seeing objects of feeble intensity. This is the reason why with the comet-seeker (a telescope of large aperture and small magnifying power) we cannot see stars, even of the first magnitude, in the daytime, when we can see them without difficulty with telescopes of much smaller apertures and greater magnifying powers. This also explains why with high magnifying powers we often discover very faint stars which are wholly invisible in the same telescope with lower powers."

The more perfect the telescope is, the more nearly will the image of a star resemble a bright point; and, according to the above, we may without hesitation always employ for the observation of fixed stars the highest magnifying powers.

10. *Spherical and Chromatic Aberration.*—A telescope of the simple construction above described would possess serious defects. All the parallel rays from an object which fall upon a simple spherical lens cannot be brought exactly to a common point in any case; and not even approximately unless the lens is small or of relatively great focal length. The image of a fixed star will, therefore, not be a well defined point, but rather an ill defined spot of light; and the images of all objects will be the more distorted the greater the objective is in proportion to the focal length. This deviation of the rays from a common point in the telescope is called the *spherical aberration.*

In the simple astronomical telescope, still another difficulty exists: for white rays of light, after they are refracted by a simple lens, are resolved into the colors of the prismatic spectrum, or of the rainbow, and, consequently, the image of any object will appear surrounded and disfigured by colored light. This arises from the different degrees of refrangibility of the different colors. The deviation of the rays of different colors from a common focus is called the *chromatic aberration.*

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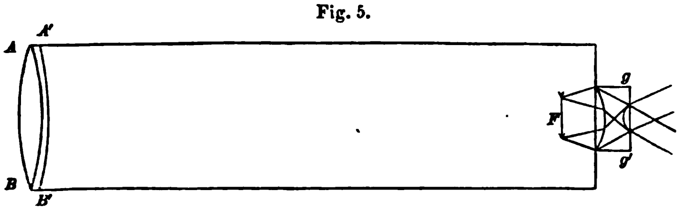
* That is, the *effect upon the eye* of the whole of the light of that portion of the sky which is visible under the magnifying power G varies nearly as $\frac{1}{G^2}$; as is evident, since the field is diminished in this ratio.

With regard to the means by which the telescope is rendered almost wholly free both from spherical and from chromatic aberration, that is, rendered both *aplanatic* and *achromatic*, it must here suffice to state, in general terms, that the result is obtained by substituting for the simple lens a compound one of which the component lenses are made of glass of different degrees of refractive and dispersive powers. There are generally two component lenses, as in Fig. 5; one of which, AB, is a biconvex lens of *crown* glass, and is that which is turned towards the object; the other, $AA'BB'$, is a meniscus or concavo-convex lens of *flint* glass. The latter kind of glass usually contains at least 33 per cent. of oxyde of lead, from which crown glass is wholly free; and both its refractive and its dispersive powers exceed those of crown glass. By giving the four spherical surfaces of the component lenses suitable curvatures, both the spherical and the chromatic aberrations produced by the crown glass lens are very nearly corrected by the flint glass lens.

Fig. 5.



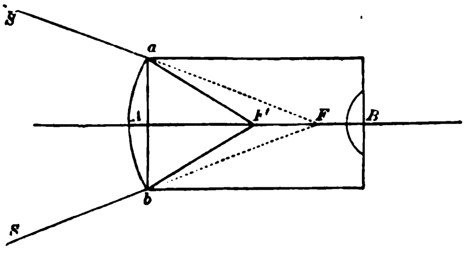
Even in the best telescopes an absolutely perfect compensation of the errors has not been reached. Some idea of the relative excellence of the instrument may readily be obtained as follows. The correction for spherical aberration is well made when the image of a star, in favorable states of the atmosphere, is a very small, well defined, round disc. Having adjusted the eye piece, by sliding it out or in, until this disc is reduced to its least dimensions and most perfectly defined, the slightest motion of the eye piece from this position, either out or in, should disturb the perfection of the image: a telescope in which the character of the image remains sensibly the same during a considerable motion of the eye piece is imperfectly corrected for the spherical aberration. The correctness of the general figure of the lens is judged of by sliding the eye piece in beyond the perfect focus, whereby the image becomes enlarged; but if the lens is symmetrical throughout, the image will remain circular, and in very perfect telescopes will present a number of complete concentric circular rings of light; a similar result should follow when the eye piece is drawn out. An imperfect, unsymmetrical lens, with the eye piece out of focus, will give an image composed of incomplete and distorted rings, or only a confused and irregular mass of variously colored light. If the glass of which the lens is composed is not perfectly homogeneous (one portion having greater refractive power than another), the images of bright stars of the first or second magnitudes will have what opticians call a *wing* on one side, which no perfection of figure or of adjustment can remove. But the defective portion of the glass may be discovered by covering up successively different parts of the lens by means of caps of variable apertures in various positions; and some improvement in the performance of the lens may be obtained by excluding this defective portion, at the expense of light.

The achromatism is judged of by pointing the telescope to some bright object, as the moon or Jupiter, and alternately pushing in and drawing out the eye piece from the place of most perfect vision: in the former case, if the lens is good, a ring of purple will appear round the edge of the image, in the latter, a ring of pale green (which is the central color of the prismatic spectrum); for these appearances show that the extreme colors of the spectrum, red and violet, are corrected.

11. *Achromatic eye pieces.*—The eye pieces now most commonly used are of two kinds: the *Huygenian* and the *Ramsden.*

The *Huygenian* eye piece consists of two plano-convex lenses of crown glass, *A* and *B* (Fig. 6), the convex surfaces of both being turned towards the object. The first lens *A* receives the converging rays *Sa*, *Sb*, coming from the object glass, before they have reached the principal focus *F* of the object glass, and brings them to a focus *F'* half-way between the two lenses

Fig. 6.



A and *B*. The focal length of the lens *B* being made equal to *BF′*, the image formed at *F′* is distinctly visible to an eye behind *B*. Since this eye piece is adapted to rays already converging, instead of diverging rays, it is commonly called the *negative* eye piece.

The *Ramsden* eye piece is shown in connection with the telescope in Fig. 5. It also consists of two plano-convex lenses; but the plane surface of the lens nearest the object is turned towards the object. The diverging rays from an image *F* are rendered less divergent by the first lens, and finally parallel by the second lens; after emerging from the latter, therefore, they are adapted for distinct vision to an eye placed behind it. This eye piece being adapted for diverging rays, like the simple double convex lens, is called the *positive* eye piece. It is universally used wherever spider threads are placed in the focus of the object glass for the purposes of measurement, as in the transit instrument, &c.; for the permanency of the position of these threads is of the first importance, and this could not be insured unless the threads were so placed as to be independent of any motion of the eye piece. Threads are, however, often placed in the focus of a *Huygenian* eye piece merely to mark the centre of the field, as in the eye pieces of the telescopes of a sextant.

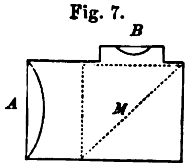
The optical qualities of the *Huygenian* eye piece are, however, superior to those of the *Ramsden*, the spherical aberration being more perfectly corrected; and it is, therefore, preferred for the mere examination of celestial objects when no measurements are to be made.

Neither of these eye pieces changes the apparent position of the image, which therefore remains inverted. Achromatic eye pieces designed to show objects in their erect positions usually consist of four lenses. They are used chiefly for land objects, and only in small telescopes. The great loss of light from the additional lenses is an insuperable objection to them for astronomical purposes.

The lenses composing the eye piece are fixed, at the proper distance from each other, in a separate tube, which has a sliding motion in another tube fixed to the telescope, so that it can be pushed in or drawn out and thus adapted for different eyes. For near-sighted persons it must be pushed in; for far-sighted persons, drawn out.

12. *Diagonal eye pieces.*—When a telescope is directed towards an object near the zenith, it is always inconvenient, and often, with small instruments, impossible, for the observer to bring his eye directly under the telescope. The inconvenience is obviated by employing an eye piece which bends the rays at right angles to the optical axis of the telescope, as in Fig. 7, where the lens *A* receives the rays in the direction of the axis of the telescope and partially refracts them; they are then reflected by the plane surface *M* (placed at an angle of 45° with the axis) to the lens *B*, by which they are rendered parallel and adapted for distinct vision to the eye at *B* looking in the direction *BM*. The surface *M* may be either a plane metallic mirror, or the interior face of a right prism of glass, the section of which is shown in the figure by the dotted lines. The prism is usually preferred, as less light is lost by reflection from its interior face than from a metallic speculum.

Fig. 7.



13. *To measure the magnifying power of a telescope.—First Method.*—The magnifying power depends upon the focal lengths of the object glass and eye piece (Art. 7), and hence for the same telescope different eye pieces will give different magnifying powers. We suppose, then, that the eye piece whose magnifying power is to be found is placed upon the telescope and very carefully adjusted for distinct vision of very distant objects. If we then direct the telescope in daytime towards the open sky, we shall see near the eye piece, and a little way beyond it, a small illuminated circle, which is nothing more than the image of the objective opening of the telescope. Let the diameter of this circle be measured by a very minutely divided scale of equal parts; then *the magnifying power is equal to the quotient arising from dividing the diameter of the object glass by the diameter of this illuminated circle.** For example, let the diameter of the object glass

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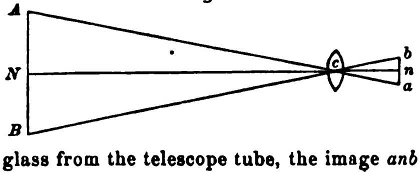
* The demonstration of this rule is not usually given in our optical works. Let *ANB*, Fig. 8, be the objective; *C* the ocular, which we can regard as in effect a single lens; *N* the middle of the objective: *n* the middle of the small illuminated circle *anb*, which is the image of the objective opening formed beyond the ocular. If we remove the object glass from the telescope tube, the image *anb* of the opening will still remain the same

Fig. 8.



be 4 inches, that of the small illuminated circle $\frac{1}{20}$ of an inch; the magnifying power is $4 \div \frac{1}{20} = 80$.

The chief difficulty in this method lies in the exact measurement of the diameter of the small illuminated circle. Various methods have been contrived for this purpose; but the most effective is by means of the instrument known as *Ramsden's Dynameter.*

Second Method (proposed by GAUSS).—If we reverse the telescope and direct the ocular towards any distant object, we shall, when looking through the objective, see the image of the object as many times diminished as we see it magnified when looking through the ocular. Select, therefore, two well defined points, lying in a horizontal line, and direct the telescope so that, looking into the objective, these points may appear to lie at about equal distances on each side of the optical axis. Then place a theodolite in front of the objective, level the horizontal circle, and bring the optical axis of its telescope nearly into coincidence with that of the larger telescope, so that looking into the objective of the latter, through the telescope of the theodolite, the selected points may be distinctly seen. Measure the apparent angular distance of the images of the points with the theodolite, by bringing the vertical thread successively upon these images and taking the difference a of the two readings of the horizontal circle. Remove the larger telescope, and measure in the same manner with the theodolite the angular distance A of the points themselves. Then the magnifying power G is given by the formula

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as when the glass is in its place. Now, it is known, from the elements of optics, that if u is the distance of a bright object from a convex lens, v the distance of the image from the lens, f the focal length of the lens, we have the equation

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

Let F be the focal length of the objective, f that of the ocular, u the distance between them; then we have $NC = u = F + f$; $Cn = v$; and, consequently,

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{F+f} = \frac{F}{f(F+f)}$$

Also,

$$\frac{AB}{ab} = \frac{NC}{nC} = \frac{F+f}{v} = \frac{F}{f}$$

But, by Art. 6, $\frac{F}{f}$ expresses the magnifying power of the telescope: hence, also, $\frac{AB}{ab}$ expresses the magnifying power, as in the method of the text.

$$G = \frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} a} \tag{4}$$

or, if the angles A and a are very small, $G = \frac{A}{a}$.

If the observed points are not very distant, we should in strictness measure the angle A by placing the theodolite at the point first occupied by the ocular; for A is the angle contained by the rays from the two points to the ocular, and a the angle contained by these rays after they have passed through the ocular and have been refracted by it.

If the telescope cannot be removed conveniently, the angle A may be obtained by measuring the linear distance D of the middle point between the two observed points from the ocular, and the horizontal linear distance d between the points; then

$$\tan \frac{1}{2} A = \frac{d}{2D} \tag{5}$$

When the latter method is practised, however, it is necessary to observe that if the telescope of the theodolite, in measuring the angle a, is inclined to the horizon by the angle I, we must employ instead of a the angle a' given by the formula

$$\sin \frac{1}{2} a' = \sin \frac{1}{2} a \cos I$$

or, with sufficient precision,

$$\tan \frac{1}{2} a' = \tan \frac{1}{2} a \cos I$$

a reduction which was unnecessary where both A and a were measured by the theodolite, since the factor cos I would enter into both numerator and denominator of (4). But the reduction may also be neglected here, if by D is understood, not the direct distance from the ocular to the observed points, but the projection of this distance on the horizontal plane, and then the formula becomes $G = \frac{d}{D \sin a}$, with sufficient precision, since a is always very small.

For accuracy, the angular distance of the points observed should be as great as can be embraced within the field of the telescope.

Example 1.—The angles A and a were directly measured with a theodolite, in the case of an equatorial telescope with a certain eye piece, and were $A = 5° 10' 30''$, $a = 3' 10''$. We have, therefore, for this eye piece,

$$G = \frac{\tan 2° 35' 15''}{\tan 0° 1' 35''} = 98.12$$

Example 2.—For verification of the preceding measure, the angle A was also obtained without the theodolite, for which purpose there was measured the distance of the observed points from the ocular, $D = 303.2$ feet, and the distance between the points, $d = 26.98$ feet. The inclination of the telescope of the theodolite was here observed to be $I = 10° 40'$, and as before by direct measure $a = 3' 10''$. We have first,

$$\tan \tfrac{1}{2} A = \frac{26.98}{606.4}$$

and hence

$$G = \frac{26.98}{606.4 \tan 1' 35'' \cos 10° 40'} = 98.30$$

The *horizontal* distance D was here 298 feet, with which, by the last formula above given, we have

$$G = \frac{26.98}{298 \sin 3' 10''} = 98.29$$

The magnifying power of this eye piece may therefore be taken at 98.2, or simply 98.

Third Method (proposed by H. B. Valz, in the *Astronomische Nachrichten*, Vol. vii). This very convenient method consists in directing the telescope towards any object of known angular diameter, and measuring the angle formed by rays from the extremities of a diameter after these rays have emerged from the eye piece. The sun, the angular diameter of which is always known, is especially adapted for the purpose. The image of the sun may be received upon a screen placed in the prolongation of the axis of the telescope with its flat surface carefully adjusted at right angles to that axis. The telescope is to remain fixed, being properly directed so that the sun shall pass over the centre of its field; and as the image passes over the screen its linear diameter d is to be measured. Also the perpendicular distance D from the middle of the eye piece to the screen. Then, if a is the true angular diameter of the sun, A the angular diameter of the image on the screen, subtended at the eye piece, we have

$$\tan \tfrac{1}{2}A = \frac{\tfrac{1}{2}d}{D}$$

and the magnifying power G, as before, is

$$G = \frac{\tan \tfrac{1}{2}A}{\tan \tfrac{1}{2}a} = \frac{d}{2D \tan \tfrac{1}{2}a} \qquad (6)$$

Fourth Method.—For small instruments, and where great accuracy is not required, the following process will answer. Let a staff, which is very boldly divided into equal parts by heavy lines, be placed vertically at any convenient distance from the telescope, for example, fifty yards. While one eye is directed towards the staff *through* the telescope, the other eye may observe the staff by looking along the outside of the tube. One division of the staff will be seen by the eye at the eye piece to be magnified, so as to cover a number of divisions of the staff, and this number, which is the magnifying power required, may be observed by the other eye looking along the tube. The staff here not being very distant, the focal adjustment of the telescope is not the same as for stars; the focal length is, in fact, somewhat greater than the "principal" focal length (Art. 3), and the magnifying power obtained is proportionally greater than that which applies to very distant or celestial objects, the rays from which are sensibly parallel. If we call the magnifying power obtained from the terrestrial object G', that for a celestial object G, F' the focal length employed, F the principal focal length, we have

$$F' : F = G' : G$$

For example, a telescope whose principal focal length was 24 inches, being directed towards a graduated staff, it was found that for distinct vision of the staff it was necessary to draw out the eye piece 0.75 inch. Then, one division of the staff seen by the eye at the eye piece was observed by means of the other eye to cover 40 divisions. Here we have $F = 24$, $F' = 24.75$, $G' = 40$, and hence

$$G = G' \cdot \frac{F}{F'} = 40 \times \frac{24}{24.75} = 38.8$$

Instead of using the divisions of a staff, which may not be sufficiently distinct, a disc of white paper may be placed against a black ground, and the size of the magnified image may be marked on the same ground by an assistant from signals made by the observer at the telescope.

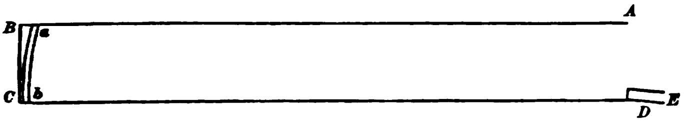
14. It was shown in Art. 7 that the magnifying power is equal to $\frac{F}{f}$, F being the focal length of the objective, and f that of the ocular. To apply this rule when the eye piece is composed of two lenses, it is necessary to find the focal length, f, of a single lens which is equivalent to the two lenses. This is effected by the formula of optics

$$f = \frac{f'f''}{f' + f'' - d}$$

in which f', f'' are the focal lengths of the component lenses, and d the distance between them. This formula, however, is but approximative (it gives f somewhat too great): it is better to measure the magnifying power directly by one of the methods above given.

15. *Reflecting telescopes.*—As these are rarely used for the purposes of *measurement*, we shall content ourselves with merely stating the forms of the two kinds which have been in most common use. The simplest, and now most commonly used, is the *Herschelian* telescope, introduced by Sir William Herschel. A polished concave speculum, ab, Fig. 9, is placed at the bottom of a tube, $ABCD$. It is ground to the form of a paraboloid, the focus of which is near the mouth of the tube; it is slightly inclined, so that the focus falls near one side of the tube, as at D, where the reflected rays from the speculum form an image which is viewed through an eye piece, E, of the usual form. The head of the observer may intercept a small portion of the rays from a celestial object to the speculum; but this is of little conse-

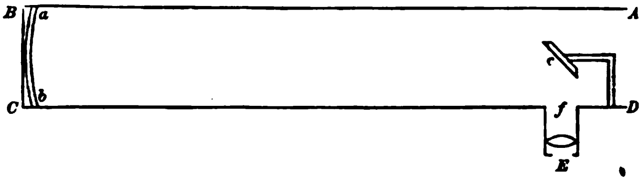
Fig. 9.



quence, as the speculum is usually very large. In Lord Rosse's Herschelian, the diameter of the speculum is six feet.

The reflecting telescope next in most common use is the *Newtonian*, which differs from the Herschelian only in receiving the reflected rays from the speculum upon a small plane mirror, *c*, Fig. 10, placed in the middle of the tube near its mouth, which reflects these rays at right angles to the axis of the tube to an eye piece at *E*. In this form, the small plane mirror intercepts a portion of the light from the object; moreover, light is lost in the double reflection; but a slight advantage is gained in having the axis of the speculum coincide in direction with the axis of the tube. The reflected rays reach the mirror *c* before they are brought to a focus: they converge after reflection to the point *f*, where is produced the image which is examined through an eye piece by the eye at *E*.

Fig. 10.



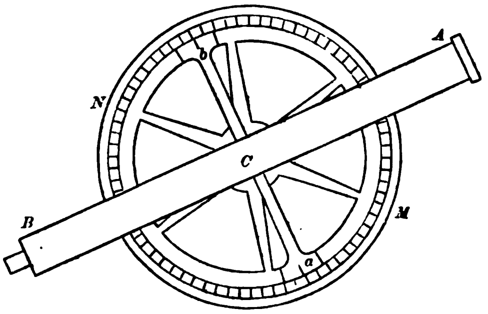
16. *Finding telescopes.*—A telescope of great focal length and high magnifying power has a very small field, in consequence of which it becomes very difficult to find a small object in the sky. This inconvenience is obviated by attaching to the outside of the tube a smaller telescope, called a *finder*, of low magnifying power and large field, with its axis adjusted parallel to that of the larger telescope. The search for the object is made with the finder (both telescopes having a common motion), and, when found, it is brought to the middle of the field of the finder; it is then somewhere in the field of the larger telescope. The middle of the field of the finder is indicated by the intersection of two coarse threads in the focus; or, still better, by four threads forming a small square, the middle point of which is the centre of the field.

CHAPTER II.

OF THE MEASUREMENT OF ANGLES OR ARCS IN GENERAL—CIRCLES—MICROMETERS—LEVEL.

17. *Graduated Circles.*—The most obvious mode in which an angle may be measured is that in which we employ a circle, or portion of a circle (constructed of metal or other durable material), the limb of which is mechanically divided into equal parts, as degrees, minutes, &c. The centre of the circle being placed at the vertex of the angle to be measured, the arc of the circumference intercepted between the two radii which coincide in direction with the sides of the angle is the required measure.* To give this mode precision when the angle is found by lines drawn to two distant points, the aid of the telescope is invoked. This is connected with the circle in various ways, according to the nature of the instrument of which it forms a part; but, in general, we may conceive it to be essentially as follows. To the tube of the telescope, *AB*, Fig. 11, is attached a pivot, *C*, at right angles to the optical axis, which turns in a circular hole in the centre of the graduated circle *MN*. An arm *aCb*, extending from the centre *C* to the graduations on the limb, is permanently attached to the telescope, and revolves with it. To measure an angle subtended by two distant objects at the point *C*, the circle is to be brought into the plane of the objects and firmly fixed. Then the telescope is

Fig. 11.



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* In the sextant and other instruments of "double reflection," the vertex of the angle to be measured is not in the centre of the arc used to measure it. See article "Sextant."

directed successively upon the two objects, and in each case the number of degrees indicated by a mark on either extremity of the arm *ab* is to be read off; the difference of the two readings, which is the number of degrees passed over by the arm, and, consequently, also by the telescope, will be the required measure of the angle. The same result is reached by permanently connecting the circle and telescope, which then revolve together, while a fixed mark near the limb of the circle serves to indicate the number of degrees through which the telescope revolves.

In order to point the telescope with ease and accuracy upon an object, a *clamp* and *tangent screw* are commonly employed. This contrivance, which may be seen upon almost every astronomical instrument, takes a great variety of forms, but in all cases the operation of it is as follows: when the telescope is *approximately* pointed upon the object by hand, it is clamped in its position by a slight motion of the clamp screw, after which the telescope admits of no motion except that which is common to it *and the clamp:* hence, by a fine screw which moves the clamp a slow delicate motion can be given to the telescope, whereby the sight-line marked by a thread in the focus is brought accurately upon the object.

The great increase of accuracy in *pointing* a telescope which is obtained by the introduction of the spider threads in its focus brings with it the necessity of a corresponding increase of accuracy in *reading off* the number of degrees and fractions of a degree on the divided limb of the circle. A single reference mark upon the extremity of an arm, as in Fig. 11, enables us to determine only the number of *entire* divisions of the limb passed over; but, as this mark will generally be found between two divisions, some additional means are required for measuring the fraction of a division. Two methods are now exclusively employed. The first of these, in the order of invention, is

THE VERNIER.*

18. Let *MN*, Fig. 12, be a portion of the divided limb of a circle; *CD* the arm which revolves with the telescope about the centre of the circle. The extremity of this arm is expanded

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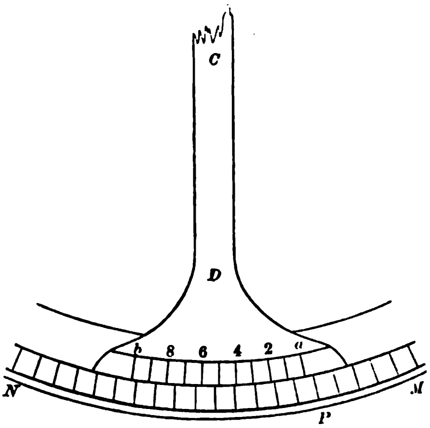
* So called after its inventor, PETER VERNIER, of France, who lived about 1630. By some it is called a *nonius*, after the Portuguese NUÑEZ or NONIUS; but the invention of the latter (who died in 1577) was quite different.

Fig. 12.



into an arc *ab*, which is concentric with the circle and is graduated into a number of divisions n which occupy the space of $n - 1$ divisions of the limb. Thus graduated, this small arc receives the name of a *vernier*. The first stroke a is the zero of the vernier, and the reading is always to be determined by the position of this zero on the limb. Let us put

d = the value of a division of the limb,
d' = the value of a division of the vernier,

then we have

$$(n - 1)\, d = nd'$$

whence

$$d' = \frac{n-1}{n}\, d$$

and

$$d - d' = \frac{1}{n}\, d \tag{7}$$

The difference $d - d'$ is called *the least count* of the vernier, which is, therefore, $\frac{1}{n}$th of a circle division. If now the zero a falls between the two circle graduations P and $P + 1$, the whole reading is Pd *plus* the fraction from P to a. To measure this fraction, we observe that if the mth division of the vernier is in coincidence with a division of the limb, the fraction is $m \times (d - d')$ or $\frac{m}{n} d$. For example, if, as in our figure, the vernier is divided into 10 equal parts, occupying the space of 9 divisions of the limb, and if the 4th division is in coincidence, the whole reading is $Pd + \frac{4}{10} d$; and if $d = 10'$ and P corresponds to $20° 20'$ (P being the 122d division from the zero of the limb), then the whole reading is $20° 20' + \frac{4}{10} \times 10' = 20° 24'$. In this case the least count is $1'$. In practice, no calculation is necessary to obtain the fraction, for this is indicated by proper numbers against the graduations of the vernier itself.

If the least count is given, to find n, we have

$$n = \frac{d}{d - d'}$$

d and $d - d'$ being, of course, expressed in the same unit. For example, if the limb is divided to 10′, and the least count is to be 10″, we have

$$d = 600''$$
$$d - d' = 10''$$

whence

$$n = 60$$

and we must make 60 divisions of the vernier equal to 59 divisions of the limb.

When a large number of divisions are made on the vernier, and the least count is very small, the graduations must be exceedingly delicate; otherwise, several consecutive divisions of the vernier may appear to be in coincidence with divisions of the limb. The reading is then to be assisted by a microscope, or *reading glass*, placed over the vernier and having a lateral motion, whereby its optical axis can be brought immediately over that division of the vernier which is in coincidence.

To increase the accuracy of a reading still more, two or more arms, each carrying a vernier, are employed, and the mean of the indications of all is taken. The effect of reading off a circle at various points, in eliminating errors of the circle, will be treated of hereafter.

The arm carrying a vernier, or the frame bearing several verniers, is often called the *alidade*. Sometimes the several verniers are attached to a circle, which then receives the name of the *alidade circle*.

19. We have assumed above that the divisions on the vernier are smaller than those on the limb. This is the most common arrangement; but we may also have them greater by making n divisions of the vernier occupy the space of $n + 1$ divisions of the limb: so that we have

$$(n + 1)\, d = nd'$$

whence the least count is, as before,

$$d' - d = \frac{1}{n}\, d$$

The only difference will be, that when the graduations of the limb proceed from right to left, those of the vernier must proceed from left to right; that is, the zero of the vernier must be the extreme left-hand stroke.

20. In case a vernier has been used which is found to be too long or too short, the reading may be corrected as follows. Let the error in its length be denoted by x, then (in the verniers of the ordinary form) we have (Art. 18)

$$(n-1)d = nd' + x$$

whence

$$d - d' = \frac{1}{n}d + \frac{x}{n} \tag{8}$$

Hence a reading in which the fraction was $m(d-d')$ becomes $\frac{m}{n}d + m\cdot\frac{x}{n}$. The *correction* of the reading is, therefore, $+\,m\cdot\frac{x}{n}$ when the vernier is too short by x; and $-\,m\cdot\frac{x}{n}$ when it is too long by x. For example, if the limb is divided to 10′ and the vernier gives 10″ (in which case $n = 60$), and we find that the vernier is too short by $x = +\,5''$, then we must add to every reading the correction $+\,m\cdot\frac{5''}{60}$; or, since every 6th graduation of the vernier gives one minute, we must add 0″.5 for every minute read on the vernier.

The actual length of the vernier is found by bringing its zero into coincidence with a division of the limb and observing where the next coincidence occurs. If this second coincidence occurs at the last division of the vernier, its length is correct; but if the coincidence occurs at $\pm\,p$ divisions from the last, it is too short or too long by p times the least count. This should be done at various points of the limb, and the mean of all the results taken, in order to eliminate the effect of accidental errors in the graduations of the limb.

The vernier is now used chiefly on small circles and portable instruments; but when the highest degree of accuracy is sought for in reading off a circle, we have recourse to

THE READING MICROSCOPE.

21. Let us conceive the arm which carried the vernier, instead of lying close to the plane of the circle, to be raised at some distance from it, and in place of the vernier let the extremity of

the arm carry a microscope AC (Plate II. Fig. 1), the optical axis of which is perpendicular to the plane of the circle MN and intersects the divisions on the limb. The telescope and circle are to be supposed to revolve together, while the microscope remains fixed. An image of the divisions is formed at the focus D of the object lens C. Two lenses, B and A, constitute a positive eye piece through which this image is viewed. HG is a micrometer, the interior of which is shown, enlarged, in Plate II. Fig. 2. A fine screw, *cc*, with a large graduated head, EF, carries the sliding frame *aa*, across which are stretched two intersecting spider threads. These threads lie exactly in the focus of the microscope, and are consequently visible at the same time with the image of the divisions of the limb. On one side of the field is a notched scale of teeth (which does not move with the cross-threads), the distance between the teeth being the same as that between the threads of the screw. The middle notch is distinguished by a hole opposite to it, and every fifth notch is cut deeper than the rest. At *i* (Fig. 1) is an index to which the divisions of the micrometer head are referred. Since one complete revolution of the micrometer head must carry the cross-threads a distance equal to the thickness of the thread of the screw, if the head is graduated into 100 parts we have the means of measuring a space equal to $\frac{1}{100}$th of the thickness of the thread of the screw. Either by making the screw very fine, or increasing the number of graduations on the head, or by both, and at the same time increasing the optical power of the microscope, we can carry this subdivision of space to almost an unlimited extent.

In order to understand the mode of reading the circle by this apparatus, let us conceive the intersection of the cross-threads to stand against the central notch, the zero of the micrometer being also exactly opposite the index. *The point of the field then occupied by the intersection of the cross-threads is to be regarded as a fixed point of reference, and, as the telescope revolves from one position to another, the number of divisions of the limb which pass by this point will be the measure of the angular motion of the telescope.* Suppose, then, the revolution has brought this point, not upon a graduation of the limb, but at a fraction of a division beyond a certain graduation P; then, to measure this fraction, we have only to move the cross-thread from the point of reference into coincidence with the graduation P, and read the number of divisions of the micrometer head. If more than one revolution of the screw is required, the whole number of revolutions will be shown by the number of notches in the field passed over by the cross-threads, and the fraction of a revolution by the micrometer head. Then, knowing the relation between a division of the micrometer head and one of the circle, the value of the required fraction is at once found. For example, suppose a division of the circle is equal to 5′, and that five revolutions of the micrometer screw just carry the cross-threads from one circle graduation to the next; and, further, that the micrometer head is divided into 60 equal parts; then each revolution of the screw represents 1′, and each division of the micrometer head represents 1″. If then we have made three whole revolutions, and the micrometer head reads 25.3, the required fraction is 3′ 25″.3. If the graduation *P* was 289° 35′, the whole reading is 289° 38′ 25″.3.

The coincidence of the point of intersection of the threads with a graduation of the limb is made in the manner shown in Fig. 2. In many of the German instruments, instead of a cross-thread, two very close parallel threads are used, the middle point between which is the point of reference, and a coincidence is made by bringing the circle division to bisect the space between them. This bisection is, of course, estimated; but it may be effected with very great accuracy where the threads are very close. Their distance should be very little greater than the breadth of the graduations of the limb. Bessel preferred the parallel threads; but it is, perhaps, doubtful whether they afford any advantage in the hands of most observers.

The spiral springs *bb* serve to make the screw bear always on the same side of the thread, so that in reverse motions of the screw there is no lost or *dead* motion, that is, revolution of the screw without a corresponding movement of the cross-threads. But, to guard against the possible existence of lost motion, the coincidence of the cross-threads with a circle division should always be produced by a motion of the micrometer head in one and the same direction.

22. *Error of Runs.*—When a reading microscope is in perfect adjustment, a whole number of the revolutions of the screw is equal to the distance of two consecutive graduations of the circle. To effect this, provision is made for lengthening or shortening the microscope tube, and also for moving the whole microscope

farther from or nearer to the circle. In this way, the magnitude of the image of a division as seen in the field can be changed until it corresponds exactly to a whole number of revolutions of the screw. For example, if a whole number of revolutions is greater than the image of a circle division, the objective lens must be brought nearer to the ocular, and at the same time the whole microscope brought nearer to the circle.

But, as changes of temperature and other causes are found to produce changes in the value of a division of the microscope, and it is not expedient to be always changing the adjustment, it is usual, after making one very exact adjustment, to let it stand, and then determine from time to time the correction of a reading for any change of value which may appear. The excess of a circle division above a whole number of revolutions is called *the error of runs*, and a proportional part of this excess must be allowed on all readings. This error is to be found by measuring several divisions in different parts of the circle and taking the mean of all the results, in order to eliminate the effect of errors in the circle graduations themselves. For example, if a division exceeds five revolutions of the screw by $+ 2''.2$, then for each minute in the fraction of a division obtained by the micrometer we must apply to the reading the correction $-\frac{2''.2}{5}$, or $- 0''.44$. The error of runs will take the negative sign, and the correction for it the positive sign, when a circle division falls short of a whole number of revolutions of the screw.

23. To increase the accuracy of a reading, several microscopes are used, having a fixed position relatively to each other, by which the fraction of a division in the reading is measured at different points of the circle and the mean of the different measures is taken. Two microscopes are placed so as to read at opposite points of the circle, that is, the angular distance of the microscopes is 180°, or differs but little from 180°; three microscopes are placed at 120°, four at 90°, &c.; or, in general, whatever the number of microscope, they are placed so as to divide the circle into equal portions. The whole degrees and minutes are read only at one of the microscopes. In large instruments, where the field of the microscope takes in but a part of a degree, the number of degrees and minutes of the nearest circle division is read off by means of an index outside the microscope, or,

indeed, wholly separate from it, the microscope being used exclusively to measure the fraction of a division.

24. The probable error of a reading of one microscope being ε, that of the mean of m microscopes ε_0, we have (Appendix, *Method of Least Squares*)

$$\varepsilon_0 = \frac{1}{m}\sqrt{m\varepsilon\varepsilon} = \frac{\varepsilon}{\sqrt{m}}$$

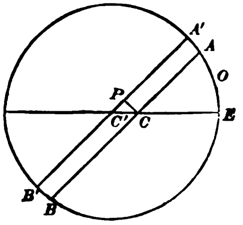
that is, the probable error of the mean varies inversely as the square root of the number of microscopes. For example, if the probable error of reading of one microscope is 1″, that of the mean of two will be $\frac{1''}{\sqrt{2}} = 0''.71$; that of four, $\frac{1''}{\sqrt{4}} = 0''.5$; that of six, $\frac{1}{\sqrt{6}} = 0''.41$, &c.; and the error will decrease but slowly as the number of microscopes increases. It would require sixteen microscopes to reduce the error to 0″.25. On this account, the advantages of increasing the number of microscopes beyond four, except in instruments of the largest class, are usually regarded as outweighed by the greater liability of the apparatus to derangement.

The use of a number of microscopes or verniers is, however, not solely to increase the accuracy of reading, but also to eliminate the errors of the circle itself, as will be seen in the following articles.

ECCENTRICITY OF GRADUATED CIRCLES.

25. The centre of the alidade is seldom, if ever, even in the best instruments, exactly coincident with the centre of the graduated arc. To investigate the effect of such eccentricity, let C (Fig. 13) be the centre of the alidade, C' that of the circle; CA a straight line joining C and the centre of one of the reading microscopes; $C'A'$ a parallel to CA. When the microscope reading is at A, the true reading is at A'. Let the diameter drawn through C and C' intersect the graduation at E, and let O be the zero of the graduation, which we will suppose is numbered from O towards A. Put

Fig. 13.



z = the microscope reading,
z' = the true reading,
$E = EO$,
e = the eccentricity CC'.

It is to be assumed that such care has been bestowed upon the centring of the instrument that e is very small, and, therefore, that the arc $AA' = z' - z$ may be regarded as equal to the perpendicular CP: so that we have, since the angle $EC'A' = z' + E$,

$$z' - z = e \sin (z' + E) \tag{9}$$

in which e must be expressed in arc. In the factor $\sin (z' + E)$ we may substitute z for z' without sensible error.

When $z' + E = \pm 90°$, we have $z' - z = \pm e$: so that e is the maximum error of a reading, and this maximum occurs when the reading is 90° from E.

26. Now, let AC and $A'C'$ be produced to meet the graduation again at the opposite points B and B', and let the alidade carry a second microscope at B. The degrees and minutes may be supposed to be obtained from the microscope A, while B is used only to give the seconds. Put

z = the division of the circle under A,
A and B = the readings of the microscopes,
z' = the true reading corresponding to A.

Then the whole reading given by A is $z + A$, and by (9) we have

$$z' = z + A + e \sin (z + E)$$

and the microscope B gives

$$180° + z' = 180° + z + B + e \sin (180° + z + E)$$

or

$$z' = z + B - e \sin (z + E)$$

The mean of the two microscopes is then

$$z' = z + \tfrac{1}{2}(A + B)$$

Hence the eccentricity is fully eliminated by taking the mean of two microscopes 180° apart. In general, an even number of microscopes are employed, which are arranged in pairs, so that the mean of each pair, and, consequently, of the whole, will be free from the eccentricity.

27. The eccentricity may also be eliminated by three microscopes or verniers, whose mutual distance is 120°. If $z + A$,

$120° + z + B$, $240° + z + C$ are the readings of the three microscopes, the true reading corresponding to A will be

$$\begin{aligned} z' &= z + A - e \sin(z + E) \\ z' &= z + B - e \sin(120° + z + E) \\ z' &= z + C - e \sin(240° + z + E) \end{aligned}$$

and since, by Pl. Trig., we have

$$\sin(120° + z + E) + \sin(240° + z + E) = -\sin(z + E)$$

the mean of these three equations is

$$z' = z + \tfrac{1}{3}(A + B + C)$$

Indeed, it will readily be inferred from the discussion in Arts. 31 and 32 that the eccentricity will be eliminated by taking the mean of any number whatever of equidistant microscopes.

28. *To find the eccentricity.*—The two opposite microscopes may not be perfectly adjusted at the distance of 180°, and hence we shall here put

$$180° + \alpha = \text{the angular distance of the microscope } B \text{ from } A;$$

and then, if we put, as before,

$$\begin{aligned} z &= \text{the division under the microscope } A, \\ A \text{ and } B &= \text{the readings of the two microscopes,} \end{aligned}$$

the true readings will be

$$\left.\begin{aligned} z' &= z + A + e \sin(z + E) \\ 180° + \alpha + z' &= 180° + z + B + e \sin(180° + z + E) \end{aligned}\right\} \quad (10)$$

for the second of which we take

$$z' = z + B - \alpha - e \sin(z + E)$$

If, therefore, we put

$$B - A = n$$

the difference of the two equations gives the equation of condition

$$n = \alpha + 2e \sin(z + E) \qquad (11)$$

in which α, e, and N are unknown. Let the values of n be obtained from the readings of both microscopes at four equidistant points of the circle, namely, z_0, $z_0 + 90°$, $z_0 + 180°$, and $z_0 + 270°$, and denote these values by n_0, n_1, n_2, n_3, respectively: then, by putting

$$P = z_0 + E,$$

we have

$$\begin{aligned} n_0 &= \alpha + 2e \sin P &&= \alpha + 2e \sin P \\ n_1 &= \alpha + 2e \sin (P + 90°) &&= \alpha + 2e \cos P \\ n_2 &= \alpha + 2e \sin (P + 180°) &&= \alpha - 2e \sin P \\ n_3 &= \alpha + 2e \sin (P + 270°) &&= \alpha - 2e \cos P \end{aligned}$$

whence

$$\left.\begin{aligned} 4e \sin P &= n_0 - n_2 \\ 4e \cos P &= n_1 - n_3 \end{aligned}\right\} \quad (12)$$

which determine both e and P, after which we have $E = P - z_0$. The value of α is evidently the mean of the values of n.

EXAMPLE.

The readings of a pair of opposite microscopes of the Repsold Meridian Circle of the U. S. Naval Academy were as follows:

z	A	B	Values of $n = B - A$
0°	+ 4″.0	− 6″.7	$n_0 = -10''.7$
90	+ 6 .9	− 13 .6	$n_1 = -20 .5$
180	+ 5 .3	− 16 .5	$n_2 = -21 .8$
270	− 1 .2	− 1 .2	$n_3 = \quad 0 .0$

From these we obtain

$$\begin{aligned} 4e \sin P &= +11''.1 & \log\ & 1.0453 \\ 4e \cos P &= -20''.5 & \log\ & n1.3118 \\ P &= 151° \, 34' & \log \tan P\ & n9.7335 \\ e &= 5''.83 & \log 4e\ & 1.3676 \end{aligned}$$

Hence, since $z_0 = 0°$, we have $E = 151° \, 34'$, and any single reading of the microscope A requires the correction for eccentricity

$$+ 5''.83 \sin (z + 151° \, 34')$$

The mean of the values of n gives $\alpha = -13''.25$, and the angular distance of the microscope B from A is 179° 59′ 46″.75.

The same process may be used for any other four equidistant points of the circle, and the mean of the various results may be taken.

29. With three nearly equidistant microscopes the eccentricity can be found from two complete readings at points 180° apart. Let the angular distances of the microscopes B and C from A be denoted by β and γ; and, z being the division under A, put $P = z + E$; then we have, for the true reading at A,

$$\begin{aligned} z' &= z + A + e \sin P \\ z' &= z + B - \beta + e \sin (P + 120°) \\ z' &= z + C - \gamma + e \sin (P + 240°) \end{aligned}$$

Subtracting the first equation from the mean of the other two, and putting

$$\tfrac{1}{2}(B + C) - A = n$$

we find

$$n = \tfrac{1}{2}(\gamma + \beta) + \tfrac{3}{2}\, e \sin P$$

and subtracting the second from the third, and putting

$$\tfrac{1}{2}(C - B) = d$$

we find

$$d = \tfrac{1}{2}(\gamma - \beta) + \tfrac{1}{2}\sqrt{3}\, e \cos P$$

If we read a second time with the microscope A over the division $z + 180°$, and obtain the readings A', B', C', we shall have

$$\begin{aligned} \tfrac{1}{2}(B' + C') - A' &= n' \\ \tfrac{1}{2}(C' - B') &= d' \end{aligned}$$

and since we shall have $180 + P$ instead of P, we shall obtain

$$\begin{aligned} n' &= \tfrac{1}{2}(\gamma + \beta) - \tfrac{3}{2}\, e \sin P \\ d' &= \tfrac{1}{2}(\gamma - \beta) - \tfrac{1}{2}\sqrt{3}\, e \cos P \end{aligned}$$

Hence

$$\begin{aligned} e \sin P &= \tfrac{1}{3}(n - n') \\ e \cos P &= \tfrac{1}{3}\sqrt{3}\,(d - d') \end{aligned}$$

which determine e and P. We find also

$$\begin{aligned} \beta &= \tfrac{1}{2}(B - A + B' - A') \\ \gamma &= \tfrac{1}{2}(C - A + C' - A') \end{aligned}$$

30. In order to determine the eccentricity with greater accuracy, and to eliminate, as far as possible, errors in reading and accidental errors of graduation, the circle may be read at a great number of equidistant points. Each reading of a pair of opposite verniers or microscopes furnishes an equation of condition of the form (11), and from all these equations the most probable value of the eccentricity will be deduced by the method of least squares. The computation according to this method is rendered extremely simple by the application of some theorems relating to periodic functions, which, on account of their utility in this and similar investigations, will be here demonstrated.

31. *Periodic Functions.*—The circumference of a circle being denoted by 2π, any commensurable fractional portion of it may be expressed by $2\pi \times \frac{p}{q} = \frac{2p\pi}{q}$, p and q being whole numbers; and the successive multiples of this fractional portion by $m \cdot \frac{2p\pi}{q}$, by supposing m to take successively the values 0, 1, 2, 3, &c. If now we consider only the multiples from $m = 0$, to $m = q - 1$, we shall have the following theorems:

THEOREM I.—*When p is not a multiple of q,*

$$\Sigma \sin m \cdot \frac{2p\pi}{q} = 0 \tag{13}$$

$$\Sigma \cos m \cdot \frac{2p\pi}{q} = 0 \tag{14}$$

but, when p is a multiple of q,

$$\Sigma \sin m \cdot \frac{2p\pi}{q} = 0 \tag{15}$$

$$\Sigma \cos m \cdot \frac{2p\pi}{q} = q \tag{16}$$

where the summation sign Σ is used to denote the sum of all the quantities of the given form *between the given limits*, namely, from $m = 0$ to $m = q - 1$.

To prove this, put

$$\cos \frac{2p\pi}{q} + \sqrt{-1} \sin \frac{2p\pi}{q} = T$$

then, by MOIVRE's formula [Pl. Trig. (440)],

$$\cos m . \frac{2p\pi}{q} + \sqrt{-1} \sin m \cdot \frac{2p\pi}{q} = T^m$$

Taking the sum of all the expressions of this form from $m = 0$, to $m = q - 1$, we have

$$\Sigma \cos m \cdot \frac{2p\pi}{q} + \sqrt{-1}\, \Sigma \sin m . \frac{2p\pi}{q} = \frac{T^q - 1}{T - 1} \tag{17}$$

But we have again, by MOIVRE's formula,

$$T^q = \cos 2p\pi + \sqrt{-1} \sin 2p\pi = 1$$

d, consequently, $T^q - 1 = 0$. The second member of the ove formula, therefore, becomes zero, unless the denominator $- 1$ is zero, that is, unless $T = 1$. Now, we can have $T = 1$ only hen $\sin\frac{2p\pi}{q} = 0$ and $\cos\frac{2p\pi}{q} = 1$, that is, only when p is a mulple of q. In all other cases we have, therefore,

$$\Sigma \cos m \cdot \frac{2p\pi}{q} + \sqrt{-1}\, \Sigma \sin m \cdot \frac{2p\pi}{q} = 0$$

nd, since the real and the imaginary terms must here be sepaately equal to zero, the first part of our theorem is established.

When $T = 1$, the second member of (17) becomes $\frac{0}{0}$, but is not really indeterminate; for, going back to the geometric progression of which this is the sum, we have

$$\frac{T^q - 1}{T - 1} = T^0 + T^1 + T^2 + \ldots\ldots + T^{q-1} = q$$

and hence, *when p is a multiple of q*, we have

$$\Sigma \cos m \cdot \frac{2p\pi}{q} + \sqrt{-1}\, \Sigma \sin m \cdot \frac{2p\pi}{q} = q$$

which establishes the second part of the theorem.

THEOREM II.—*When 2p is not a multiple of q,*

$$\Sigma\left(\sin m \cdot \frac{2p\pi}{q}\right)^2 = \tfrac{1}{2} q \qquad (18)$$

$$\Sigma\left(\cos m \cdot \frac{2p\pi}{q}\right)^2 = \tfrac{1}{2} q \qquad (19)$$

but, when 2p is a multiple of q,

$$\Sigma\left(\sin m \cdot \frac{2p\pi}{q}\right)^2 = 0 \qquad (20)$$

$$\Sigma\left(\cos m \cdot \frac{2p\pi}{q}\right)^2 = q \qquad (21)$$

For we have, for any angle x,

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2} \cos 2x$$

and, therefore,

$$\Sigma\left(\sin m \cdot \frac{2p\pi}{q}\right)^2 = \Sigma\left(\tfrac{1}{2} - \tfrac{1}{2} \cos m \cdot \frac{4p\pi}{q}\right)$$

$$= \tfrac{1}{2} q - \tfrac{1}{2} \Sigma \cos m \,.\, \frac{4p\pi}{q}$$

which, by Theorem I., gives either (18) or (20). Again

$$\Sigma\left(\cos m\cdot\frac{2p\pi}{q}\right)^2=\Sigma\left[1-\left(\sin m\cdot\frac{2p\pi}{q}\right)^2\right]$$
$$=q-\Sigma\left(\sin m\cdot\frac{2p\pi}{q}\right)^2\cdot$$

which gives either (19) or (21).

THEOREM III.—*For all integral values of p and q we have, from $m=0$ to $m=q-1$,*

$$\Sigma\sin m\cdot\frac{2p\pi}{q}\cos m\cdot\frac{2p\pi}{q}=0 \qquad (22)$$

for this is the same as the quantity

$$\tfrac{1}{2}\,\Sigma\sin m\cdot\frac{4p\pi}{q}=0$$

32. Now, let the circle be read off by a pair of opposite microscopes, A and B, at any number of equidistant points. The circle is thus divided into a number of equal parts, each of which may be denoted by $\frac{2\pi}{q}\cdot$ If the first reading corresponds to the division z_0, the subsequent readings will correspond to $z_0+\frac{2\pi}{q}$, $z_0+2\cdot\frac{2\pi}{q}$, $z_0+3\cdot\frac{2\pi}{q}$, &c. to $z_0+(q-1)\frac{2\pi}{q}\cdot$ Each reading furnishes an equation of condition of the form (11), giving, therefore, the following system, where $P=z_0+E$:

$$n_0=a+2e\sin P$$
$$n_1=a+2e\sin\left(P+\frac{2\pi}{q}\right)$$
$$n_2=a+2e\sin\left(P+\frac{4\pi}{q}\right)$$
" " "
" " "
$$n_{q-1}=a+2e\sin\left(P+\frac{2(q-1)\pi}{q}\right)$$

which are all included in the general form

$$n_m=a+2e\sin\left(P+\frac{2m\pi}{q}\right)$$

m being taken from 0 to $q-1$.

Developing the sine in the second member, we have

$$n_m=a+2e\sin P\cos\frac{2m\pi}{q}+2e\cos P\sin\frac{2m\pi}{q}$$

this form, the three unknown quantities are α, $e \sin P$, and s P. The final equation in each unknown quantity, according the method of least squares, is to be found by multiplying h equation of condition by the coefficient of the unknown antity in that equation, and adding together the products. is process gives, by the aid of the theorems of the preceding icle (observing that here $p = 1$),

$$\left.\begin{aligned} q\alpha &= \Sigma\, n_m \\ qe \sin P &= \Sigma\left(n_m \cos \frac{2m\pi}{q}\right) \\ qe \cos P &= \Sigma\left(n_m \sin \frac{2m\pi}{q}\right) \end{aligned}\right\} \qquad (23)$$

These formulæ embrace, as a particular case, the solution already given in Art. 28 for $q = 4$.

EXAMPLE.

The following values of $n = B - A$ were obtained from the readings of two opposite microscopes of the meridian circle of the U. S. Naval Academy:

z	n	z	n	z	n	z	n
0°	− 10″.7	90°	− 20″.5	180°	− 21″.8	270°	− 0″.0
10	11 .6	100	20 .7	190	18 .3	280	1 .3
20	12 .8	110	21 .0	200	16 .4	290	2 .4
30	14 .7	120	21 .2	210	11 .8	300	4 .5
40	16 .3	130	22 .8	220	7 .8	310	5 .1
50	17 .3	140	24 .7	230	4 .3	320	7 .4
60	18 .5	150	23 .4	240	1 .9	330	9 .4
70	18 .1	160	22 .5	250	− 2 .0	340	11 .7
80	19 .7	170	22 .3	260	+ 0 .3	350	11 .6

We have here $q = 36$, and $\frac{2\pi}{q} = 10°$: so that $\frac{2m\pi}{q}$ is successively 0°, 10°, 20°, &c. We find, first, by taking the sum of all the values of n,

$$36\,\alpha = -\,476''.2 \qquad \alpha = -\,13''.23$$

and hence the distance of the microscope B from A was 179° 59′ 46″.77.

To find $qe \sin P$, we multiply each n by the cosine of the angle to which it belongs, and add the products. In like manner,

$qe \cos P$ is found by multiplying each n by the sine of the a
to which it belongs, and adding the products.* We thus f
the following table, in which, for brevity, we put n
and $n \sin z$ for the quantities denoted in our formulæ (23)
$n_m \cos \frac{2m\pi}{q}$ and $n_m \sin \frac{2m\pi}{q}$.

z	$n \cos z$	$n \sin z$	z	$n \cos z$	n sin
0°	− 10″.70	− 0″.00	180°	+ 21″.80	+ 0″.
10	− 11 .42	− 2 .01	190	+ 18 .02	+ 3 .
20	− 12 .03	− 4 .38	200	+ 15 .41	+ 5 .
30	− 12 .73	− 7 .35	210	+ 10 .22	+ 5 .
40	− 12 .49	− 10 .48	220	+ 5 .98	+ 5 .
50	− 11 .12	− 13 .25	230	+ 2 .76	+ 3 .
60	− 9 .25	− 16 .02	240	+ 0 .95	+ 1 .
70	− 6 .19	− 17 .01	250	+ 0 .68	+ 1 .
80	− 3 .42	− 19 .40	260	− 0 .05	− 0 .
90	0 .00	− 20 .50	270	0 .00	0 .
100	+ 3 .59	− 20 .39	280	− 0 .23	+ 1 .
110	+ 7 .18	− 19 .73	290	− 0 .82	+ 2 .
120	+ 10 .60	− 18 .36	300	− 2 .25	+ 3 .
130	+ 14 .66	− 17 .47	310	− 3 .28	+ 3 .
140	+ 18 .92	− 15 .88	320	− 5 .67	+ 4 .
150	+ 20 .26	− 11 .70	330	− 8 .14	+ 4 .
160	+ 21 .14	− 7 .70	340	− 10 .99	+ 4 .
170	+ 21 .96	− 3 .87	350	− 11 .42	+ 2 .
Sums	+ 28 .96	− 225 .50		+ 32 .97	+ 53 .

$36e \sin P = + 28''.96 + 32''.97 = + 61''.93$ log 1.
$36e \cos P = - 225.50 + 53.04 = - 172.46$ log n2.
$P = 160° 15'$ log tan P n9.
$e = 5''.09$ log $36e$ 2.

Then, since $z_0 = 0°$, we have $E = P$, and each reading of microscope A requires the correction, for eccentricity,

$$+ 5''.09 \sin (z + 160° 15')$$

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* The several products may be taken by inspection from a traverse table, by e ing the table with the angle z as a "bearing" and with n as a "distance," and t out the corresponding "difference of latitude" and "departure," which wi respectively, the products required in forming $qe \sin P$ and $qe \cos P$.

ELLIPTICITY OF THE PIVOT OF THE ALIDADE.

33. If the pivot of the alidade is the horizontal axis of a vertical circle, as in the case of some meridian circles, or if, as in other cases, the alidade is fixed to a pier while the pivot of the horizontal axis of the circle revolves in a V, then any defect in the pivot, which renders a section at right angles to its axis other than a circle, will cause the centre of the alidade to vary its distance from the centre of the graduated circle during a revolution of the instrument. If the section of the pivot is any *regular* figure, the variations in the readings of a single microscope may be regarded as a function of the division (z) which is under the microscope, and the correction of this reading may be denoted by $\varphi(z)$. The correction of the reading of the opposite microscope must be $-\varphi(z)$. In order to investigate the form of the pivot without involving the errors of eccentricity or of graduation, let us denote the correction of the division z for both these errors by $\psi(z)$, and that of the division $180° + z$, which is under the opposite microscope, by $\psi(180° + z)$. Then, A and B being the readings of the microscopes, and $180° + \alpha$ their constant distance from each other, we have

$$\begin{aligned} z' &= z + A \quad + \quad \varphi(z) + \psi(z) \\ z' &= z + B - \alpha - \varphi(z) + \psi(180° + z) \end{aligned}$$

whence

$$0 = B - A - \alpha - 2\varphi(z) - \psi(z) + \psi(180° + z)$$

Now, let the division $180° + z$ be brought under the microscope A, and let A' and B' be the microscope readings; then we have the true reading z'' by the equations

$$\begin{aligned} z'' &= 180° + z + A' \quad + \varphi(180° + z) + \psi(180° + z) \\ z'' &= 180 \;\, + z + B' - \alpha - \varphi(180° + z) + \psi(z) \end{aligned}$$

whence

$$0 = B' - A' - \alpha - 2\varphi(180° + z) + \psi(z) - \psi(180° + z)$$

therefore, if we put

$$\tfrac{1}{2}(B - A + B' - A') = n'$$

we have

$$n' = \alpha + \varphi(z) + \varphi(180° + z) \tag{25}$$

the errors of eccentricity and of graduation being wholly eliminated. The form of the function φ is yet to be determined; since, however, it necessarily returns to the same value after one complete revolution, we may assume for it a general periodic series, namely:

$$\varphi(z) = f' \sin(z + F') + f'' \sin(2z + F'') + f''' \sin(3z + F''') + \&c.$$

in which f', F', f'', F'', f''', F''', &c. are constants. Hence also

$$\varphi(180° + z) = -f' \sin(z + F') + f'' \sin(2z + F'') - f'' \sin(3z + F''') + \&c.$$

and

$$\varphi(z) + \varphi(180° + z) = 2f'' \sin(2z + F'') + 2f^{\text{iv}} \sin(4z + F^{\text{iv}}) + \&c. \quad (26)$$

The combination of two readings 180° apart gives, therefore, the equation of condition

$$n' = \alpha + 2f'' \sin(2z + F'') + 2f^{\text{iv}} \sin(4z + F^{\text{iv}}) + \&c. \quad (27)$$

If we have read the circle at $2q$ equidistant points, so that the number of such equations is q, then the values of z are successively $0, \frac{\pi}{q}, \frac{2\pi}{q}, \ldots\ldots \frac{(q-1)\pi}{q}$, and the general form of the equation of condition is

$$n'_m = \alpha + 2f'' \sin\left(m \cdot \frac{2\pi}{q} + F''\right) + 2f^{\text{iv}} \sin\left(m \cdot \frac{4\pi}{q} + F^{\text{iv}}\right) + \&c. \quad (28)$$

m being taken from 0 to $q - 1$. If we treat these equations by the method of least squares, we shall readily find, by the aid of the theorems of Art. 31,

$$\left.\begin{aligned}
q\alpha &= \Sigma n'_m \\
q f'' \sin F'' &= \Sigma\left(n'_m \cos m \cdot \frac{2\pi}{q}\right) \\
q f'' \cos F'' &= \Sigma\left(n'_m \sin m \cdot \frac{2\pi}{q}\right) \\
q f^{\text{iv}} \sin F^{\text{iv}} &= \Sigma\left(n'_m \cos m \cdot \frac{4\pi}{q}\right) \\
q f^{\text{iv}} \cos F^{\text{iv}} &= \Sigma\left(n'_m \sin m \cdot \frac{4\pi}{q}\right) \\
\&c. \qquad & \qquad \&c.
\end{aligned}\right\} \quad (29)$$

EXAMPLE.

To investigate the form of the alidade pivot of the meridian circle, in the example of Art. 32, the readings there given are combined as follows:

z	$B - A$	$B' - A'$	n'	z	$B - A$	$B' - A'$	n'
0°	— 10″.7	— 21″.8	— 16″.25	90°	— 20″.5	— 0″.0	— 10″.25
10	11 .6	18 .3	14 .95	100	20 .7	1 .3	11 .00
20	12 .8	16 .4	14 .60	110	21 .0	2 .4	11 .70
30	14 .7	11 .8	13 .25	120	21 .2	4 .5	12 .85
40	16 .3	7 .8	12 .05	130	22 .8	5 .1	13 .95
50	17 .3	4 .3	10 .80	140	24 .7	7 .4	16 .05
60	18 .5	1 .9	10 .20	150	23 .4	9 .4	16 .40
70	18 .1	— 2 .0	10 .05	160	22 .5	11 .7	17 .10
80	19 .7	+ 0 .3	9 .70	170	22 .3	11 .6	16 .95

Since here $q = 18$, the sum of the values of n' gives

$$18\alpha = -238''.10 \qquad \alpha = -13''.23$$

Then, with the aid of a traverse table, we find the values of $n' \cos 2z$ and $n' \sin 2z$, as below:

z	$n' \cos 2z$	$n' \sin 2z$	z	$n' \cos 2z$	$n' \sin 2z$
0°	— 16″.25	— 0″.00	90°	+ 10″.25	+ 0″.00
10	— 14 .05	— 5 .12	100	+ 10 .34	+ 3 .76
20	— 11 .18	— 9 .38	110	+ 8 .96	+ 7 .52
30	— 6 .68	— 11 .48	120	+ 6 .43	+ 11 .13
40	— 2 .09	— 11 .87	130	+ 2 .42	+ 13 .74
50	+ 1 .88	— 10 .64	140	— 2 .79	+ 15 .81
60	+ 5 .10	— 8 .83	150	— 8 .20	+ 14 .20
70	+ 7 .70	— 6 .46	160	— 13 .10	+ 10 .99
80	+ 9 .12	— 3 .32	170	— 15 .93	+ 5 .80
Sums	— 26 .40	— 67 .10		— 1 .62	+ 82 .95

$18 f'' \sin F'' = - \; 28''.02$ $\qquad$ $\log n1.4475$

$18 f'' \cos F'' = + \; 15\,.85$ $\qquad$ $\log \; 1.2000$

$F'' = \; 299° \, 30'$ $\qquad$ $\log \tan F'' \; n0.2475$

$f'' = \; 1''.79$ $\qquad$ $\log 18 f'' \; 1.5078$

In the same manner, we find, from the sums of the products $n' \cos 4z$ and $n' \sin 4z$,

$$\begin{aligned} 18 f^{\text{iv}} \sin F^{\text{iv}} &= +\ 0''.15 \\ 18 f^{\text{iv}} \cos F^{\text{iv}} &= +\ 2''.00 \\ F^{\text{iv}} &= \quad 4^\circ\, 17' \\ f^{\text{iv}} &= \quad 0''.11 \end{aligned}$$

Hence we have

$$\varphi(z) + \varphi(180^\circ + z) = 3''.58 \sin(2z + 299^\circ\, 30') + 0''.22 \sin(4z + 4^\circ\, 17') \quad (30)$$

The term in $4z$ is so small that we may suppose that it proceeds from the accidental errors of reading, and *irregularities* of the pivot, and we may, therefore, disregard it, as well as the subsequent terms in $6z$, &c.

Bessel has shown* that if the section of a pivot which rests in a V is an ellipse, the centre of this ellipse will, as the instrument revolves, move in the arc of a circle the centre of which is the angular point of the V; that during a complete revolution the centre of the ellipse describes this arc four times,—twice forwards and twice backwards; and that the effect of this motion upon the reading of a single microscope is expressed by a term depending upon $2z$.

Hence, the last term of (30) being neglected, the remaining term may be regarded as the effect of ellipticity of the pivot, and, since we must then have $\varphi(z) = \varphi(180^\circ + z)$, it follows that

$$\varphi(z) = 1''.79 \sin(2z + 299^\circ\, 30') \quad (31)$$

Upon the hypothesis that the pivot is elliptical, the observed values of n' should satisfy the equation (27), which in the present case becomes

$$n' = -\ 13''.23 + 3''.58 \sin(2z + 299^\circ\, 30')$$

at least within the errors of reading. To show that this hypothesis explains the observations in the present case sufficiently well, the following comparison is made, in which the value of n' computed by the preceding formula is denoted by C, the observed value by O, the residual error, or $O - C$, by v.

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* *Astronomische Beobachtungen auf der Sternwarte in Königsberg*, Vol. I. p. xii.

z	*O*	*C*	*v*	*vv*	*z*	*O*	*C*	*v*	*vv*
0°	— 16″.25	— 16″.85	+ 0″.10	0.0100	90°	— 10″.25	— 10″.11	— 0″.14	0.0196
10	14 .95	15 .55	+ 0 .60	.3600	100	11 .00	10 .91	— 0 .09	.0081
20	14 .60	14 .48	— 0 .12	.0144	110	11 .70	11 .98	+ 0 .28	.0784
30	13 .25	13 .26	+ 0 .01	.0001	120	12 .85	13 .20	+ 0 .35	.1225
40	12 .05	12 .03	— 0 .02	.0004	130	13 .95	14 .43	+ 0 .48	.2304
50	10 .80	10 .95	+ 0 .15	.0225	140	16 .05	15 .51	— 0 .54	.2916
60	10 .20	10 .15	— 0 .05	.0025	150	16 .40	16 .31	— 0 .09	.0081
70	10 .05	9 .71	— 0 .34	.1156	160	17 .10	16 .75	— 0 .35	.1225
80	9 .70	9 .70	0 .00	.0000	170	16 .95	16 .76	— 0 .19	.0361

If we denote the mean error of a single observed value of n' by ε, we have (Appendix, *Method of Least Squares*), q being the number of observations,

$$\varepsilon = \sqrt{\left(\frac{\Sigma(vv)}{q-1}\right)} = \sqrt{\frac{1.4428}{17}} = 0''.29$$

and this quantity also expresses the mean error of a single reading of one microscope of this instrument. This mean error of a reading was also found by comparing a number of successive readings of the same microscope on the same division, which gave $0''.36$: so that the agreement of the above computed and observed values of n' is even closer than is necessary to sustain the hypothesis of an elliptical form of the pivot. It is also evident that the addition of the term $0''.22 \sin(4z + 4^\circ\ 17')$ of (30) would but slightly reduce the mean error of n'.

34. The error introduced by the ellipticity of the pivot, like that produced by the eccentricity of the circle, is fully eliminated by taking the mean of the readings of a pair of opposite microscopes. If, however, the arms of the alidade, carrying the microscopes, do not preserve a constant inclination to the horizon during a revolution of the instrument, the readings of both microscopes will be increased or diminished by the whole amount of the change of inclination, and, consequently, their mean will involve the same error. A level placed on the alidade is usually employed to determine these changes of inclination, and the readings are finally corrected according to its indications.

ERRORS OF GRADUATION.

35. Errors of graduation of a divided circle may be either *regular* or *accidental*.

The regular or periodic errors are those which recur at regular intervals according to some law, and which may, therefore, be expressed as functions of the reading itself. Even the error of eccentricity, above considered, may be treated as such a periodic error of graduation, since its effect upon the reading (z) is the same as if the graduation everywhere required the correction $e \sin (z + E)$. The sum of all the corrections for such periodic errors, regarded as a function of the reading (z), and denoted by $\psi(z)$, must have the general form

$$\psi(z) = u' \sin (z + U') + u'' \sin (2z + U'') + u''' \sin (3z + U''') + \&c. \quad (32)$$

in which u', U', u'', U'', &c. are constants. The shorter the period of any error, the higher is the multiple of z in the term representing it.

Now, let the circle be read by q microscopes at q equidistant points, namely, at all the points expressed by

$$z_m = z + m \cdot \frac{2\pi}{q}$$

m being taken successively 0, 1, 2, 3 ..... $(q - 1)$, and z being the reading of the first microscope; then we shall have, for the correction of any one of these microscopes, the general expression

$$\psi(z_m) = u' \sin \left(z + U' + m \cdot \frac{2\pi}{q} \right) + u'' \sin \left(2z + U'' + m \cdot \frac{4\pi}{q} \right) + \&c.$$

The discussion of this series will be abridged if we express it under the following general form:

$$\psi(z_m) = \Sigma_p u^{(p)} \sin \left(pz + U^{(p)} + m \cdot \frac{2p\pi}{q} \right)$$

in which p is successively 1, 2, 3, &c., and Σ_p denotes the sum of all the terms thus found. Developing the sine, this gives

$$\psi(z_m) = \Sigma_p u^{(p)} \sin (pz + U^{(p)}) \cos m \cdot \frac{2p\pi}{q} + \Sigma_p u^{(p)} \cos (pz + U^{(p)}) \sin m \cdot \frac{2p\pi}{q}$$

The mean of the q microscopes will, therefore, require the correction

$$\frac{1}{q} \sum_{m=0}^{m=q-1} \psi(z_m) = \frac{1}{q} \Sigma_p \left[u^{(p)} \sin (pz + U^{(p)}) \cdot \sum_{m=0}^{m=q-1} \cos m \cdot \frac{2p\pi}{q} \right]$$

$$+ \frac{1}{q} \Sigma_p \left[u^{(p)} \cos (pz + U^{(p)}) \cdot \sum_{m=0}^{m=q-1} \sin m \cdot \frac{2p\pi}{q} \right]$$

But we have (Art. 31), from $m = 0$ to $m = q - 1$, $\Sigma \sin m \cdot \frac{2p\pi}{q} = 0$ in all cases; and also $\Sigma \cos m \cdot \frac{2p\pi}{q} = 0$, except when p is a multiple of q, or $p = rq$, in which case this latter sum is equal to q. Hence all the terms of the above expression which do not vanish are expressed by the formula

$$\frac{1}{q} \sum_{m=0}^{m=q-1} \psi(z_m) = \Sigma_{rq}\, u^{(rq)} \sin (rqz + U^{(rq)}) \tag{33}$$

r being successively the integers 1, 2, 3.....; whence the following important theorem: *The terms of the periodic series not eliminated by taking the mean of q equidistant microscopes are those only which involve the multiples of qz.*

Thus, the mean of two microscopes requires a correction of the form

$$u'' \sin (2z + U'') + u^{\text{iv}} \sin (4z + U^{\text{iv}}) + \&\text{c.};$$

the mean of three microscopes, the correction

$$u''' \sin (3z + U''') + u^{\text{vi}} \sin (6z + U^{\text{vi}}) + \&\text{c.};$$

the mean of four microscopes, the correction

$$u^{\text{iv}} \sin (4z + U^{\text{iv}}) + u^{\text{viii}} \sin (8z + U^{\text{viii}}) + \&\text{c.}$$
$$\&\text{c.} \qquad\qquad \&\text{c.}$$

36. The values of the terms of the periodic series which are eliminated by means of a number of microscopes may be found from the readings of these microscopes themselves. Thus, for two microscopes, the readings of which at the divisions z and $z + 180°$ are A and B, and whose angular distance is $180° + \alpha$, we have

$$\begin{aligned} z' &= z + A \qquad\;\; + \psi(z) \qquad\qquad + \varphi(z) \\ z' &= z + B - \alpha + \psi(z + 180°) - \varphi(z) \end{aligned}$$

in which $\varphi(z)$ is the correction for the form of the pivot (Art. 33). Hence, putting $B - A = n$, we have

$$n = \alpha + \psi(z) - \psi(z + 180°) + 2\varphi(z)$$

But we have

$$\psi(z) = u' \sin (z + U') + u'' \sin (2z + U'') + u''' \sin (3z + U''') + \&\text{c.}$$

and hence, substituting $z + 180°$ for z,

$$\psi(z + 180°) = -u' \sin(z + U') + u'' \sin (2z + U'') - u''' \sin(3z + U''') + \&\text{c.}$$

For $\varphi(z)$ we have already found the form $f''\sin(2z+F'')$, and therefore the value of n becomes

$$n = \alpha + 2u'\sin(z+U') + 2f''\sin(2z+F'') + 2u'''\sin(3z+U''') + \&c. \quad (34)$$

The readings being made for successive values of z expressed generally by

$$z_m = m\cdot\frac{2\pi}{q}$$

we have q equations of condition of the form

$$n_m = \alpha + 2u'\sin\left(m\cdot\frac{2\pi}{q} + U'\right) + 2f''\sin\left(m\cdot\frac{4\pi}{q} + F''\right) + \&c. \quad (35)$$

m being taken equal to 0, 1, 2, 3 .... $q - 1$, successively. The solution of these equations by the method of least squares gives

$$\begin{array}{lll}
q\alpha = \Sigma n_m & & \\
qu' \sin U' & = \Sigma\left(n_m \cos m\cdot\frac{2\pi}{q}\right) & = \Sigma\,(n_m \cos z_m) \\
qu' \cos U' & = \Sigma\left(n_m \sin m\cdot\frac{2\pi}{q}\right) & = \Sigma\,(n_m \sin z_m) \\
qf'' \sin F'' & = \Sigma\left(n_m \cos m.\frac{4\pi}{q}\right) & = \Sigma\,(n_m \cos 2z_m) \\
qf'' \cos F'' & = \Sigma\left(n_m \sin m\cdot\frac{4\pi}{q}\right) & = \Sigma\,(n_m \sin 2z_m) \\
qu''' \sin U''' & = \Sigma\left(n_m \cos m.\frac{6\pi}{q}\right) & = \Sigma\,(n_m \cos 3z_m) \\
qu''' \cos U''' & = \Sigma\left(n_m \sin m\cdot\frac{6\pi}{q}\right) & = \Sigma\,(n_m \sin 3z_m) \\
\&c. & \&c. & \&.
\end{array}$$

Example.

The values of n given on page 45 for thirty-six readings of the Meridian Circle of the Naval Academy give, by the preceding formulæ, $\alpha = -13''.23$ and

$$\begin{array}{lll}
U' = 160^\circ\,15', & F'' = 299^\circ\,30', & U''' = 68^\circ\,19' \\
u' = 5''.09, & f'' = 1''.79, & u''' = 0''.69
\end{array}$$

The difference of the readings of the two microscopes A and B of this circle is therefore represented by the formula

$$\begin{aligned} n = -13''.23 + 10''.18\sin(z + 160^\circ\,15') + 3''.58\sin(2z + 299^\circ\,30') \\ + 1''.38\sin(3z + 68^\circ\,19') \end{aligned}$$

of which the terms in z and $2z$ of course agree with those before found for the eccentricity and for the ellipticity of the pivot of the alidade.

If now we compute the values of n by this formula for every 10°, we shall find that they agree with the observed values given on page 45 within quantities which in almost every instance are less than 1″. From this agreement we may presume that this circle is very accurately graduated throughout.

37. In a similar manner, the terms of the periodic series which do not involve the multiples of $4z$ can be found from the readings of four microscopes. If A, C, B, D are these readings at the divisions z, $z + 90°$, $z + 180°$, $z + 270°$ respectively, and if $180° + \alpha$ is the distance of the microscope B from A, while $180° + \gamma$ is that of D from C, then the mean of the readings of A and B gives

$$\begin{aligned} z' &= z + \tfrac{1}{2}(A + B) - \tfrac{1}{2}\alpha + \tfrac{1}{2}[\psi(z) + \psi(z + 180°)] \\ &= z + \tfrac{1}{2}(A + B) - \tfrac{1}{2}\alpha + u'' \sin(2z + U'') + u^{\text{iv}} \sin(4z + U^{\text{iv}}) + \&c. \end{aligned}$$

and, consequently (exchanging z for $z + 90°$), the mean of the readings of C and D gives

$$z' = z + \tfrac{1}{2}(C + D) - \tfrac{1}{2}\gamma - u'' \sin(2z + U'') + u^{\text{iv}} \sin(4z + U^{\text{iv}}) - \&c.$$

Taking the difference of these equations, and putting

$$\begin{aligned} n &= \tfrac{1}{2}(C + D) - \tfrac{1}{2}(A + B) \\ \beta &= \tfrac{1}{2}(\gamma - \alpha) \end{aligned}$$

we have the equation of condition

$$n = \beta + 2u'' \sin(2z + U'') + 2u^{\text{vi}} \sin(6z + U^{\text{vi}}) + \&c. \qquad (36)$$

and from the q equations of this form we derive β, u'', U'', &c. by the process already employed.

The terms in z and $3z$ may be found from either pair of microscopes as in the preceding article.

38. *The accidental errors* of graduation are those which follow no regular law, and may with equal probability occur at any given division with either the positive or the negative sign. An error of this kind in any division is to be regarded as peculiar to that division, and, therefore, as having no analytical connection with other errors of the same kind. The use of a number of

microscopes tends to reduce the effect of such errors, without entirely eliminating them; for (as in Art. 24) if ε is the probable accidental error of a division, the probable accidental error in the mean of m microscopes will be $\frac{\varepsilon}{\sqrt{m}}$.

The general character of the graduation, as to its freedom from accidental errors, may be judged of by comparing the values of the n of the preceding articles, computed from the terms of the periodic series, with their observed values. The differences will be composed of both errors of reading and accidental errors, which may be separated by employing an independent determination of the probable error of reading. Thus, if we have $n = B - A$, and have found the probable error of an observed value of n to be ε, and then, if we put

$\varepsilon_1 =$ the probable error of a single reading,
$\varepsilon_2 =$ " " " " division,

the probable error of either A or B will be $\sqrt{(\varepsilon_1^2 + \varepsilon_2^2)}$, and that of $B - A$ will be $\sqrt{2(\varepsilon_1^2 + \varepsilon_2^2)}$, whence

$$\varepsilon^2 = 2(\varepsilon_1^2 + \varepsilon_2^2)$$

which will determine ε_2 when ε and ε_1 have been found.

39. The accidental error of any division of the circle may be directly found by means of an additional microscope which can be set and securely clamped at any given distance from the regular or fixed microscopes. Let us denote this movable microscope by M, and let it be proposed to determine the error of the division z. Bring the division 0° under the microscope A, and clamp the movable microscope M over the division z. Let the true angular distance of M from A (which is as yet unknown) be denoted by $z + \mu$, and let the readings of the two microscopes, referred to the divisions 0 and z respectively, be called A and M, then, z denoting the nominal value and z' the true value of the arc from 0 to z, we shall have

$$z + \mu = z' + M - A$$

and the correction of the graduation z will be

$$z' - z = \mu - (M - A)$$

or rather, since every division (and, therefore, 0° included) may be regarded as in error, this will be the difference of the corrections of the graduations 0 and z, and we may write

$$\varphi(z) - \varphi(0) = \mu - (M - A) \tag{37}$$

in which $\varphi(z)$ denotes the total correction of a division for both periodic and accidental errors. The periodic errors being known from previous investigation, the accidental error may be separated.

Now, to find the constant distance μ, we resort to the well known method of *repetition*. First, bring any arbitrarily selected division Z under the microscope A, then $Z + z$ will be under M; let the readings of the two microscopes be A' and M' respectively. Then bring the division $Z + z$ under A, and, consequently, the division $Z + 2z$ under M, and let the readings be A'' and M''. In this way, let m repetitions be made, the microscope A being successively placed upon the divisions Z, $Z + z$, $Z + 2z, \ldots\ldots. Z + (m - 1)z$, and M successively upon $Z + z$, $Z + 2z$, $Z + 3z, \ldots\ldots Z + mz$; then we have, as in (37),

$$\begin{aligned}
\varphi(Z + z) - \varphi(Z) &= \mu - (M' - A') \\
\varphi(Z + 2z) - \varphi(Z + z) &= \mu - (M'' - A'') \\
\varphi(Z + 3z) - \varphi(Z + 2z) &= \mu - (M''' - A''') \\
\cdots\cdots\cdots &\quad \cdots\cdots\cdots \\
\varphi(Z + mz) - \varphi(Z + (m - 1)z) &= \mu - (M^{(m)} - A^{(m)})
\end{aligned}$$

The mean of all these equations is

$$\frac{1}{m}[\varphi(Z + mz) - \varphi(Z)] = \mu - \frac{1}{m}\Sigma(M - A)$$

If the number m is large, the mth part of the difference of the accidental errors of the extreme divisions Z and $Z + mz$ may be regarded as evanescent, and then, if we regard the first member as composed only of the periodic errors already found, we shall have

$$\mu = \frac{1}{m}\Sigma(M - A) + \frac{1}{m}[\psi(Z + mz) - \psi(Z)] \tag{38}$$

where the function ψ denotes a periodic error, as in Art. 35. If this process be repeated a number of times, each time commencing at a different division, the mean of all the values of μ may be regarded as entirely free from the effect of the accidental errors of the first and last divisions. Thus, μ being found, the correction of the division (z) becomes known by (37).

If z is an aliquot part of the circumference $= \frac{2\pi}{m}$, we shall have

$\varphi(Z+mz)=\varphi(Z)$, since we have returned to the same division; and the value of μ is then rigorously

$$\mu=\frac{1}{m}\Sigma(M-A)$$

Thus, the fixed microscopes themselves, whose distance is $\frac{2\pi}{q}$, may be at once employed in this manner (without an additional microscope) to determine the errors of the divisions whose mutual distance is $\frac{2\pi}{q}$. If then we have four fixed microscopes and one movable one M placed at the distance z from A, we shall be able to find: 1st, the errors of the four cardinal divisions 0°, 90°, 180°, and 270°, by the fixed microscopes; 2d, the errors of the divisions z, $90°+z$, $180°+z$, $270°+z$, by placing the microscope A successfully upon 0°, 90°, 180°, and 270°, and reading M; 3d, the errors of the divisions $90°-z$, $180°-z$, $270°-z$, and $360°-z$, by placing M successively upon 90°, 180°, 270°, and 360°, and reading A. Thus, after the errors of the four cardinal divisions are known, the operation just described gives the errors of eight divisions. A second operation with the microscope M at the distance z_1 from A gives in like manner the errors of eight more divisions, $\pm z_1$, $90°\pm z_1$, $180°\pm z_1$, $270°\pm z_1$; and, moreover, the errors of the divisions $\pm z\pm z_1$, $90°\pm z\pm z_1$, $180°\pm z\pm z_1$, $270°\pm z\pm z_1$, by placing the microscope A over $\pm z$, $90°\pm z$, &c. successively while M is over $\pm z+z_1$, $90°\pm z+z_1$, &c., or placing M over $\pm z$, $90°\pm z$, &c. successively while A is over $\pm z-z_1$, $90°\pm z-z_1$, &c. By judiciously combining all the observations of this kind, the corrections of each degree of the circle may be found.

In order to eliminate the effect of changes in the angular distance of the fixed and movable microscopes occurring during the observations and produced chiefly by changes of temperature, it is proper to repeat each series of observations at a given distance z *backwards*, commencing this repetition by placing the movable microscope M over the last division $Z+mz$ and the fixed one A over $Z+(m-1)z$, and so returning to the first assumed division Z. Also the readings on the eight divisions to be determined should be made several times, say, once before the first or *forward* repetition series, again, between the two repetition series, and finally, after the second or *backward* repetition series. Thus, the whole operation will embrace

1st. Observations on the eight divisions,
2d. Repetition series *forwards*,
3d. Observations on the eight divisions,
4th. Repetition series *backwards*,
5th. Observations on the eight divisions.

By this symmetrical arrangement, the mean of the three determinations of the errors of the eight divisions corresponds to the mean state of the apparatus as found from the mean of the two repetition series.[*]

THE FILAR MICROMETER.

40. For the measurement of small angles, not greater than the angular breadth of the field of the telescope, graduated circles may be wholly dispensed with, and a micrometer attached to the eye end of the telescope may be substituted with great advantage both in respect of accuracy and facility of manipulation. Indeed, for many purposes to which the micrometer is adapted, divided circles are entirely out of the question; for example, the measurement of the angular distance between the two components of a double star.

Micrometers, however, are very frequently used in combination with graduated circles; as in the meridian circle.

41. *The filar micrometer* is the same in principle as the micrometer employed in the reading microscope (Art. 21), only more elaborate and complete when intended to be used at the focus of a large telescope. It is variously constructed, according to the instrument with which it is to be connected. A very common form which involves the essential features of all the others is sketched in Plate II. Fig. 3, where the outside plate and the eye piece are removed and the field of view exhibited. The plate *aa* is permanently attached to the eye end of the telescope tube at right angles to the optical axis. The plate *bb*, carrying the thread *mm*, slides upon *aa*, and is moved by the screw *B*. The plate *cc*, carrying the thread *nn*, slides upon *bb*, and is moved by the screw *C*. The threads are at right angles to the

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[*] This process, which is due to BESSEL, will be found more fully discussed in the *Königsberg Observations*, Vol. VII., and in the *Astron. Nach.*, Nos. 481 and 482. See also C. A. F. PETERS, *Untersuchung der Theilungsfehler des Ertelschen Verticalkreises der Pulkowaer Sternwarte* (St. Petersburg, 1848); and HANSEN in the *Astron. Nach.*, No. 388.

direction of the motion produced by the screws. Their distance apart is changed only by the screw *C*, which carries a large graduated head, by means of which this distance is measured. The screw *B* merely shifts the whole apparatus *bb*, so that the threads may be carried to any part of the field of view. A notched scale in the field of view, the notches of which are at the same distance apart as the threads of the screw *C*, is attached either to the plate *bb*, or to the plate *cc* (in the figure, to the latter); in either case the number of notches between the threads indicates the whole number of revolutions of the screw by which the threads are separated, while the graduated head of *C* indicates the fraction of a revolution. Finally, at least one thread is stretched across the middle of the field at right angles to the micrometer threads: sometimes three or more equidistant and parallel threads; these are usually attached to the plate *bb*. In micrometer measures the thread *mm* usually remains fixed while *nn* moves: the former is therefore usually called the *fixed* thread, and the latter the *movable* thread. The threads at right angles to these are called *transverse* threads; sometimes *transit* threads.

That portion of the telescope to which the micrometer is immediately attached is a tube which both slides and revolves within the main tube of the telescope, so that (by sliding) the plane of the threads may be accurately placed in the focus of the object glass, and (by revolving) the threads may be made to take any required direction.

To measure directly the angular distance between two objects whose images are seen in the field, we have first to revolve the whole micrometer until the middle transverse thread passes through the two objects; then, bringing the fixed thread upon one of the objects and the movable thread upon the other, the distance is at once obtained in revolutions and parts of a revolution of the micrometer screw. This measure is then to be reduced to seconds of arc, for which purpose the angular value of a revolution of the screw must be known.

42. *To find the angular value of a revolution of the micrometer screw.*—This value evidently depends not only upon the distance of the threads of the screw, but also upon the focal length of the telescope, since the greater the focal length, the larger will be the image of any given object.

A FIRST METHOD of finding the value of the screw is, therefore, to measure the focal length, F, of the object glass, and the distance, m, between the threads of the screw (which is done by counting the number of threads to an inch); then, if R denotes the angular value of a revolution, we have

$$\tan \tfrac{1}{2}R = \frac{\tfrac{1}{2}m}{F} \qquad \text{or} \qquad R = \frac{m}{F \sin 1''} \tag{39}$$

as is evident from Fig. 2, p. 13, where we may suppose dl, at the focus of the lens AB, to be the space through which the micrometer thread is moved by a revolution of the screw, and the angular breadth of the object DL, of which dl is the image, to be $DCL = lCd$, and $Cm = F$, $dl = m$.

43. SECOND METHOD.—Measure with the micrometer any previously known angle A, and let M be the number of revolutions of the screw in the measure; then, assuming that the middle point of A is observed in the middle of the field,

$$\tan R = \frac{2 \tan \tfrac{1}{2}A}{M} \qquad \text{or, nearly,} \qquad R = \frac{A}{M} \tag{40}$$

The sun's apparent horizontal diameter (see Vol. I. Art. 134) may be used for the angle A, if the field is sufficiently large to embrace the whole image of the sun, which, however, is the case only with small instruments, or with low magnifying powers.

The constellation of the *Pleiades* furnishes pairs of stars at various distances, suited to instruments of various capacities: and BESSEL determined their distances with very great accuracy with a view to this as well as other applications.*

The angle A in (40) is the *apparent* angular distance measured, so that, when two stars are employed, their apparent distance must be computed by subtracting the correction for refraction, for which see Chapter X.

44. THIRD METHOD.—Point the telescope at a star, and let the micrometer be revolved so that the transverse thread will coincide with the apparent path of the star in its diurnal movement, and the fixed micrometer thread will represent a declination circle. Place the movable thread at any number M of revolutions

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* BESSEL'S *Astronomische Untersuchungen*, Vol. I. p. 209.

from the fixed thread, and note the times of transit of the star over these threads by the sidereal clock, the telescope remaining fixed during the whole observation. Denote the sidereal interval between these times by I, the declination of the star by δ, the true angular interval of the threads by i; then (as will be proved in the theory of the transit instrument) we shall find i by the formula

$$\sin i = \sin I \cos \delta \tag{41}$$

or, when the star is not within 10° of the pole,

$$i = I \cos \delta \tag{41*}$$

after which the value of a revolution of the screw in seconds of arc is found by the formula

$$R = \frac{15 i}{M} = \frac{15\, I \cos \delta}{M} \tag{42}$$

For extreme precision, the correction for refraction should be applied to i; but if the observations are made near the meridian the correction will rarely be appreciable.

We may in this process dispense with the use of the fixed thread by setting the movable thread successively at different points in the field, and noting the times of transit of the star over it together with the number of revolutions of the screw between the successive positions. In this way the regularity of the screw may be tested throughout its whole length. If the star is very near the pole, each observation should be compared with that made near the middle of the field, and the true intervals computed by the formula $\sin i = \sin I \cos \delta$.

This method is applicable in all cases where the micrometer can be revolved so as to place the fixed and movable threads in the direction of a declination circle. If the telescope is equatorially mounted, this can be done in all positions of the instrument, and the star may be in any part of the heavens; but a slow moving star near the meridian is to be preferred, if we wish to avoid the correction for refraction.

The times of transit are supposed to be observed by a sidereal clock, the rate of which if it is large should be allowed for. If the time is noted by a mean time clock, the mean intervals are to be converted into sidereal intervals (Vol. I. Art. 49).

45. If the micrometer is attached to an instrument designed only for the measurement of zenith distances, or differences of zenith distance (as in the case of the Zenith Telescope), the movable threads being always perpendicular to a vertical circle, we can still employ this method of transits, by observing the pole star, or any star near the pole, at the time of its *greatest elongation.* At this time the vertical circle of the star is tangent to its diurnal circle, and, consequently, the micrometer thread will coincide in direction with this declination circle, as required in the preceding method. If the instrument is not moved in azimuth during the star's transit through the field, the formula for computing the interval i from the sidereal interval I is still, as in the transit instrument, $\sin i = \sin I \cos \delta$; but it must be observed that this formula here applies strictly only to the case where the thread is at one time at the point of greatest elongation, and therefore each observation should be compared with that taken nearest the computed time of elongation. To find this time, we first find the hour angle t of the star by the formula (Vol. I. Art. 18)

$$\cos t = \cot \delta \tan \varphi$$

in which φ is the latitude of the place of observation; and then, α being the star's right ascension, we have

$$\text{Sid. T. of gr. elongation} = \alpha \pm t$$

the lower sign for the eastern elongation.

If the instrument is slowly moved in azimuth as the star crosses the field, so as to make each observation of a transit in the middle of the field, the vertical distances between the different positions of the movable thread are, rigorously, differences of zenith distance, and the formula for the transit instrument is no longer strictly applicable. I shall show, however, that it is practically sufficiently exact. Let the zenith distance, hour angle, and azimuth of the star at the elongation be denoted by z_0, t_0, and A_0 respectively; those for any observation by z, t, A; and let A_0 and A be reckoned from the elevated pole. At the time of the observation, the star, the zenith, and the pole form an oblique spherical triangle, and we have the general relations

$$\cos \delta \cos t = \cos \varphi \cos z - \sin \varphi \sin z \cos A$$
$$\cos \delta \sin t = \sin z \sin A$$

At the elongation the triangle becomes right angled at the star, and we have

$$\cos t_0 = \cos z_0 \sin A_0$$

$$\sin t_0 = \frac{\sin z_0}{\cos \varphi} = \frac{\cos z_0 \cos A_0}{\sin \varphi}$$

From these we deduce

$$\begin{aligned} \cos \delta \sin t_0 \cos t &= \sin z_0 \cos z - \cos z_0 \sin z \cos A_0 \cos A \\ \cos \delta \cos t_0 \sin t &= \cos z_0 \sin z \sin A_0 \sin A \end{aligned}$$

the difference of which gives

$$\begin{aligned} \cos \delta \sin (t - t_0) &= - \sin z_0 \cos z + \cos z_0 \sin z \cos (A_0 - A) \\ &= \sin (z - z_0) - 2 \cos z_0 \sin z \sin^2 \tfrac{1}{2} (A_0 - A) \end{aligned}$$

where, if we neglect the last term and denote $t - t_0$ by I, and $z - z_0$ by i, we have the formula for the transit instrument. To obtain an expression for this last term, we take the relations

$$\begin{aligned} \sin z \cos A &= \cos \varphi \sin \delta - \sin \varphi \cos \delta \cos t \\ \sin z \sin A &= \cos \delta \sin t \end{aligned}$$

and combine them with

$$\cos A_0 = \sin \delta \sin t_0$$

$$\sin A_0 = \frac{\cos \delta}{\cos \varphi} = \frac{\sin \delta \cos t_0}{\sin \varphi}$$

whence

$$\begin{aligned} \sin z \sin (A_0 - A) &= \sin \delta \cos \delta - \sin \delta \cos \delta \cos (t - t_0) \\ &= \sin 2\delta \sin^2 \tfrac{1}{2} (t - t_0) \end{aligned}$$

Thus $\sin (A_0 - A)$ is very nearly proportional to the square of $\sin \frac{1}{2} (t - t_0)$, and is, consequently, so small that we may put $\sin \frac{1}{2} (A_0 - A) = \frac{1}{2} \sin (A_0 - A)$ in the last term of the above formula. We may also in so small a term put z_0 for z. Making these substitutions, and writing I and i for $t - t_0$ and $z - z_0$, we find

$$\sin i = \sin I \cos \delta + \tfrac{1}{2} \cot z_0 \sin^2 2\delta \sin^4 \tfrac{1}{2} I \qquad (43)$$

Since not only $\sin \frac{1}{2} I$ is a small quantity, but also $\sin 2\delta$, it is evident that the last term will be inappreciable in all practical cases. Thus, for the pole star, $\delta = 88° 30'$ and $I = 30^m = 7° 30'$, this term is only $0''.0052 \cot z_0$.

For either method of observation, therefore, we can regard the formula $\sin i = \sin I \cos \delta$ as entirely rigorous.

But in either method we must correct the computed interval i for refraction. This computed interval is the difference of the true zenith distances at the two instants of transit, and the micrometer interval M represents the difference of the apparent zenith distances at these instants; hence, if r and r_0 are the refractions for the zenith distances z and z_0, we shall have

$$R = \frac{i - (r - r_0)}{M} = \frac{z - z_0 - (r - r_0)}{M}$$

If we put

Δr = the difference of refraction for 1′ of zenith distance,

we shall have

$$r - r_0 = (z - z_0)\,\Delta r$$

or, very nearly,

$$r - r_0 = MR\,\Delta r$$

and, consequently,

$$R = \frac{i}{M} - R\,.\,\Delta r \qquad (44)$$

The value of Δr may be taken from the refraction table for the zenith distance at the elongation, which will be found by the formula

$$\cos z_0 = \frac{\sin \varphi}{\sin \delta}$$

An example of this method will be given in the chapter on the Zenith Telescope.

46. **Fourth Method.**—The angular distance of two threads in the focus of a telescope may be directly measured with a theodolite. We have seen (Art. 4) that the rays which diverge from the focus and fall upon the object glass emerge from this glass in parallel lines. If then these emerging rays be received by the lens of another telescope, they will be converged by the latter lens to its principal focus, where they will form an image of the point from which they diverged. Hence, if two telescopes are placed with their optical axes in the same straight line and with their objectives turned towards each other, we may in either telescope see the images of threads at the principal focus of the other. If our second telescope is connected with a

vertical or horizontal circle, as in the theodolite, the circle may be used to measure the angular distance of the threads in the first.

First.—If the micrometer threads are horizontal, that is, perpendicular to the vertical plane (as in the meridian circle when the micrometer is arranged to measure differences of zenith distance or of declination), the telescopes may have any inclination to the horizon, and the angular distance of two threads will be directly measured by moving the theodolite telescope in the vertical plane and bringing its cross-thread successively into coincidence with the images of the two micrometer threads. Denoting the difference of readings of the vertical circle in the two positions by A, and the number of revolutions of the micrometer screw between the threads by M, we have $\tan R = \frac{2 \tan \frac{1}{2} A}{M}$, or, very nearly, $R = \frac{A}{M}$.

Secondly.—If the micrometer threads are parallel to a vertical plane (as in the meridian circle when the micrometer is arranged to measure differences of right ascension), the theodolite is placed as before, and the angular distance of the threads is measured with the horizontal circle. But, in this case, if the telescopes are inclined to the horizon by the angle γ (which is obtained from the vertical circle of the theodolite), the angular distance A, read on the horizontal circle, will exceed that of the threads in the ratio $1 : \cos \gamma$ (see the theory of the altitude and azimuth instrument): so that we shall then have $R = \frac{A \cos \gamma}{M}$.

This ingenious method was suggested by GAUSS.*

47. FIFTH METHOD.—When the telescope is connected with a graduated vertical circle and its micrometer is arranged to measure differences of zenith distance, the value of the screw may be found by means of this vertical circle as follows. Let the telescope be directed towards the nadir and looking into a basin of mercury immediately under it. The rays which diverge from a thread in the focus of a telescope emerge from the objective in parallel lines; they are therefore reflected by the mercury in

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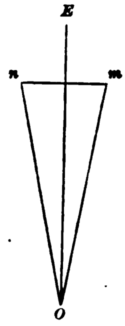
* In 1823, *Astron. Nach.*, Vol. II. p. 371. RITTENHOUSE had previously (in 1785) pointed out the practicability of observing the threads of one telescope through another directed towards the objective of the first, in the *Transactions of the American Philosophical Society*, Vol. II. p. 181.

parallel lines, so that they must be converged by the objective again to the focus, where they form an image of the thread. It is evident that the distance of the reflected images of two micrometer threads will be the same as that of the threads themselves. Let then EO, Fig. 14, be a vertical line drawn through the centre O of the objective, and suppose the fixed and movable threads n and m to be at the same angular distance from EO, on opposite sides of it, or $EOn = EOm$. Then the rays from n, after passing through the objective, form a system of rays parallel to nO, and, after reflection from the mercury (the surface of which is perpendicular to EO), form a system of rays parallel to Om, and therefore the reflected image of n is seen at m. For the same reason, the reflected image of m is seen at n. Now let the telescope be revolved through an angle equal to EOn, so as to make the line nO a vertical line; then the image of n will be found in the vertical line, and will, consequently, be seen in coincidence with n itself. And if the telescope is revolved in the opposite direction through an angle equal to EOm, the image of m will be brought into coincidence with itself. Hence the whole angular motion (A) of the telescope, as measured by the vertical circle, between the two positions in which n and m are seen in coincidence with their own reflected images, respectively, is the required angular distance of the threads; and, the number of revolutions of the micrometer screw between them being M, we have, as in other cases, $R = \frac{A}{M}$.

Fig. 14.



We may, however, dispense with the use of the fixed thread in this process. Let the movable thread be placed in any part of the field, bring it into coincidence with its reflected image by revolving the telescope, and read the circle. Then place it in any other part of the field, bring it into coincidence with its reflected image, and read the circle. The thread having been moved through M revolutions, and the difference of the circle readings being A, we find R as before.

In order that the reflected images of the threads may be visible, it is found necessary to throw light *down* the tube, that is, *from* the ocular. For this purpose, one of the eye pieces (called a *collimating* or *nadir eye piece*) is furnished with a reflector, placed at an angle of 45° with the optical axis, which receives light from a lamp held on one side and reflects it down the tube. This reflector is sometimes placed within the eye piece, between the two lenses; the light is then received through an aperture in the side of the eye tube, and the reflector, if made of metal, is perforated in the centre in order that the field may be visible. A better plan is to place a small piece of very thin mica outside the eye piece, between the outer lens and the eye, and at an angle of 45° with the axis. The mica, being transparent, does not interfere with the view of the field, and is at the same time a very perfect reflector. This plan has the advantage that the mica reflector may be temporarily applied to any of the eye pieces in actual use.

A mercury reflector used, as in this case, to give reflected images of the threads, we shall hereafter designate as a *mercury collimator.*[*]

48. *Effect of temperature upon the value of a revolution of the micrometer screw.*—Changes of temperature affect the angular value of a revolution of the screw in two ways: *first*, by changing the absolute length of the screw itself; *secondly*, by changing the figure of the objective, and thereby also the focal length. Perhaps we should add, also, the almost evanescent change in the focal length resulting from a change in the refractive power of the glass. The whole effect, however, is very small, and may be assumed to be proportional to the change of temperature: so that, if R_0 is the value of a revolution of the screw for an assumed temperature τ_0, R the value for any given temperature τ, we have

$$R_0 = R + R(\tau - \tau_0)x = R[1 + (\tau - \tau_0)x] \qquad (4$$

in which x is to be determined so as to satisfy the observed values of R at different temperatures as nearly as possible, which done by the method of least squares.

EXAMPLE.—Suppose the following values of R have bee observed:

$R =$	26″.557,	26″.532,	26″.529,	26″.500,	26″.498,
for $\tau =$	10°	30°	40°	62°	75° (Fahr.)

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[*] The use of the mercury collimator in connection with the nadir eye piece was introduced by BOHNENBERGER in 1825: v. *Astron. Nach.*, Vol. IV. p. 327.

and it is proposed to determine R_0 for $\tau_0 = 50°$. We shall have the equations

$$\begin{aligned}
R_0 &= 26''\ 557\,(1 - 40x) \\
R_0 &= 26\ .532\,(1 - 20x) \\
R_0 &= 26\ .529\,(1 - 10x) \\
R_0 &= 26\ .500\,(1 + 12x) \\
R_0 &= 26\ .498\,(1 + 25x)
\end{aligned}$$

Let us assume $R_0 = 26.5 + y$; these equations become

$$\begin{aligned}
1062x + y - 0''.057 &= 0 \\
531x + y - 0\ .032 &= 0 \\
265x + y - 0\ .029 &= 0 \\
-\ 318x + y + 0\ .000 &= 0 \\
-\ 662x + y + 0\ .002 &= 0
\end{aligned}$$

Hence, by the usual process in the method of least squares, we find the normal equations

$$\begin{aligned}
2019398x + 878y - 86''.535 &= 0 \\
878x + \ \ 5y - \ \ 0\ .116 &= 0
\end{aligned}$$

whence

$$x = +\ 0.0000355 \qquad\qquad y = +\ 0''.017$$

and, consequently, $R_0 = 26''.517$, and

$$R = \frac{26''.517}{1 + 0.0000355\,(\tau - 50°)}$$

As the coefficient of $\tau - 50°$ is so small, we may take

$$\begin{aligned}
R &= 26''.517\ [1 - 0.0000355\,(\tau - 50°)] \\
&= 26''.517 + 0''.000941\,(50° - \tau)
\end{aligned}$$

This gives for the values of R at the observed temperatures,

$R =$	$26''.555$,	$26''.536$,	$26''.526$,	$26''.504$,	$26''.493$
for $\tau =$	$10°$	$30°$	$40°$	$62°$	$75°$

which agree with the observed values within the probable errors of such determinations.

49. *The position filar micrometer.*—When a filar micrometer is attached to an equatorially mounted telescope, there is usually combined with it a small graduated circle, the plane of which is parallel to that of the micrometer threads, by means of which

the angle which these threads, or the transverse threads, make with a declination circle may be ascertained. The micrometer then serves to measure not only the distance between two stars, but also their *angle of position*; that is, the angle which the arc joining the two stars makes with a declination circle.

The index error of the circle, or its reading for the position angle zero, is best obtained with the telescope in the meridian. Let the micrometer be revolved until the movable thread is perpendicular to the meridian, which will be the case when a star of small declination remains upon the thread throughout its passage across the field. The transverse thread will then represent the meridian, and in all other positions of the telescope, if the equatorial adjustment is good, will represent a declination circle.* If the reading of the position circle is then P_0, and the micrometer is afterwards revolved so that its transverse thread passes through two stars in the field, and the reading becomes P, the apparent position angle of the stars is

$$p = P - P_0 \tag{46}$$

All position angles should be read from 0 to 360° in the same direction. I shall always suppose them to be reckoned from the north through the east.

50. I shall briefly notice some other micrometers hereafter (Chapter X.). What has been given in relation to the filar micrometer was necessary in this place on account of the connection of this instrument with nearly every form of telescope.

THE LEVEL.

51. The spirit level may here be classed among the instruments for measuring small angles, inasmuch as its use in astronomy is not so much to make a given line absolutely level as to measure the small inclination of the line to the horizon. It consists of a glass tube, ground on the interior to a curve of large radius, and nearly filled with alcohol or sulphuric ether. (Water would freeze and burst the tube). The bubble of air occupying the space left by the fluid will always stand at the

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* See, however, Chapter X. in case the adjustment of the equatorial telescope is not quite exact.

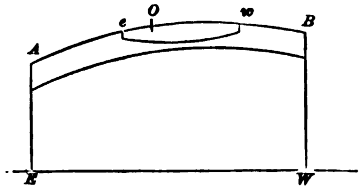
highest point of the curve of the tube; and therefore any change of the relative elevation of the two ends of the tube must be followed by a corresponding change in the position of the bubble. This position of the bubble, therefore, which is read off by means of a scale, or by graduations marked on the tube itself, serves to measure all changes of inclination within the extreme ranges of the arc of the curve employed. The larger the radius of the curve, the more sensitive will the level be. There is, however, obviously a practical limit to the radius, which is determined by the kind of instrument to which the level is to be applied and the degree of accuracy aimed at.

In order to apply the level to the horizontal axis of an instrument, it is either mounted upon two legs, the distance apart of which is nearly equal to the length of the axis; and these legs terminate in Vs, so that the level bears only at two points of the cylindrical pivots of the axis, in which case it is called a *striding* level: or it hangs from the axis by arms, which are recurved and terminate in inverted Vs; and it is then called a *hanging* level.

Plate II., Fig. 4, represents a common form of the striding level, and Fig. 5 is an end view of the legs. The tube *ef* is in this level covered by a larger glass tube *abcd*, to protect the fluid from sudden changes of temperature. These are secured to a bar *AB*, usually a hollow brass cylinder, which is connected with the legs by screws *s* and *t*, which serve to adjust the relation of the level tube to the line of bearing of the Vs of the feet, as will be explained hereafter.

52. In order to investigate the method of using the level, let us first suppose *EW*, Fig. 15, to be a truly horizontal line on which the level *AB* rests. Let *O* be the zero of the graduations; *e* and *w* the ends of the bubble. Let the length of the bubble be $2l$. If the legs *AE* and *BW* were perfectly equal, and *O* were in the middle of *AB*, the readings of *w* and *e* from *O* would be exactly the same, and each equal to l. But, if *BW* is the longer leg, the bubble will stand nearer to *B* by a number x of divisions; and if at the same time the zero *O* stands nearer to *A* than to *B*,

Fig. 15.



at a distance of y divisions from the middle, then the readings will be

$$\begin{aligned} &\text{at } w, && l + x + y, \\ &\text{at } e, && l - x - y. \end{aligned}$$

If now W is raised so that EW becomes inclined to the horizon by the angle b, the bubble will stand nearer to the end B by a number z of divisions, so that the whole readings at w and e will be

$$\left.\begin{aligned} w &= l + x + y + z \\ e &= l - x - y - z \end{aligned}\right\} \quad (47)$$

To eliminate the errors x and y, let the level now be reversed, so that the end A stands over W and B over E. The errors x and y will both change sign; but, the line EW being inclined as before, the readings of the ends of the bubble towards W and E, respectively, will be

$$\left.\begin{aligned} w' &= l - x - y + z \\ e' &= l + x + y - z \end{aligned}\right\} \quad (48)$$

From the equations (47) and (48) we deduce

$$\left.\begin{aligned} \tfrac{1}{2}(w - e) &= x + y + z \\ \tfrac{1}{2}(w' - e') &= -(x + y) + z \end{aligned}\right\} \quad (49)$$

whence

$$z = \tfrac{1}{2}\left[\tfrac{1}{2}(w - e) + \tfrac{1}{2}(w' - e')\right]$$

or

$$z = \frac{(w + w') - (e + e')}{4} \quad (50)$$

whence the practical rule: *Place the level on the line whose inclination is to be measured, and read the divisions at the ends of the bubble; reverse the level, and read again. Add together the two readings lying towards one end of the line, and also the two readings lying towards the other end of the line. One-fourth the difference of these sums is the measure of the inclination. The line is elevated at that end which gives the greatest sum of readings.*

This gives the inclination expressed in divisions of the level; the value of the angle b corresponding to z divisions is known when the angular value d of a division is known, so that

$$b = dz \quad (51)$$

53. The errors x and y are inseparable; we can only find their sum, which is

$$x + y = \frac{(w - e) - (w' - e')}{4} \tag{52}$$

If the errors of the level could be regarded as constant, the value of $x + y$ thus found would enable us to dispense with the reversal of the level, since either of the equations (49) would then determine z; but such constancy is never to be assumed.

54. For greater accuracy, the level may be read a number of times in each position, taking care to lift it up after each reading, so that each observation may be independent of the others. The sums of all the readings at each end of the bubble are to be formed, and the difference of these sums divided by the whole number of readings. The number of readings in the two positions must be equal.

EXAMPLE 1.

A level on the axis of a transit instrument was read as follows:

	W.	E.	$w - e$
1st Position	29.1	31.2	$-\ 2.1$
2d "	35.4	24.9	$+\ 10.5$
	64.5	56.1	4) $-\ 12.6$
	56.1		$x + y = -\ 3.15$ = error of the level.
	4) 8.4		
	$z =$ 2.1		

The value of a division was $d = 1''.25$; and hence

$$b = dz = 2''.63$$

which is the elevation of the west end of the axis.

EXAMPLE 2.

The following readings were obtained with the same instrument:

	W.	E.
1st Position	29.0	31.3
2d "	35.4	24.9
2d "	35.6	24.6
1st "	29.2	31.0
	129.2	111.8
	111.8	
	8) 17.4	
	$z =$ 2.18	$b = 2''.72$

By taking the first and last observations in the same position of the level, as in this example, any small change in the level itself, occuring during the observations, is eliminated.

55. The zero of the level is, however, not always placed near the middle of the tube; it may be at one end and the divisions numbered consecutively through the whole length of the tube. In this case, we have only to find the reading corresponding to the middle of the bubble in each position of the level: the half difference of these readings will evidently be the required inclination. It will be necessary, in the record of the observation, to note the position of the ends of the level, or to indicate in some manner the direction in which the divisions increase, which is usually effected most readily by a conventional use of the algebraic sign, as in the following

EXAMPLE.

A level which is graduated from the end *A* towards the end *B* reads as follows when placed on the axis of a transit instrument:

	W.	E.	Reading of middle of bubble.	or thus:
A east	+ 64.0	+ 13.5	+ 38.75	+ 77.5
B "	— 10.1	— 60.7	— 35.40	— 70.8
			2) + 3.35	4) + 6.7
			$z = +1.675$	$z = +1.675$

Since in the case of a transit instrument we wish to find the *elevation* of the *west* end (a negative elevation being interpreted as a depression), we here mark the level readings with the positive sign when they increase towards the west, and with the negative sign when they increase towards the east. The value of z will then be obtained, with its proper sign, by simply taking the mean of all the readings, as in the last column above.

56. In the above examples, the diameters of the two pivots of the axis on which the level rests are assumed to be the same. When this is not the case, a correction becomes necessary, which will be considered in its place under "Transit Instrument," Chapter V.

57. *To find the value of a division of the level.*—This is most readily done by means of a simple instrument called a *level-trier*. A horizontal bar is supported by two feet at one end and by a single foot-screw at the other. The level is placed on the bar, and the number of turns of the foot-screw necessary to carry the bubble over any given number of divisions is observed. The angular value of a turn of the foot-screw is known from the distance of its threads and the length of the bar. The head of the screw is graduated so that a fraction of a turn may be noted. We can also determine the value of a division by attaching the level tube to a vertical circle and noting the number of seconds on the circle corresponding to a motion (of the circle and level together) which carries the bubble over a given number of divisions. Thus, suppose we read the ends A and B of a level thus attached to a circle, and also read the circle itself, as follows:

A	B	Circle.
5.0	40.2	0° 0′ 40″.
41.3	3.8	0 1 25 .3
36.3	36.4	45 .3

$$\text{(mean) } 36.35d = 45''.3$$

$$d = 1''.246$$

When the level is applied to a telescope which is provided with a micrometer, the value of the divisions of the level may be found from those of the micrometer. An example of this method will be given in connection with the Zenith Telescope, Chapter VIII.

58. *To find the radius of curvature of a level.*—Let n be the length of a division in linear units, d the value of a division in arc, found as above; then the radius will be

$$r = \frac{n}{d \sin 1''}$$

Suppose that in the level of the preceding article we have $n = 0.103$ inch, then we find, for this level, $r = 17051$ inches, or 1421 feet.

59. *The value of a division of a level may be affected by changes of temperature.*—This will be discovered by taking observations for determining this value at two temperatures as different as pos-

sible. The proper value to be used for any intermec perature will then be found by interpolation.

60. *It is also possible that the radius of curvature of dif tions of the tube may be different.*—This, of course, is a radi in the construction of the instrument: its effect is tc ferent angular values to divisions of equal absolute different portions of the tube. The existence of sucl will be discovered by determining the value of a divi pendently at various points; and it is proper to exami levels in this manner. A level thus defective should b as unfit for any refined observation; but, if no other ca a careful investigation might determine a system of c to be applied to the different readings.

61. It remains to be shown how to effect the mechanic ment of the level. 1st. The bubble should stand *nea* middle of the tube when the level stands upon any l line. This is quickly brought about by finding the err level $= x + y$, (as in Example 1, Art. 54) and then tu screws t, t', Plate II. Fig. 5, until the bubble has move this quantity in the proper direction. 2d. The axis o should be parallel to the line joining the angle of the feet, and, consequently, parallel to the axis of an instr which it rests. This is tested by slightly revolving o the level on the axis of the instrument, so that the thrown out of a perpendicular on either side. If the a level tube is not parallel to the line joining the fee *cross-wise* with respect to that line, this revolution will bubble to change its position, and it will be easy to se direction the correction must be made. The adjustmen by the screws s, s'.

CHAPTER III.

INSTRUMENTS FOR MEASURING TIME.

62. *Chronometers.*—The chronometer is merely a very perfect watch, in which the balance wheel is so constructed that changes of temperature have the least possible effect upon the time of its oscillation. Such a balance is called a *compensation* balance. A chronometer may be well compensated for temperature and yet its *rate* may be gaining or losing on the time it is intended to keep: the compensation is good when changes of temperature do not affect the rate. It is not necessary that a chronometer's rate should be zero (or even very small, except that a small rate is practically *convenient*); it is sufficient if the rate, whatever it is, remains constant. The indications of a chronometer at any instant require a correction for the whole accumulated error up to that instant. If the correction is known for any given time, together with the rate, the correction for any subsequent time is known. The methods of finding these quantities are given in Vol. I., Chapter V.

63. *Winding.*—Most chronometers are now made to run either eight days or two days. The former are wound every seventh day, the latter daily, so that in case the winding should be forgotten for twenty-four hours the chronometers will still be found running. But it is of importance that they should be wound regularly at stated intervals; otherwise an unused part of the spring comes into action, and an irregularity in the rate may result.

Chronometers are wound with a given number of half turns of the key. It is well to know this number, and to count in winding, in order to avoid a sudden jerk at the last turn: still the chronometer should always be wound *as far as it will go*, that is, until it resists further winding. This resistance is produced not by the end of the chain, but by a catch provided to act at the proper time and thus protect the chain.

When a chronometer has stopped, it does not again start immediately after being wound up. It is necessary to give the whole instrument a quick rotatory movement, by which the balance wheel is set in motion. This must be done with care, however, and with little more force than is necessary to produce the result; afterwards the chronometer must be guarded from all sudden motions.

The hands of a chronometer can be moved without injury to the instrument, so that it may be set proximately to the true time. It is, however, not advisable to do this often.

64. *Transporting.*—Chronometers transported on board ship should be placed as near the centre of motion as possible, and allowed to swing freely in their gimbals, so that they may preserve a horizontal position. They should also be kept as nearly as possible in a uniform temperature.

When transported by land, the chronometer should no longer be allowed to swing in its gimbals, but is to be fastened by a clamp provided for the purpose; for the sudden motions which it is then liable to receive would set it in violent oscillation in the gimbals, and produce more effect than if allowed to act directly.

Pocket chronometers should be kept at all times in the same position: consequently, if actually carried in the pocket during the day, they should be suspended vertically at night.

It has been found that the rates of chronometers have been affected by masses of iron in their vicinity, indicating a magnetic polarity of their balances. Such polarity may exist in the balance when it first comes from the hands of the maker, or it may be acquired by the chronometer standing a long time in the same position with respect to the magnetic meridian. In order to avoid any error that might result from this polarity (whether known or unknown), it will be well to keep the chronometers always in the same position. Hence, they should not be removed from the ship to be *rated;* but their rates should be found after they are placed in the position they are to occupy.

The rate of a chronometer when transported is seldom the same as when at rest. The travelling rate is found by comparing the observations taken at the same place before and after the journey, or from observations at two places whose difference of longitude is perfectly well known. A list of well determined "differences of longitude" is given in RAPER's *Practice of Navigation*, for the use of navigators in finding the sea rates of their chronometers. (See Vol. I. Art. 258).

65. *Correction for temperature.*—An absolutely perfect compensation for temperature in chronometers is hardly to be expected. It has been found* that the average temperature compensation of chronometers is of such a nature as to cause the instrument to lose on its daily rate when exposed to a temperature either above or below a certain point for which the compensation is most perfect. Professor BOND found for a large number of chronometers that if ϑ_0 be the temperature of best compensation, ϑ that of actual exposure, the rate may be expressed for a range of 20° above and below ϑ_0 by the formula

$$m = m_0 + k(\vartheta - \vartheta_0)^2 \tag{53}$$

in which k is a constant, and has, with rare exceptions, a positive sign, and m_0 and m are the *rates* at the temperatures ϑ_0 and ϑ, respectively; losing rates being positive.

M. LIEUSSON, from a very extended examination of the performance of chronometers on trial at the Observatories of Greenwich and Paris, finds that the rate varies both with the temperature and with the age of the oil with which the pivots are lubricated. The thickening of the oil tends to diminish the amplitude of the vibration of the balance, and thus produces an acceleration of the chronometer. This acceleration is almost exactly proportional to the time, so that for any time t the rate may be found by the complete formula

$$m = m_0 + k(\vartheta - \vartheta_0)^2 - k't \tag{54}$$

in which k' is the daily change of rate resulting from the gradual thickening of the oil. The constants k and k' will be different for every chronometer, and are determined by experiment for each instrument.

66. *Comparison of Chronometers.*—When one or more chronometers are to be regulated by means of astronomical observa-

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* LIEUSSON, Récherches sur les variations de la marche des pendules et des chronomètres; Paris, 1854. G. P. BOND, in his report on the longitude in the Report of the Superintendent U. S. Coast Survey for 1854, App. p. 141.

tions, these observations are made with but one of them, and the corrections of all the others are found by comparing them with this. On board ship the chronometers are never brought on deck; but the observations are made with a watch (often called a "hack-watch"), which is compared with the chronometer either before or after, or both before and after, the observations. The double comparison is necessary where extreme precision is required, in order to eliminate any difference of the rates of the watch and chronometer.

EXAMPLE.

An observation is recorded by a hack-watch at the time $10^h\ 12^m\ 13^s.3$, and the following comparisons are made with the chronometer. Required the time of the observation by the chronometer.

Chron.	$8^h\ 17^m\ 0^s.$	$8^h\ 27^m\ 0^s.$
Watch	10 8 9.5	10 18 8.0
Reduction	—1 51 9.5	—1 51 8.0

Here the watch loses $1^s.5$ in 10^m: hence, in 4^m, the time from the first comparison to the observation, it loses $1^s.5 \times \frac{4}{10}$ or $0^s.6$, so that the difference at the time of the observation is $1^h\ 51^m\ 8^s.9$: therefore we have

Watch time of obs.	=	$10^h\ 12^m\ 13^s.3$
Reduction to chron.	= —	1 51 8.9
Chron. time of obs.	=	8 21 4.4

Comparison by coincident beats.—When two chronometers are compared which keep the same kind of time, and both of which beat half seconds, it will mostly happen that the beats of the two instruments are not synchronous, but one will fall after the other by a certain fraction of a beat, which will be pretty nearly constant, and must be estimated by the ear. This estimate may be made within half a beat, or a quarter of a second, without difficulty, but it requires much practice to estimate the fraction within $0^s.1$ with certainty. But if a mean time or *solar* chronometer is compared with a *sidereal* chronometer, their difference may be obtained with ease within *one-twentieth* of a second. Since 1^s sidereal time is less than 1^s mean time, the beats of the sidereal chronometer will not remain at a constant fraction behind those of the solar chronometer, but will gradually gain n them, so that at certain times they will be coincident. Now, f the comparison is made at the time this coincidence occurs, here will be no fraction for the ear to estimate, and the differnce of the two instruments *at this time* will be obtained exactly. The only error will be that which arises from judging the beats o be in coincidence when they are really separated by a small raction; and it is found that the ear will easily distinguish the eats as not synchronous so long as they differ by as much as $^{s}.05$; consequently the comparison is accurately obtained within that quantity. Indeed, with practice it is obtained within $0^{s}.03$, or even $0^{s}.02$. Now, since 1^{s} sidereal time = $0^{s}.99727$ mean time, the sidereal chronometer gains $0^{s}.00273$ on the solar chronometer in 1^{s}; and therefore it gains $0^{s}.5$ in 183^{s}, or very nearly in 3^{m}. Hence, once every three minutes the two chronometers will beat together.* When this is about to occur, the observer begins to count the seconds of one chronometer, while he directs his eye to the other; when he no longer perceives any difference in the beats, he notes the corresponding half seconds of the two instruments.

EXAMPLE.

A solar and a sidereal chronometer were compared by coincident beats, as follows:

Solar chron.	4^{h} 16^{m} 0^{s}.		4^{h} 19^{m} 10^{s}.
Sidereal "	1 3 11.5		1 6 22.
Difference	3 12 48.5		3 12 48.

Here the interval between the two comparisons being about 3^{m}, the sidereal chronometer has gained a beat. In order to judge of the accuracy of the comparisons, let us reduce the second to the time of the first. The solar interval is, by the solar chronometer, 3^{m} 10^{s}; the corresponding sidereal interval is, by the tables, 3^{m} $10^{s}.52$; the second comparison reduced to the time of the first stands as follows:

Solar chron.	4^{h} 16^{m} 0^{s}.
Sid. "	1 3 11.48
Difference	3 12 48.52

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* They will either beat together, or at least their beats will both fall within a space of time equal to one-half of $0^{s}.00273$.

that is, it agrees with the first comparison within $0^s.02$. Suppose that at the second comparison the time when the beats were coincident was mistaken, and the observer made his comparison 10^s later; he would have had 10^s more on each chronometer, and consequently would have put down the comparison thus:

Solar chron.	$4^h\ 19^m\ 20^s.$
Sid. "	$1\ \ 6\ \ 32\ .$

The mean interval between the comparisons would have been $3^m\ 20^s$, and the equivalent sidereal interval is $3^m\ 20^s.55$, so that this second comparison reduced to the time of the first would have stood thus:

Solar chron.	$4^h\ 16^m\ \ 0^s.$
Sid. "	$1\ \ 3\ \ 11\ .45$
Difference	$1\ \ 12\ \ 48\ .55$

that is, the two comparisons would still have agreed within $0^s.05$. The observer can in this way satisfy himself by a few trials that the two chronometers can really be compared within $0^s.05$ with certainty.

When two solar chronometers are to be compared together, it will be most accurately done by comparing each with a sidereal chronometer by coincident beats, and reducing the comparisons as follows:

EXAMPLE.

Two solar chronometers *A* and *B* are compared with a sidereal chronometer *C*, as below:

	C $6^h\ 13^m\ 20^s.$	*A*	$4^h\ 40^m\ 10^s.5$
	C $6\ \ 15\ \ 15\ .$	*B*	$5\ \ 21\ \ 13\ .$
Sid. interval	$1\ \ 55\ .$	=	$1\ \ 54\ .69$ solar
	B reduced to time of *A*	=	$5\ \ 19\ \ 18\ .31$
	Difference of *A* and *B*	=	$0\ \ 39\ \ \ 7\ .81$

The intermediate chronometer used for comparison is not necessarily a sidereal one. It may be a mean time chronometer which does not beat half seconds; for example, a pocket chronometer which beats 13 times in 6 seconds. In this case each beat of the pocket chronometer is worth $\frac{6}{13}$, and therefore differs from that of a chronometer beating half seconds by $\frac{1}{26}$ of a second.

The inaccuracy of a coincidence cannot exceed this quantity, and the comparison may, therefore, also be made within $\frac{1}{28}$ of a second.

67. *Probable error of an interpolated value of a chronometer correction.*—When the corrections ΔT and $\Delta T'$ for the times T and T' are given, the correction for any other time $T + t = T' - t'$ is found by interpolation. Denoting the rate by δT, and the required correction by x, we have

$$\text{either } x = \Delta T + t\,.\,\delta T \qquad \text{or} \qquad x = \Delta T' - t'\,.\,\delta T$$

Now, granting that the given quantities ΔT and $\Delta T'$ are perfectly correct, the interpolated values of x will also be correct if there are no *accidental* irregularities in the going of the chronometer. But such accidental irregularities certainly exist, and tend to diminish the *weight* to be assigned to any interpolated value of the correction. If the mean (accidental) error in a unit of time is ε, the mean error in the interval t is, by the theory of least squares, $\varepsilon\sqrt{t}$, and the weight is inversely proportional to the square of this error, that is, inversely proportional to t. We shall have then

$$x = \Delta T + t\,.\,\delta T \text{ with the weight } \frac{k}{t}$$

$$x = \Delta T' - t'\,.\,\delta T \quad \text{"} \quad \text{"} \quad \text{"} \quad \frac{k}{t'}$$

in which k is an undetermined constant.

Multiplying each value by its weight, and dividing the sum by the sum of the weights (according to the usual process in the method of least squares), we have

$$x = \frac{t'\,.\,\Delta T + t\,.\,\Delta T'}{t + t'}, \text{ with the weight } = k\left(\frac{t + t'}{tt'}\right)$$

$$\text{or with the mean error } = \varepsilon\sqrt{\frac{tt'}{t + t'}} \qquad (55)$$

This error is zero either for $t = 0$ or $t' = 0$, and is a maximum for $t = t'$, that is, when the correction is found for the middle time between the two given times T and T'.

68. If, however, the chronometer has accelerated or retarded uniformly, the error will obtain a different expression. Let the

rate at the time T be δT and at the time T' be $\delta' T$. The acceleration in a unit of time is

$$\delta'' T = \frac{\delta' T - \delta T}{t + t'} \tag{56}$$

The rate at the middle instant between T and $T + t$ is $\delta T + \frac{1}{2} t . \delta'' T$; and at the middle instant between T' and $T' - t'$ it is $\delta' T - \frac{1}{2} t' . \delta'' T$; hence we have

$$x = \Delta T + t\,(\delta T + \tfrac{1}{2} t . \delta'' T) = \Delta T + t . \delta T + \tfrac{1}{2} t^2 . \delta'' T$$
$$x = \Delta T' - t'\,(\delta' T - \tfrac{1}{2} t' . \delta'' T) = \Delta T' - t' . \delta' T + \tfrac{1}{2} t'^2 . \delta'' T$$

Multiplying the first by t', the second by t, and dividing the sum of the products by $t + t'$, we have

$$x = \frac{t' . \Delta T + t . \Delta T'}{t + t'} - tt' . \frac{\delta' T - \delta T}{t + t'} + \tfrac{1}{2} tt' . \delta'' T$$

or

$$x = \frac{t' . \Delta T + t . \Delta T'}{t + t'} - \tfrac{1}{2} tt' . \delta'' T \tag{57}$$

whence it appears that the error of the value obtained by simple interpolation, or upon the supposition of a uniform rate, is $\frac{1}{2} tt' . \delta'' T$, and this error is also a maximum for the middle instant between T and T', when $t = t'$, and vanishes for $t = 0$ or $t' = 0$.

69. Every chronometer has, moreover, its own peculiariti which render the application of any formula for weight more less uncertain. Struve found that, for the greater number o the chronometers which he tried, the mean error of an interp lated value of their corrections could be expressed by the empir cal formula $\varepsilon \cdot \frac{tt'}{t + t'}$, differing from the above theoretical formul by the omission of the radical sign. (*Expédition Chronométrique* p. 101.)

70. *Clocks.*—The astronomical clock is provided with a compensation pendulum, by which the effect of temperature is even more completely eliminated than in chronometers. The only forms in use are the *Harrison* (the *gridiron*) and the *mercurial* pendulum.

In the gridiron pendulum the rod is composed (in part) of a number of parallel bars of steel and brass, so connected together that the expansion of the steel bars produced by an increase of temperature tends to depress the "bob" of the pendulum, the greater expansion of the brass bars tends to raise it, so that when the total lengths of the steel and brass bars have been properly adjusted a perfect compensation occurs, and the centre of oscillation remains at a constant distance from the point of suspension. The rate of the clock, so far as it depends upon the length of the pendulum, will therefore be constant.

In the mercurial pendulum, the weight which forms the bob in other cases is replaced by a cylindrical glass vessel nearly filled with mercury. With an increase of temperature the rod lengthens, but the mercury expanding must rise in the cylinder, so that when the quantity of mercury is properly proportioned to the length of the rod the centre of oscillation remains at the same distance from the point of suspension. If a clock is to be exposed to sudden changes of temperature, the gridiron pendulum will be preferable to the mercurial, as the large body of mercury will obtain the temperature of the air more slowly than the thin metal rods.

In setting up the clock the chief point to be observed is that its alternate beats are exactly equal. The pendulum usually carries a pointer at its lower extremity which indicates upon an arc below the pendulum the extent of a vibration. Let the pendulum be drawn towards one side gently, until a tooth of the escapement wheel is just freed, and mark the point of the arc at which this occurs; then let the pendulum be drawn towards the other side, and mark the point of the arc at which a tooth escapes. Find the middle point A of the included arc. Then let the pendulum come to rest in a vertical position: if the pointer is on A the adjustment is correct, and the vibrations on each side will be isochronous; if not, the clock case must be moved until the vertical pendulum is directed exactly towards A. The equality of the vibrations may also be tested by the electro-chronograph, hereafter described.

What has been said above respecting the comparison of chronometers will apply, with scarcely any modification, to that of clocks, or of a clock with a chronometer.

In the observatory, a clock regulated to *sidereal* time is the indispensable companion of the transit instrument. The standard or *normal* clock of an observatory is carefully mounted upon a stone pier which is disconnected from the walls or floors of the building, and also protected as much as possible from changes of temperature. For the latter purpose it is sometimes imbedded in a stone pier, in an air-tight compartment below the surface of the ground. STRUVE found that the changes of barometric pressure, by varying the resistance which the air opposes to the motions of the pendulum, caused a variation in the rate of the normal clock of the Pulkowa Observatory of 0ˢ.32 for a variation of one English inch of the barometer.*

71. *The electro-chronograph.*—This contrivance may be regarded as an appendage of the astronomical clock, and bearing the same relation to it that the reading microscope bears to a divided circle; for its chief use is to subdivide the seconds of the clock, and thus to measure micrometrically the smallest fractions of time. In order to effect this micrometric subdivision, the clock beats are converted from audible into visible signals, which are recorded on paper by means of an electro-magnet. The instant of the occurrence of any phenomenon is also registered by a visible signal on the same paper, and thus referred to the preceding clock beat with great precision. This general statement covers a great variety of special contrivances leading to the same end. We shall here treat only of those which, thus far, have been most used.

72. The simplest form of register is that known on our telegraphic lines as MORSE'S, in which a fillet of paper is reeled off at a uniform velocity by means of a train of wheels moved by a weight. The fillet passes over a small cylinder and just under a hard steel point, or pen (as it is called, for brevity), which is so connected with the armature of an electro-magnet that whenever the electric circuit of the galvanic battery is established, the pen is pressed upon the paper and leaves a visible mark. The wire from one pole of the battery which passes around the electro-magnet does not return directly to the other pole, but first passes through the clock, where, by a contrivance presently to be described, the circuit is broken and restored at every second. The Morse fillet in running off, therefore, receives an impression every second, and thus becomes graduated into spaces representing seconds. These spaces are greater or less according to the

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* *Description de l'observatoire astronomique central de Poulkova*, p. 220.

velocity with which the paper runs off; an inch per second is even more than sufficient, as it is easy to divide an inch into fifty parts by a scale, even without the aid of a magnifier.

It is of importance that the paper should run off with a uniform velocity; at least, no sudden changes of velocity should occur. In the Morse register this regularity is maintained by an ordinary fly-wheel. In the *spring-governor*, invented by the Messrs. BOND, a fly-wheel and pendulum are both used. The pendulum secures the condition that the seconds shall be of the same length, while the fly is supposed to maintain a uniform motion during the second. In this and in other chronographic instruments there is substituted for the fillet a sheet of paper wrapped about a cylinder which makes one revolution per minute. As the cylinder revolves, a fine screw causes it to move also in the direction of its length, so that the pen records in a perpetual spiral, and when the paper is removed from the cylinder the successive minutes are found recorded in successive parallel lines. One such sheet will contain the record of upwards of two hours' work. This cylindrical register is preferable to the Morse fillet for most chronographic purposes, on account of the convenience with which the sheets may be read off and filed away for subsequent reference.

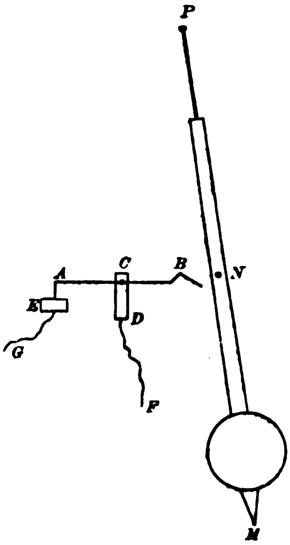
In SAXTON'S cylindrical register the movement is regulated by a combination of the crank motion with the vibration of two pendulums.

Professor MITCHEL employed a circular disc upon which the successive minutes occupied concentric circles, each of which was graduated into seconds with great precision by connection with the clock.

73. The connection of the clock with the register is made in one of two ways; either so as to *break* the circuit every second, or so as to *make* it.

The method most used of causing the clock to break the circuit is that suggested by Mr. SAXTON, of the Coast Survey. *ACB*, Fig. 16, is a small and very light "tilt-hammer," usually made of platinum wire, mounted upon a pivot *C*, so that the end *A* shall slightly preponderate and rest upon a platinum plate *E*. The end *B* is bent into an obtuse angle. The wire *F* from one pole of the galvanic battery is constantly connected with the tilt-hammer through the metallic support *D*. Another wire *G* is

Fig. 16.



connected with the plate *E*, and gc first to the electro-magnet of the regist and thence to the other pole of the b tery. This apparatus is placed in t clock case in front of the pendulum *P*, with the vertex of the angle *B* in a v tical line below the point of suspensi *P*. A small pin *N* projecting from t pendulum rod passes over the angle at each vibration of the pendulum, an by thus depressing the end *B* of the ti hammer, raises the end *A* from the pla *E* and breaks the circuit, which oth wise is complete through the connecti of the portion *AC* of the tilt-hamm with both the wires *F* and *G*. The i terval of time during which the circ is broken will be longer or shorter acco ing as the pin *N* strikes the sides of the angle *B* farther from nearer to its vertex. It may be adjusted so that the break sh last but one-twentieth of a second, or for a shorter time required.

Now, if the pen of the register is kept pressed upon the pap by the attraction of the electro-magnet, it is clear that the brea produced by the clock will produce corresponding breaks in t continuous line made by the pen, and the paper will be grad ated into seconds, thus:

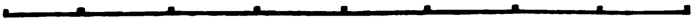
———— ———— ———— ———— ———— ———— ———— ————

But if the pen is pressed upon the paper by a spring acti *against* the attraction of the magnet, then each break produc by the clock will give a corresponding short mark on the pap with an intervening blank, so that the paper will be graduat into seconds, thus:

–   –   –   –   –   –   –   –

The first of these methods is commonly preferred.

In the cylindrical registers a pen carrying ink is used, and tl breaking of the circuit by the clock does not cause the pen rise from the paper, but moves it laterally; in this case the paper is graduated into seconds, thus:

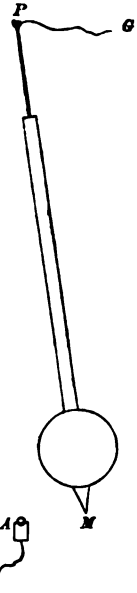


Dr. **Locke** also employed a tilt-hammer for breaking the circuit; but the hammer was worked by the teeth of a wheel placed on the axis of the escapement wheel of the clock.

At the Washington Observatory, the record on the paper of the cylindrical registers has also been made by fine punctures produced by a needle point. The needle has a little play which prevents its resisting the motion of the cylinder during the time required for the needle to enter and leave the paper.

74. The most simple method by which the pendulum *makes* the circuit at each beat is also the suggestion of Mr. Saxton. A small globule of mercury is placed just below the pendulum, as at *A*, Fig. 17, upon a metallic support which by the wire *F* is in connection with one pole of the battery. Another wire *G* is connected with the metallic support of the pendulum rod at *P*, and is connected with the other pole of the battery through the electro-magnet. A fine point *m* upon the extremity of the pendulum passes through the globule at each vibration and establishes the electric circuit, for a small fraction of a second, through the pendulum itself. The effect will be to graduate the paper in one of the above mentioned ways according to the arrangement of the register.

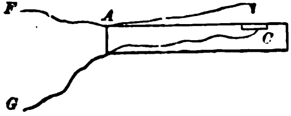
Fig. 17.

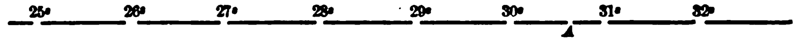


75. Having thus obtained a graduated visible time-scale, its application to the exact recording of an astronomical observation is very simple. We have only to let one of the wires in connection with the magnet pass, on its way to the battery, *through the hand of the observer*, where the circuit may be broken and restored at pleasure. A small piece of apparatus called a *signal-key* is used for this purpose. It consists of a piece of wood, five or six inches in length, Fig 18, on which is fastened a metallic spring *AB*, which by a very slight pressure of the finger can be brought into contact with a metallic

plate at *C.* Conceive the wire in its circuit from the magnet to the battery to be severed at the key; let one end *F* be connected with the spring *AB*, the other end *G* with the plate *C.* The continuity of the wire may be regarded as restored whenever the spring is pressed into contact with the plate *C.* This constitutes a *make-circuit key.* It is easy to see how the arrangement may be reversed, so that by pressing the spring the continuity of the wire is interrupted, constituting a *break-circuit key.* Now, whenever the observer taps on his key he will produce upon his graduated time scale a mark similar to that of the clock, but mostly distinguishable from it. For example, on a Morse-fillet, and with a break-circuit key, we have

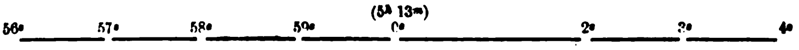
Fig. 18.



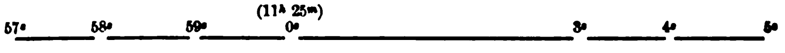


Here, at *A*, is a record of an astronomical observation occurring between the 30th and 31st second. By a scale of equal parts, we find the distance of *A* from 30^s is 0.61 of the distance from 30^s to 31^s, and hence the instant of the observation is $30^s.61$.

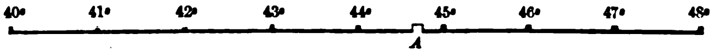
In order to identify the seconds on the register, a peculiar mechanical contrivance (which need not be described here) is employed, by means of which one of the breaks is omitted at the beginning of each minute of the clock; thus, for example:



The observer has only to identify the minute and write it on the fillet, as in this example. For greater security, sometimes, every fifth minute is also distinguished by the omission of two consecutive breaks, thus:



A record on a cylindrical register stands thus:



where the observation *A* occurs at $44^s.71$. The observer's signal is generally distinguishable from the clock signals, as in this example, by its form.

In all the forms of recording it must be observed that the *beginning* of the break, or dot, marks the point of time recorded.

In order to read off the record with the greatest convenience, a glass scale is used, on which are etched eleven equidistant parallel lines, dividing the second of the chronograph into tenths; the hundredths are obtained by estimation. (Plate I. Fig. 3.)

When the length of a second on the register is greater than the perpendicular distance of the extreme lines of the scale, we have only to place the scale obliquely on the line of seconds, always causing their extreme lines to pass through two consecutive second dots. Sometimes the lines on the scale are made divergent; it is then always applied so that the line of seconds shall be perpendicular to the middle line of the scale, and at the point where the distance of the extreme lines is equal to the length of the second. (Plate I. Fig. 2.)

76. When the pen of the chronograph is made to press upon the paper by the attraction of the electro-magnet upon its armature, a certain small fraction of time elapses after the closing of the circuit (by the clock or by the observer) before the signal is actually impressed upon the paper. This time is called the *armature time*. If it were certainly constant, and the same for the clock signals and for those of the observer, it would have no effect upon the difference of time between any two recorded phenomena. But the armature time probably varies both with the strength of the battery and the length of the wire through which the electric current passes. The variable error which would thus be introduced into our results is avoided, or at least very much reduced in magnitude, by employing *break-circuit* signals exclusively; for the interval of time between the *breaking* of the circuit and the *cessation* of the action of the magnet is probably smaller and more constant than that between the *making* of the circuit and the *commencement* of the action of the magnet.

77. To give the reader a just appreciation of the degree of accuracy attained in the recording of time by the chronograph, full size specimens of the records on three different kinds of registers are given in Plate I. Figs. 4 and 5 are specimens of clock signals as recorded on a Morse-Fillet and Saxton's Cylindrical Register used on the United States Coast Survey. Fig. 6 is a specimen of clock signals and a number of actual

observations of stars' transits recorded on **Bond's** Spring-Governor Register, which has been obligingly furnished by Professor G. P. **Bond**. Figs. 2 and 3 exhibit in full size the manner in which the glass scales for reading these records are ruled. Fig. 1 exhibits the reticule of a transit instrument, provided with twenty-five transit threads, for determining the longitude by the electric telegraph. (Vol. I., p. 344).

CHAPTER IV.

THE SEXTANT, AND OTHER REFLECTING INSTRUMENTS.

78. **The sextant**, of all astronomical instruments, is the most especially adapted to the purposes of the navigator and the scientific explorer, as it is at once portable and extremely simple of manipulation, requires no fixed support, and furnishes its data with the least expenditure of the time of the observer. Being held in the hand, and having small dimensions, the extreme accuracy of fixed instruments is not to be expected from it, but in the hands of a practised observer the precision of the results obtained with it is often surprising.*

79. The optical principle upon which the sextant and other reflecting instruments are founded is the following: "If a ray of light suffers two successive reflections in the same plane by two plane mirrors, the angle between the first and last directions of the ray is twice the angle of the mirrors."

Let M and m, Fig. 19, be the two mirrors. Since the direct and reflected rays are always found in a plane perpendicular to the reflecting surface,—called the *plane of reflection*,—it follows that, after two successive reflections from two surfaces, the last direction of the ray will be found in the same plane as the first only when the plane of reflection is perpendicular to both mirrors. In the diagram, let the plane of reflection be that of the paper, the lines *M* and *m* being the intersections of this plane with the surfaces of the mirrors. Let *AM* be the direct ray falling upon the mirror *M*, which we shall first suppose to lie in the direction *MC*; let *Mm* be the direction of the ray after the first reflection, and *mE* its direction after the second reflection. Draw *MB* parallel to *Em*, *MP* perpendicular to *MC*, and *Mp* perpendicular to the mirror *m*. The angle *AMB* is the difference of the first and last directions of the ray. The angle *PMp* is the same as the angle contained by the mirrors, being obviously equal to *MCm*. We have, therefore, to prove that $AMB = 2PMp$.

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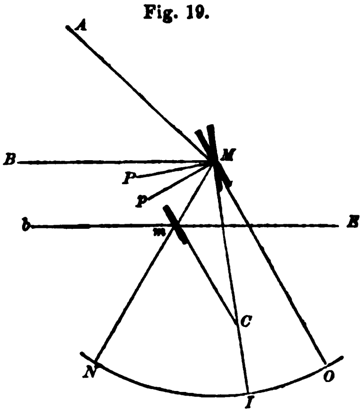
* The first *inventor* of the sextant (or quadrant) was **Newton**, among whose papers a description of such an instrument was found after his death; not, however, until after its re-invention by **Thomas Godfrey**, of Philadelphia, in 1730, and, perhaps, by **Hadley**, in 1731.

Fig. 19.



If we conceive a perpendicular drawn at *m*, parallel to *Mp*, we easily see that *pMm* is equal to the angle of incidence of the ray *Mm* falling upon *m*, and *pMB* is equal to the angle of reflection of the same ray; and since these angles, by a principle of Optics, are equal, we have

$$pMm = pMB = PMp + PMB$$

But, on the same principle, we have

$$PMm = PMA = AMB + PMB$$

The difference of these two equations gives

$$PMp = AMB - PMp$$

whence

$$AMB = 2PMp$$

80. In order to apply this principle, let the mirror *M* be attached to an index arm *MCI*, which revolves upon a pivot at *M* in the centre of a graduated arc *OIN*, and let *m* be permanently secured in a fixed position at right angles to the plane of this arc. Let *MO* be the direction of the central mirror and of the index arm when it is parallel to the fixed mirror *m*, and let the graduation of the arc commence at *O*. In this position, an incident ray *BM* from a distant object *B* will be reflected first to *m* and then in the direction *mE*, which will be parallel to the

first direction *BM*. If then the object is so distant that two rays from it, *BM* and *bm*, falling upon the two mirrors, will be sensibly parallel, an observer's eye at *E* will receive both the direct ray *bm* and the reflected ray *mE* at the same time. Hence the observer will see two images of the same object—a direct and a reflected image—in *coincidence*.

In the next place, let the mirror *M* be revolved into the position *MCI*, in which a ray *AM* from a second object *A* is reflected finally into the line *mE*. The observer now sees the direct image of the object *B* in apparent coincidence with the reflected image of the object *A*. The angular distance *AMB* of the two objects is then equal to twice the angle of the mirrors, that is, to twice *MCm* or to twice *OMI*. The arc *OI*, which measures this angle, is then the measure of one-half the angular distance of the objects. If the arm *MI* carries a vernier at *I*, the exact value of the arc will be obtained. In order to avoid the necessity of doubling this value after reading, a half degree of the arc is numbered as a whole degree: thus, an arc of 60° is divided into 120 equal parts, each of which is reckoned as a degree. As the index arm *MI* cannot pass beyond the position *MmN*, where it comes against the fixed mirror, it is not found practicable, in this form of the instrument, to extend the arc *OD* much beyond 60°, and it is from this circumstance that the instrument derives its name.

81. Plate III. Fig. 1 represents the most common form of the sextant constructed upon these principles.

The frame is of brass, constructed so as to combine strength with lightness; the graduated arc, inlaid in the brass, is usually of silver, sometimes of gold, or platinum. The divisions of the arc are usually 10′ each, which are subdivided by the vernier to 10″. The handle *H*, by which it is held in the hand, is of wood. The mirrors *M* and *m* are of plate glass, silvered. The upper half of the glass *m* is left without silvering, in order that the direct rays from a distant object may not be intercepted. To give greater distinctness to the images, a small telescope *E* is placed in the line of sight *mE*. It is supported in a ring *KK*, which can be moved by means of a screw in a direction at right angles to the plane of the sextant, whereby the axis of the telescope can be directed either towards the silvered or the transparent part of the mirror. This motion changes the plane of reflection, which, however, remains always parallel to the plane of the sextant: the use of the motion being merely to regulate the relative brightness of the direct and reflected images.

The vernier is read with the aid of a glass *R* attached to an arm which turns upon a pivot *S*, and is carried upon the index bar.

The *index glass M*, or central mirror, is secured in a brass frame, which is firmly attached to the head of the index bar by screws *a*, *a*, *a*. This glass is generally set perpendicular to the plane of the sextant by the maker, and there are no adjusting screws connected with it.

The fixed mirror *m* is usually called the *horizon glass*, being that through which the horizon is observed in taking altitudes. It is usually provided with screws by which its position with respect to the plane of the sextant may be rectified.

At *P* and *Q* are colored glasses of different shades, which may be used separately or in combination, to defend the eye from the intense light of the sun.

I shall first treat of those common adjustments of the sextant which the observer is obliged to attend to in the ordinary use of the instrument, and shall afterwards treat fully of its mathematical theory.

82. *Adjustment of the index glass.*—The reflecting surface of the glass must be perpendicular to the plane of the sextant. The simplest test of its perpendicularity is the following. Set the index near the middle of the arc; then, placing the eye very nearly in the plane of the sextant, and near the index glass, observe whether the arc seen directly and its reflected image in the glass appear to form one continuous arc, which will be the case only when the glass is perpendicular. The glass leans *forward* or *backward* according as the reflected image appears *too high* or *too low*. It may be corrected by putting a piece of paper under one edge of the plate by which the glass is secured to the index arm, first loosening the screws *a*, *a*, *a* (Pl. III. Fig. 1) for that purpose. Or we may make the adjustment, as it is done by the instrument makers, by removing the glass and filing down one of the metallic points against which the glass bears when secured in its frame.

83. *Adjustment of the horizon glass.*—This must also be perpen-

dicular to the plane of the sextant. The index glass having been previously adjusted, if by revolving it (by means of the index arm) there is found one position in which it is parallel to the horizon glass the latter must also be perpendicular to the plane of the sextant. The test of this parallelism is the following. Put in the telescope, and direct it towards a star. Move the index until the reflected image of the star appears to pass the direct image. If one image passes exactly over the other, it will be possible to bring both into exact coincidence, so as to form but a single image; and it is evident that when this coincidence takes place the mirrors must be parallel. If one image passes on either side of the other, the horizon glass needs adjustment.

The perpendicularity of the horizon glass may also be tested as follows. Hold the instrument so that its plane shall be nearly vertical, and bring the direct and reflected images of the sea horizon into coincidence. Then incline the instrument until its plane makes but a small angle with the horizon; if the images still coincide, the two glasses are parallel: consequently, if the index glass is perpendicular to the plane of the sextant, the horizon glass is also in adjustment.

Any distant and well defined terrestrial object may be substituted for the star or the sea horizon. A star, however, is to be preferred; and one of the third magnitude will afford greater precision than the brighter ones.

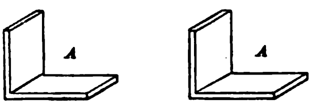
84. *Adjustment of the telescope.*—The sight-line of the telescope must be parallel to the plane of the sextant. Two parallel wires or threads are placed in the telescope, which are to be made parallel to the plane of the sextant by revolving the sliding tube containing them; then all contacts or coincidences of images are to be made midway between these two wires. The sight-line of the sextant telescope is, therefore, a line drawn through the optical centre of the object lens and the middle point between these parallel threads.

Select two objects from 100° to 120° apart, as the sun and moon, and bring the reflected image of one into contact with the direct image of the other, at the thread nearest the plane of the instrument; then move the instrument so as to throw the images upon the other thread; if the contact remains perfect, the line of sight midway between the threads is parallel to the plane of the sextant. If the limbs of the two objects appear to *separate* on the thread farthest from the instrument, the object end of the telescope *droops* towards the sextant; otherwise it rises.

It is to be observed that when the telescope is adjusted and two images are brought into contact at either thread, they will not be in contact in the middle of the field, but will there overlap; consequently, the reading of the sextant will be less for a contact in the true sight-line in the middle of the field than for one on either side. If the telescope is out of adjustment, the middle of the field is no longer in the true sight-line, and the contacts observed there give angles which are too great. The correction for a given inclination of the telescope will be investigated in a subsequent article.

This adjustment may also be examined as follows. Place the sextant horizontally on a table, and place two small metallic sights A, A (Fig. 20) on the arc. At a distance of at least 15 or 20 feet, let a well defined mark be placed so as to be in the same straight line with the upper edges of the sights, and in such a position that it may also be seen through the telescope. The top edges of the sights should be at the same distance from the plane of the sextant as the axis of the telescope. The threads of the telescope being made parallel to the plane of the sextant, the mark should be seen in the middle between them.

Fig. 20.



The adjustment of the telescope when necessary is effected by means of two small opposing screws in the ring which carries it.

85. *The index correction.*—Having made the preceding adjustments, it is necessary to find the point of the graduated arc at which the zero of the vernier falls when the two mirrors are parallel; for all angles measured by the instrument are reckoned from this point (Art. 80). If this point is to the left of the actual zero of the scale by a quantity r, all readings in the arc will be too great by r; if it is to the right of the actual zero, all readings will be too small by the same quantity. If we wish the reading to be zero when the mirrors are parallel, we must place the zero of the vernier on the zero of the arc, and then revolve the horizon glass about a vertical line, until the direct

and reflected images of the same object coincide. Some instruments are provided with a pair of opposing screws by which this revolution can be effected; but in others no such adjustment is possible. In fact, the adjustment is unnecessary, as we can always determine the correction to be applied to our readings to reduce them to what they would be if the adjustment were made. This *index correction* is found as follows:

1st. *By a star.*—Bring the direct and reflected images of a star into coincidence, and read off the arc. The index correction is numerically equal to this reading, and is positive or negative according as the reading is on the right or the left of the zero. For example, the direct and reflected images of a star being in coincidence, we read on the arc 5′ 20″; then, calling the index correction x, we have

$$x = -5' 20''.$$

In another sextant the direct and reflected images of a star being in coincidence, we read on the extra arc 2′ 40″; then

$$x = +2' 40''.$$

This method may be used with the sea-horizon instead of a star, but not with great precision.

2d. *By the sun.*—Measure the apparent diameter of the sun by first bringing the upper limb of the reflected image to touch the lower limb of the direct image; and again by bringing the lower limb of the reflected image to touch the upper limb of the direct image. Denote the readings in the two cases by r and r'; then, if s = the apparent diameter of the sun and R is the reading of the sextant when the two images are in coincidence, we have

$$r = R + s$$
$$r' = R - s$$

whence

$$R = \tfrac{1}{2}(r + r')$$

and the index correction is $x = -R$. The practical rule derived from this is as follows. If the reading in either case is *on* the arc, mark it with the *negative* sign; if *off* the arc (*i.e.* on the extra arc), mark it with the *positive* sign; then the index correction is one-half the algebraic sum of the two readings. For example, we have read as follows:

$$\begin{array}{rl}\text{On the arc} & -\ 31'\ 20''\\ \text{Off the arc} & \underline{+\ 33\ \ 10}\\ & +\ \ 1\ \ 50\\ x = & +\ \ 0'\ 55''\end{array}$$

We have $s = \frac{1}{2}(r - r')$: hence, if the observations are good, we ought to find that half the algebraic difference of the readings is equal to the sun's diameter as given in the Ephemeris on the day of the observation. But, in order that this comparison may be a good criterion, we should measure the sun's *horizontal* diameter, which is not sensibly affected by refraction. (Vol. I. Art. 134.)

In order to obtain the index correction with the greatest precision, the mean of a number of measures of the sun's diameter should be taken.

Example.—March 15, 1858, the following measures of the sun's horizontal diameter were taken:

	On the arc.	Off the arc.
	— 31′ 20″	+ 33′ 10″
	" 10	" 0
	" 15	" 20
	" 25	" 15
	" 20	" 10
	" 20	" 10
Means	— 31 18.3	+ 33 10 .8
		— 31 18 .3
		$x = +$ 56″.3

Observed sun's diameter, $s = 32'\ 14''.6$

By the Ephemeris, $s = 32\ \ 13\ .3$

86. *To measure the angular distance of two objects with the sextant*:—Place the threads of the telescope parallel to the plane of the instrument. Direct the telescope towards the fainter of the two objects, and revolve the sextant about the sight-line until its plane produced passes through the other object, observing to have the index glass on the side towards this object. Then move the index until the reflected image of the second object is nearly in contact with the direct image of the first; clamp the index, and make an exact contact (at the middle point between the threads) by means of the tangent screw. The reading of the arc will be the *instrumental distance:* applying to this the index correction according to its sign, the result will be the *observed distance.*

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In order to make a good observation, it is important that the two images whose contact is observed should be equally bright. Hence, we direct the telescope towards the fainter object, so that it may be the brighter one which suffers the double reflection. But in observing the distance of the moon from a star it will generally be found that, even after the double reflection, the image of the moon is so bright that the star will appear very indistinct unless the telescope is raised (by the screw for that purpose) so that the sight-line is directed through the transparent part of the horizon glass; for then, a portion of the reflected rays from the moon being lost, the intensity of its light is rendered more nearly equal to that of the star. When the distance of the sun and moon is observed, the telescope is usually directed towards the moon, and the intensity of the sun's rays is diminished by putting one or more of the colored shades between the index and horizon glasses. It will be found necessary in this case also to regulate the distance of the telescope from the plane of the instrument, in order to give the image of the moon the same intensity as that of the sun. It is a common error of inexperienced observers with the sextant to have the images too bright. It is essential to a good observation, 1st, that the images be well defined by carefully adjusting the focus of the telescope; 2d, that they be so faint as not in the least to fatigue the eye, yet perfectly distinct; 3d, that their intensities should be as nearly as possible equal.

In the case of the moon and a star, we observe the distance of the star from that point of the moon's bright limb which lies in the great circle joining the star and the moon's centre. To ascertain that this point has actually been brought into contact with the star, the sextant must be slightly revolved or *vibrated* about the sight-line (which is directed towards the star), thus causing the moon to sweep by the star; the limb of the moon should appear to graze the star as it passes, or, rather, the limb should pass through the centre of the star's light, for in the feeble telescope of the sextant the star does not appear as a well defined point.

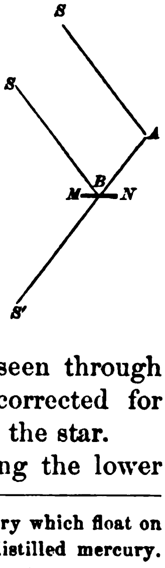
In the case of the moon and a planet we bring the reflected image of the moon's limb to the *estimated* centre of the planet.

In the case of the moon and the sun, the contact of the nearest limbs is observed, vibrating the instrument as above stated, and making the limbs just touch as they pass each other.

It facilitates the observation of lunar distances to set the index approximately upon the angular distance before commencing the observation. The approximate distance for a given time may be found from the Ephemeris (see Vol. I. Art 65); the distance thus found is in the case of the sun and moon to be diminished by the sum of the semidiameters of the two bodies (say 32′), and in the case of the moon and a star or planet it is to be diminished or increased by the moon's semidiameter (say 16′), according as the bright limb is nearer to or farther from the star than the moon's centre. This proceeding is also a check against the mistake of employing the wrong star.

87. *To observe the altitude of a celestial body with the sextant and artificial horizon.*—The artificial horizon is a small rectangular shallow basin of mercury, over which is placed a roof, consisting of two plates of glass at right angles to each other, to protect the mercury from agitation by the wind. The mercury affords a perfectly horizontal surface which is at the same time an excellent mirror.* If MN (Fig. 21) is the horizontal surface of the mercury, SB a ray of light from a star, incident upon the surface at B, BA the reflected ray, then an observer at A will receive the ray BA as if it proceeded from a point S' whose angular depression MBS' below the horizontal plane is equal to the altitude SBM of the star above that plane. If then SA is a direct ray from the star, parallel to SB, an observer at A can measure with the sextant the angle $SAS' = SBS' = 2SBM$, by bringing the image of the star reflected by the index glass into coincidence with the image S' reflected by the mercury and seen through the horizon glass. The instrumental measure, corrected for index error, will be double the apparent altitude of the star.

Fig. 21.



The sun's altitude will be measured by bringing the lower limb of one image to touch the upper limb of the other. Half the corrected instrumental reading will be the apparent altitude of the sun's *lower* or *upper* limb, according as the *nearest* or *farthest* limbs of the direct and reflected suns were brought into contact. For examples, see Vol. I. Arts. 145, 151, &c.

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* Observers are sometimes annoyed by impurities in the mercury which float on its surface, and imagine that it is important to have very pure distilled mercury. I have found it preferable to use mercury amalgamated with tin (a few square inches of tin foil added to the mercury of an ordinary horizon will answer). When the mercury is poured out, a scum of amalgam will cover its surface: this scum can be drawn to one side of the basin with a card or the smooth edge of a folded piece of paper, leaving a perfectly bright reflecting surface, entirely free even from the minutest particles of dust.

In observations of the sun with the artificial horizon, the eye is protected by a single dark glass over the eye piece of the telescope, thereby avoiding the errors that might possibly exist in the dark glasses attached to the frame of the sextant.

The glasses in the roof placed over the mercury should be made of plate glass with perfectly parallel faces. If they are at all prismatic, the observed altitude will be erroneous. The error may be removed by observing a second altitude with the roof in reversed position, and, in general, by taking one-half of a set of altitudes with the roof in one position and the other half with the roof in the reverse position. It is easily proved that the error in the altitude produced by the glass will have different signs for the two positions: so that the mean of all the altitudes will be free from this error.

Instead of the mercurial horizon, a glass plate is sometimes used, standing upon three screws, by means of which it is levelled, a small spirit level being applied to the surface to test its horizontality. The lower surface of the plate is blackened, so that the reflexion of the celestial object takes place only at the upper surface.

88. In the observation of the altitude of a star with the artificial horizon, it requires some practice to find the image of the star reflected from the sextant mirrors; and sometimes, when two bright stars stand near each other, there is danger of employing the reflected image of one of them for that of the other. A very simple method of avoiding this danger, by which the observation is also facilitated, has been suggested by Professor KNORRE, of Russia.* From very simple geometrical considerations it is readily shown that at the instant when the two images of the same star—one reflected from the artificial horizon, the other from the sextant mirrors—are in coincidence, the inclination of the index glass to the horizon is equal to the inclination of the sight-line of the telescope to the horizon glass, and is,

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* *Astron. Nach.*, Vol. VII. p. 262.

therefore, a *constant angle*, which is the same for all stars. If, therefore, we attach a small spirit level to the index arm, so as to make with the index glass an angle equal to this constant angle, the bubble of this level will play when the two images of the star are in coincidence in the middle of the field of view. With a sextant thus furnished, we begin by directing the sight line towards the image in the mercury; we then move the index until the bubble plays, taking care not to lose the image in the mercury; the reflected image from the sextant mirrors will then be found in the field, or will be brought there by a slight vibratory motion of the instrument about the sight line.

It is found most convenient to attach the level to the stem which carries the reading glass, as it can then be arranged so as to revolve about an axis which stands at right angles to the plane of the sextant, and thus be easily adjusted. This adjustment is effected by bringing the two images of a known star, or of the sun, into coincidence, then, without changing the position of the instrument, revolving the level until the bubble plays.

89. Observations on shore may be rendered more accurate by means of a stand to which the sextant can be attached, and which is so arranged that the sextant can be placed in any required plane and there firmly held. The manipulation must be learned from the examination of the stands themselves, which are made in various forms.

90. On account of the feeble power of the sextant telescope and consequent imperfect definition of the sun's limb, the apparent diameter of the sun is somewhat increased. This error, however, may be removed by taking the mean of two sets of altitudes, one of the lower limb and one of the upper limb.

91. *To measure an altitude of a celestial object from the sea horizon.* —Direct the telescope towards that part of the horizon which is beneath the object. Move the index until the image of the object reflected in the sextant mirrors is brought to touch the horizon at the point immediately under it. To determine this point, the observer should move the instrument round to the right and left (by a swinging motion of the body, as if turning on his heel), and at the same time vibrate it about the sight line, taking care to keep the object in the middle of the field of view;

the object will appear to sweep in an arc the lowest point of which must be made to touch the horizon, by a suitable motion of the tangent screw.

In general, altitudes for determining the time should be taken when the altitude varies most rapidly; and this is near the prime vertical. (See Vol. I. Arts. 143 and 149.) If the object is the sun, the lower limb is usually brought to touch the horizon; if the moon, the bright limb.

The apparent altitude of the point observed is found by correcting the sextant reading for the index error, and subtracting the dip of the horizon. (Vol. I. Art. 127.) To obtain the apparent altitude of the sun's or moon's centre, we must also add or subtract the *apparent* semidiameter. (Vol. I. Art. 135.)

92. As the sea horizon is often enveloped in mist, even when the celestial bodies are visible, various attempts have been made to obtain an artificial horizon adapted for use on shipboard. The simplest apparatus heretofore proposed for the purpose is that of Capt. BECHER, of the English Navy. "Outside the horizon glass of the sextant is a small pendulum about an inch and a half long, suspended in oil (in order to check its sudden oscillations); to the pendulum is attached a horizontal arm, carrying at the inner end a slip of metal which is seen in the field of the telescope at the usual focus, and whose upper edge when it coincides with a given line is the true horizon. The error is easily determined by a known altitude, and is the same for all altitudes. The apparatus, which is in a very compact form, is easily attached to any reflecting instrument, and is shipped and unshipped at pleasure. A lamp is attached for observing at night."* With this apparatus, when the motion of the ship is not too great, an altitude can be obtained within 5′ by a practised observer; and this is often sufficient.

93. *Method of observing equal altitudes with the sextant.*—Some observers set the sextant at pleasure, and note two instants, namely, the contact of the nearest and farthest limbs of the two images of the sun (one from the sextant, and the other from the mercurial horizon), both morning and evening, without touching

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* RAPER'S *Practice of Navigation*, 2d edition, p. 151. It does not appear, however, how the slip of metal behind the horizon glass could be distinctly seen in the field of the *telescope*. A plain tube must be used.

the index in the mean time. With a star they obtain but one observation on each side of the meridian. This practice is designed to secure the condition that the altitudes observed before and after meridian shall be absolutely identical, which may not be the case of the index if the sextant is moved and brought back again to the same reading. The errors to be feared, however, from not setting the index correctly on a given reading, are, in general, so much less than errors of observation, that it is better to sacrifice this merely theoretical consideration for the sake of multiplying the observations. The following method will be found convenient in practice.

1st. *For the sun.*—In the morning, bring the lower limb of the sun, reflected from the sextant mirrors, and the upper limb of that reflected from the mercury, into approximate contact; move the 0 of the vernier forward (say about 10′ or 20′) and set it on a division of the limb; the images will now appear *overlapped*, and will be *separating;* wait for the instant of contact: note it by the chronometer, and immediately set the vernier on the next division of the limb, that is, 10′ in advance; note the instant of contact again, and proceed in the same manner for as many observations as are thought necessary. If the sun rises too rapidly, let the intervals on the limb be 20′.

Now, find (roughly) the time when the sun will be at the same altitude in the afternoon, and just before that time set the vernier on the last altitude noted in the morning (of course employing the same sextant); the images will be *separated*, but will be *approaching;* wait for the instant of contact; note it by the chronometer; set the vernier *back* to the next division of the limb (10′ or 20′, as the case may be); note the contact again, and so proceed until all the A.M. altitudes have been again noted as P.M. altitudes.

If, instead of noting the times directly by the chronometer, a watch is employed (compared with the chronometer both before and after each observation), it will generally be found necessary to allow for its gain or loss on the chronometer, so as to obtain the exact difference between the two at the instant of observation.

The mean of all the A.M. chronometer times and the mean of all the corresponding P.M. times are regarded as two simple observations of the same altitude, and the computation proceeds from these according to the method and example of Vol. I. Art. 140.

2d. *For a star.*—Set the sextant, and note the *coincidences* of the two images of the star in the same manner as the contacts of the sun's limbs are oberved.

In selecting stars for this observation, it is to be observed that the nearer the zenith the star passes, the less may the elapsed time be; and when the star passes exactly through the zenith, the two altitudes may be taken within a few minutes of each other. But with the ordinary sextants altitudes near 90° cannot be taken with the artificial horizon, as the double altitude is then nearly 180°. The prismatic sextants and circles of Pistor and Martins are adapted for measuring angles of all magnitudes up to 180°, and are, therefore, especially suitable for these observations.

94. *To examine the colored glasses.*—The two faces of any one of the colored glasses, or shades, may not be parallel. The glasses then act like prisms with small refracting angles, which change the direction of the rays passing through them, and, consequently, vitiate the angles measured. To examine them, measure the sun's diameter with a suitable combination of shades; then invert one of the shades, turning it about on an axis perpendicular to the plane of the sextant, and repeat the measure; the half difference of the two measures will be the error produced by that shade. A number of measures must, of course, be taken in both positions of the shade, in order to eliminate accidental errors of observation.

In order to save the necessity of this examination, the shades are so arranged in Pistor and Martins' sextants that they may be instantaneously reversed. We have then only to take one-half of a set of observations with one position of the shades, and the other half with the reverse position, and take the mean of all the measures, in order fully to eliminate the errors of these glasses.

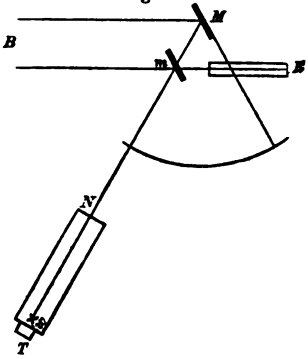
95. *To find the constant angle between the sight line and the perpendicular to the horizon glass.*—A knowledge of the value of this angle will be useful in following out the theory of the errors of the sextant in the subsequent articles. It varies in different instruments, and must be found for each by a special examination. Let the sextant be placed on a firm horizontal support; direct the sight line towards a distant object B, Fig. 22, and bring the two images of the object into coincidence. The mirrors M and m are then parallel; and, if we put

β = the angle between the sight line and the perpendicular to the horizon glass,

we have

$$BMm = MmE = 2\beta$$

Fig. 22.



We have, therefore, only to find some means of measuring the angle BMm. Leaving the sextant in its present position, place a theodolite in the line Mm produced, with its telescope TN on a level with the sextant mirrors and looking into the index glass; adjust it so that the image of B reflected from M shall be seen upon the cross-wire w in the focus. Rays from w passing through the object glass N emerge in parallel lines, as if from an infinitely distant object lying in the direction MNT. Bring the sextant telescope to look into the theodolite telescope, and reflect the image of B to the *cross-wire:* the reading of the sextant corrected for the index error is the measure of the angle BMm, or of 2β. If the object is not very distant, the angle subtended by the distance Mm at the object may be appreciable. This angle may be called the *sextant parallax*, and denoted by p. We shall have

$$BMm = 2\beta - p$$

When the object and its reflected image are in coincidence, let the reading be R, and let x be the true index correction for an infinitely distant object; then we have

$$R + x = -p \tag{58}$$

and when the object is reflected to the cross-wire of the theodolite, let the sextant reading be R'; then we have

$$R' + x = 2\beta - p \tag{59}$$

and from these two equations,

$$R' - R = 2\beta \tag{60}$$

By this method I found for one of TROUGHTON'S sextants, at the Naval Academy, $2\beta = 33° 6'$.

96. *The sextant parallax* for an object at a known distance found with the aid of the angle β. Let

f = the distance of the index and horizon glasses,
d = the distance of the object from the index glass.

The perpendicular drawn from M upon mE is equal to $f \sin 2$ and for the angle p at the object, subtended by this perpendicul we have

$$\sin p = \frac{f \sin 2\beta}{d} \qquad \text{or} \qquad p = \frac{f \sin 2\beta}{d \sin 1''} \qquad (6$$

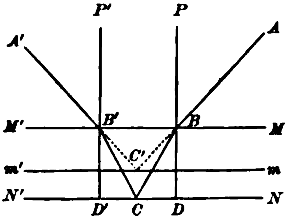
From this formula we may find a rough value of β when p h been determined for a near object by means of (58) and f and are carefully measured.

The distance of an object for which the sextant parallax w be $1''$ will be found by the equation $d = f \sin 2\beta \operatorname{cosec} 1''$. the sextant mentioned in the preceding article we have $f =$ inches, whence $d = 5.33$ miles.

In measuring horizontal angles between terrestrial objec the effect of the sextant parallax may be eliminated by det mining the index correction from the object which is se directly through the horizon glass. This index correction w involve the parallax, and, when applied to the sextant readi of the angular distance between the objects, will give the an subtended by the objects at the centre of the sextant. The s tant must, of course, remain in the same position in the meas of the angle and the determination of the index correction.

97. *To determine the error produced by a prismatic form of the in glass.*—Let us first consider the case o glass with parallel faces. Let MM', N Fig. 23, be the parallel faces, of whi NN' is silvered. An incident ray AB refracted by the glass at B, and takes t direction BC; at C it is reflected i CB'; and at B' it is refracted into B If we put

Fig. 23.



m = the index of refraction for glass,
φ = the angle of incidence ABP,
ϑ = the angle of refraction DBC,
$\varphi' = A'B'P'$,
$\vartheta' = D'B'C$,

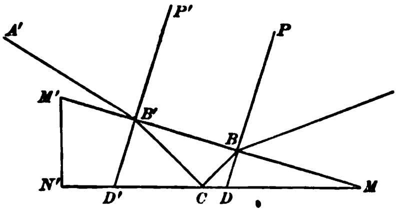
we have, by Optics,

$$\sin\varphi = m \sin\vartheta$$
$$\sin\varphi' = m \sin\vartheta'$$

But when the faces MM' and NN' are parallel, the normals BD and $B'D'$ are also parallel; moreover, the incident ray BC upon NN', and the reflected ray CB', make equal angles with DD': hence, also $\vartheta = \vartheta'$, and, consequently, $\varphi = \varphi'$. If AB and $A'B'$ are produced to meet in C', we see that $A'B'$ has the same direction that it would have had if it had been reflected directly from the plane surface $mC'm'$ parallel to MM' or to NN'. The refraction which the ray suffers in passing through the glass, therefore, produces no error when the surfaces of the glass are parallel. It may here be remarked, also, that it is not necessary that the reflecting surface of the mirror should stand exactly over the centre of the arc of the sextant.

Let us next consider the case of a glass whose faces are not parallel, as $M'B$, $N'D$, Fig. 24, which, produced to meet in M, form a prism $MM'N'$. Let us assume that these faces are perpendicular to the plane of the sextant, and, consequently, that the refracting edge of the prism is also perpendicular to this plane. The incident and reflected rays will be found in a plane parallel to that of the sextant. The ray being traced through the glass, we shall have, as before, employing the same notation,

Fig. 24.



$$\left.\begin{aligned}\sin\varphi &= m \sin\vartheta\\ \sin\varphi' &= m \sin\vartheta'\end{aligned}\right\} \qquad (62)$$

but here ϑ and ϑ' are no longer equal. If we put

$$M = \text{the angle of the prism} = M'MN'$$

we shall evidently have

$$90^\circ - \vartheta = CBB' = BCD + M$$
$$90^\circ - \vartheta' = CB'B = B'CD' - M$$

and, since $BCD = B'CD'$, the difference of these equations gives

$$\vartheta' - \vartheta = 2M \qquad (63)$$

From (62) and (63), φ, m, and M being given, we can determine φ', or the difference $\varphi' - \varphi$. From (62) we deduce

$$\cos \tfrac{1}{2}(\varphi + \varphi') \sin \tfrac{1}{2}(\varphi' - \varphi) = m \cos \tfrac{1}{2}(\vartheta + \vartheta') \sin \tfrac{1}{2}(\vartheta' - \vartheta)$$

whence, by (63),

$$\sin \tfrac{1}{2}(\varphi' - \varphi) = m \sin M \cdot \frac{\cos \tfrac{1}{2}(\vartheta + \vartheta')}{\cos \tfrac{1}{2}(\varphi + \varphi')}$$

As M is always a very small angle, approximate values may be employed in the second member of this equation: it will be sufficient to take

$$\sin \tfrac{1}{2}(\varphi' - \varphi) = m \sin M \cdot \frac{\cos \vartheta}{\cos \varphi}$$

or

$$\varphi' - \varphi = 2mM \sec \varphi \sqrt{1 - \frac{\sin^2 \varphi}{m^2}}$$

which may be reduced to the form

$$\varphi' - \varphi = 2M \sqrt{1 + (m^2 - 1) \sec^2 \varphi}$$

or, finally, by putting

$$q^2 = m^2 - 1$$

to the form

$$\varphi' - \varphi = 2M \sqrt{1 + q^2 \sec^2 \varphi} \tag{64}$$

The error varies with φ, and consequently with the angle measured. If

$\gamma =$ the angle given by the sextant,

we have, in Fig. 19, $PMm = PMp + pMm$, or

$$\varphi = \tfrac{1}{2}\gamma + \beta \tag{65}$$

The whole error in the measured angle will be the difference of the errors produced at the reading γ and at the zero point of the sextant; and at the zero point we have $\varphi = \beta$. Hence the error will be the difference of the values of (64) for $\varphi = \tfrac{1}{2}\gamma + \beta$ and $\varphi = \beta$, so that, if γ' denotes the true value of the angle, we shall have

$$\gamma - \gamma' = 2M \left[\sqrt{1 + q^2 \sec^2(\tfrac{1}{2}\gamma + \beta)} - \sqrt{1 + q^2 \sec^2 \beta}\right] \tag{66}$$

For glass we have usually $m = 1.55$, and hence $q^2 = 1.4025$. If $M = 10''$, $\beta = 10^\circ$, and $\gamma = 120^\circ$, we shall find $\gamma - \gamma' = 41''$.

The effect of the error in the glass is evidently less for small values of β than for large ones. Moreover, the smaller the angle β, the larger the angle which can be measured with the sextant, for all reflection from the index glass ceases when $\varphi = 90°$, and this value gives by (65) $\gamma = 180° - 2\beta$ as the limit of possible measures with the instrument.

The preceding investigation is confined to the case in which both faces of the glass are perpendicular to the sextant plane; but it suffices to show the nature of the effect produced. This case is, moreover, that in which the effect is greatest.

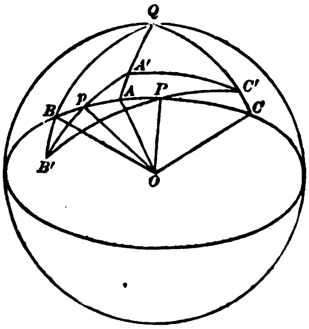
The glass reflects from its outer face as well as from its silvered face, though in a less degree. If the faces are parallel, the rays from a distant object reflected from the two faces will be parallel after leaving the glass; they will, therefore, be converged to the same focus in the telescope and produce but a single image of the object. But if the glass is prismatic there will be two images, a fainter image superposed upon the stronger one and not quite coincident with it. The effect will be to give an image with an indistinct outline; a star will present a somewhat enlarged or elongated image. We can, therefore, very readily determine whether the glass is prismatic by examining the reflected image of a star when the index is set upon a reading of about 120°.

The best makers will reject a glass that does not stand this test. If, however, an instrument is found to be defective in this respect, we may determine the error produced by it as follows. After carefully adjusting the instrument and finding its index correction, measure a large angle between two well defined terrestrial objects. Then take out the index glass and invert it (so that the edge, which was before uppermost, may now be next the plane of the instrument), readjust the instrument, determine the new index correction, and again measure the angle between the two objects. Half the difference of the two measures will be the error in either measure produced by the glass. The same process repeated for a number of angles of various magnitudes will furnish a table of errors, from which the error for any particular angle may be obtained by interpolation.

98. *A prismatic form of the horizon glass* affects all angles, the index correction included, by the same quantity, and therefore produces no error in the results.

99. *To determine the error produced by a small inclination of the sight line to the plane of the sextant.*—The directions of lines in space are most clearly represented by points on the surface of a sphere described about an assumed centre with an arbitrary radius (Vol. I. Art. 1). The radii drawn parallel to any given lines in space will intersect each other under the same angles as those lines, and these angles will be measured by the arcs of great circles joining the extremities of the radii on the surface of the sphere. Let us here take the centre of the sextant arc as the centre of such a sphere. Let O, Fig. 25, be that centre, OP the direction of the perpendicular to the index glass, Op that of the perpendicular to the horizon glass. The points P and p are the poles of the great circles whose planes are parallel to those of the glasses, and may be called, briefly, the poles of the index glass and horizon glass, respectively. Let OA be the direction of the sight line. When the instrument is perfectly adjusted, the lines OP, Op, and OA are in the same plane, which is parallel to that of the sextant. The course of a ray which reaches the eye will be most readily followed by tracing it backwards from the eye. Thus, the ray OA coinciding with the sight line is reflected from the horizon glass in the direction BO, so that $pB = pA$. It is then reflected from the index glass in the direction OC, so that $PB = PC$; and OC is therefore the direction of an object whose image is reflected to the eye in the same direction, AO, in which another object is seen directly. Hence AOC, or AC, is the angular distance of the objects. From this construction we obtain easily $AC = 2Pp$, which is the fundamental property of the sextant (Art. 79).

Fig. 25.



But if the sight line is inclined to the plane of the instrument, it meets the sphere in a point A' not in the great circle Pp. The inclination is measured by the arc AA' perpendicular to Pp, which is a part of the arc $QA'A$ drawn through A' and the pole Q of the great circle. The point Q may be called the pole of the sextant plane. Tracing the ray OA' backwards, we observe that the plane of reflexion from the horizon glass is represented by the great circle $A'pB'$, determined by the ray and the normal Op, so that if we take $pB' = pA'$, the reflected ray takes the direction $B'O$. The plane of reflexion from the index glass will be represented by the great circle $B'PC'$, and by taking $PC' = PB'$, OC' will be the direction of the reflected ray. Hence, $A'C'$ will be the true angular distance of the two objects observed in contact; while AC or $2Pp$ will be the angle given by the sextant. Let

γ = the angle given by the sextant = AC,
γ' = the true angle = $A'C'$,
i = the inclination of the sight line = AA'.

It is evident that $CC' = BB' = AA'$, and therefore $QA'C'$ is an isosceles triangle of which the angle $Q = \gamma$, the side $A'C' = \gamma'$, and the side QA' or $QC' = 90° - i$. If then we divide this triangle into two rectangular ones by a perpendicular from Q, we obtain

$$\sin \tfrac{1}{2}\gamma' = \cos i \sin \tfrac{1}{2}\gamma \qquad (67)$$

for which, as i is always very small, we may take the approximate equation*

$$\gamma' - \gamma = -i^2 \sin 1'' \tan \tfrac{1}{2}\gamma \qquad (67^*)$$

According to the second method of adjustment in Art. 84, if the mark is placed at a distance of 20 feet, and if the error of its position in a vertical direction is not more than ½ an inch (which is a large error in such a case), the telescope adjusted to it will have an inclination which will be found by the equation $\sin i = \frac{0.5}{20 \times 12}$, which gives $i = 7'\ 10''$. Taking this value of i, the formula (67*) gives $\gamma' - \gamma = -0''.897 \tan \frac{1}{2}\gamma$, and for $\gamma = 120°$, $\gamma' - \gamma = -1''.5$. The error may therefore be regarded as evanescent when ordinary care has been bestowed upon the adjustment. When the error exists, the observed angles are always too great.

100. If the contact of the images of two objects is made on either side of the middle of the field of the telescope, the actual sight line is inclined, although the axis of the telescope may be parallel, to the sextant plane.

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* This approximate equation can be deduced from (67) or taken directly from Sph. Trig. (112).

The inclination of this actual sight line can be estimated by the aid of the angular distance of the threads. To find this distance, place the threads at right angles to the plane of the sextant, bring the direct image of a distant, well defined line on one thread, and the reflected image on the other thread, and read the arc; then move the index until the images have exchanged places on the threads, and again read the arc; the half difference of the two readings is the angular distance of the two threads.

Let this distance of the threads be denoted by δ, and suppose an angle γ is observed by making the contact at a distance $n\delta$ from one of the threads (the fraction n being estimated at the time of making the observation); then the inclination of the actual sight line to the true sight line corresponding to the middle point between the threads will be $i = \frac{1}{2}\delta - n\delta$, with which value of i, the correction of the observed angle γ, will be found by (67*).

The distance δ in the best sextant telescopes will not exceed 30′. When the instrument is held in the hand, we cannot make all contacts exactly in the middle of the field; but, if we assume that we can always make them at a distance greater than $\frac{1}{3}\delta$ from either thread (which a little practice will enable us to do), we shall always have $i < \frac{1}{6}\delta$, or $i < 5'$, and hence the correction $\gamma' - \gamma < 0''.44 \tan \frac{1}{2}\gamma$. For any tolerably good observer, therefore, this correction will be practically insensible.

At the same time, however, we see the importance of making the contacts as near to the middle of the field as possible, since the error always has the same sign and all the measured angles are liable to be too great. If a contact is made on either thread, and we have $\delta = 30'$, the error in γ will be $3''.93 \tan \frac{1}{2}\gamma$, or $6''.8$ for $\gamma = 120°$.

101. The distance δ of the threads may also be used to find the inclination of the axis of the telescope, or rather of the true sight line. Measure an angular distance of 120° or more, between two well defined objects; bring the images in contact first on one thread and then on the other (the threads being placed parallel to the plane of the instrument), and let the readings on the arc be γ and γ_1. Then, γ' being the true reading in either case, and i the inclination of the true sight line, we have

$$\gamma' - \gamma = -\left(\frac{\delta}{2} - i\right)^2 \sin 1'' \tan \tfrac{1}{2}\gamma$$

$$\gamma' - \gamma_1 = -\left(\frac{\delta}{2} + i\right)^2 \sin 1'' \tan \tfrac{1}{2}\gamma_1$$

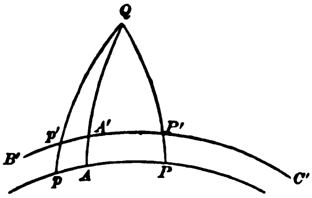
whence, taking $\tan \frac{1}{2}\gamma = \tan \frac{1}{2}\gamma_1$ in the second members,

$$i = \frac{\gamma_1 - \gamma}{2\delta \sin 1''} \cot \tfrac{1}{2}\gamma \qquad (68)$$

It is evident that, when i is positive, the greater measure is γ_1, taken on the thread nearest the plane of the instrument, and $\frac{\delta}{2} + i$ is the distance from this thread to the point in the field which represents a direction parallel to the plane of the sextant. Hence the first method of adjusting the telescope given in Art. 84.

102. *To find the error produced by a small inclination of the index glass.*—The horizon glass, being adjusted by means of the index glass (Art. 83), may be supposed to have the same inclination. Let pP (Fig. 26) be the great circle of the sextant plane; let the poles of the mirrors be at P' and p', and put

Fig. 26.



l = the inclination of the index glass = PP' = that of the horizon glass = pp'.

If we suppose that the sight line is adjusted by the first method of Art. 84, it will be found in a plane perpendicular to both mirrors, and its direction will be represented by a point A' in the great circle $p'P'$. The direct ray from the eye to an object A' will be reflected in the direction B', and thence to C', these points all lying in the same great circle; $A'C'$ will be the true distance γ' of the objects observed, and $p'P' = \frac{1}{2}\gamma'$ will be the true angle of the mirrors, while $pP = \frac{1}{2}\gamma$ will be the angle given by the sextant reading. In the isosceles triangle $P'Qp'$, we have the angle $p'QP' = \frac{1}{2}\gamma$ and $Qp' = QP' = 90° - l$; and, dividing it into two right triangles by a perpendicular from Q, we obtain

$$\sin \tfrac{1}{2}\gamma' = \cos l \sin \tfrac{1}{2}\gamma \qquad (69)$$

whence, very nearly,

$$\gamma' - \gamma = -2 l^2 \sin 1'' \tan \tfrac{1}{2}\gamma \qquad (69^*)$$

By the method of adjusting the index glass given in Art. 82, it may easily be placed within 5′ of its true position, and for $l = 5' = 300''$, and $\gamma = 120°$, this formula gives $\gamma' - \gamma = -0''.5$. Hence, with ordinary care, this error will also be practically insignificant.

The inclination of the sight line, in this solution, is variable with the angle measured. Denoting it by $i' = AA'$, we readily find, by the aid of a perpendicular from Q upon $p'P'$,

$$\tan i' = \tan l \cdot \frac{\cos(\tfrac{1}{2}\gamma - \beta)}{\cos \tfrac{1}{2}\gamma} \qquad (70)$$

in which $\beta = Ap$; or

$$i' = l \sec \tfrac{1}{2}\gamma \cos(\tfrac{1}{2}\gamma - \beta) \qquad (70^*)$$

103. If, however, the sight line is not determined as above supposed, but has a constant inclination to the plane of the sextant, denoted by i, its inclination to the plane of reflection $p'P'$ will be $i' - i$, and the additional error produced by this inclination will be found by (67*) to be

$$-(i' - i)^2 \sin 1'' \tan \tfrac{1}{2}\gamma$$

Combining this with (69*), the complete formula is

$$\gamma' - \gamma = -2 l^2 \sin 1'' \tan \tfrac{1}{2}\gamma - [l \sec \tfrac{1}{2}\gamma \cos(\tfrac{1}{2}\gamma - \beta) - i]^2 \sin 1'' \tan \tfrac{1}{2}\gamma$$

which can be put under the form

$$\gamma' - \gamma = -2 \sin 1'' \tan \tfrac{1}{2}\gamma \left[l^2 + \sec \tfrac{1}{2}\gamma \, [l \cos(\tfrac{1}{2}\gamma - \beta) - i \cos \tfrac{1}{2}\gamma]^2 \right] \qquad (71)$$

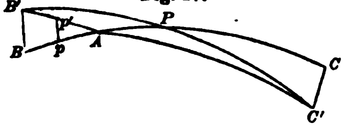
which agrees with ENCKE's formula in the Berlin *Jahrbuch* for 1830, p. 292.

Taking, as an extreme case, $l = 5'$, $i = \mp 5'$, $\gamma = 120°$, $\beta = 30°$, this gives $\gamma' - \gamma = -4''.0$.

104. *To find the error produced by a small inclination of the horizon glass.*—Assuming that the index glass and the telescope are in adjustment, let the pole of the horizon glass be at p', Fig. 27, the pole of the index glass being at P, and the sight line directed towards A in the plane of the sextant. The ray from the eye towards A is reflected to B' in the arc Ap', so that $p'B' = p'A$,

and thence to C', which is at the distance $CC' = BB'$ from the great circle pPC. $AC = \gamma$ is the angle given by the sextant; and $AC' = \gamma'$ is the true angular distance between the two objects whose images are observed in contact. Putting

Fig. 27.



$$k = \text{the inclination of the horizon glass} = pp',$$
$$m = CC' = BB', \qquad \beta = Ap,$$

we have from the triangles App' and ABB', very nearly,

$$m = 2k \cos \beta$$

and, from the triangle $AC'C$,

$$\cos \gamma' = \cos m \cos \gamma$$

whence

$$\gamma' - \gamma = \tfrac{1}{2} m^2 \sin 1'' \cot \gamma = 2k^2 \sin 1'' \cos^2 \beta \cot \gamma \qquad (72)$$

This error is sensible only for small values of γ. For $\gamma = 0$ the expression becomes infinite; for in fact it is inapplicable in this case, since when the horizon glass is inclined it is impossible to make a contact of two images of the *same point*. But in the determination of the index correction by the sun, the limbs of the two images will be brought into contact alternately on each side of the true zero point of the arc, and we shall have $\gamma = \pm 0° 32'$. For this case, with $\beta = 30°$ and $k = 30''$ (which ought to be the maximum error in the adjustment by Art. 83), we find $\gamma' = \gamma = \pm 0''.7$; and even this error is eliminated from the index correction itself. For all angles greater than 0° 32′ the error is wholly inappreciable.

105. *To find the eccentricity of the sextant.*—As the arc of the sextant is limited, the method of determining whether the centre about which the index arm revolves is coincident with the centre of the graduations by means of two verniers 180° apart (Art. 28) is not applicable. We can find the eccentricity only by comparing various angles measured with the sextant with their known values found by some other means. Thus, the angular distances of a number of terrestrial points situated in a horizontal plane may be accurately determined with a good theodolite and then also measured with the sextant.

Or we may measure with the sextant the distance of two well known fixed stars and compare it with the apparent distance computed from their right ascensions and declinations. The refraction, however, must be taken into account, which may be done in either of two ways. 1st, The true distance of the stars will be found as in the case of the moon and a star, Vol. I. Art. 255. Then the apparent distance will be found by the formulæ (448) and (449) of Vol. I., in which we must for this case suppose h', H', d' to be the true altitudes and distance, and h_1, H_1, d_1 to be their apparent values affected by refraction. The altitudes will be computed by Art. 14, Vol. I., the local time, and consequently the hour angles of the stars, being given.

2d. We may compute the zenith distances and parallactic angles of the stars for the time of the observation by Vol. I. Art. 15, and then the refraction in right ascension and declination by Art. 120. We shall then have the apparent right ascensions and declinations, from which the apparent distance will be directly computed by the method of Vol. I. Art. 255.

Now, let γ be the sextant reading, x the index correction (here supposed to be unknown, as we must regard the zero point as likewise affected by the eccentricity), γ' the true value of the measured angle, e the eccentricity; then, since the readings of the sextant are double the true arcs, we have, by (9),

$$\gamma' - (\gamma + x) = 2e \sin(\tfrac{1}{2}\gamma' + E)$$

or, putting $n = \gamma' - \gamma$,

$$x + 2e \cos E \sin \tfrac{1}{2}\gamma' + 2e \sin E \cos \tfrac{1}{2}\gamma' = n \tag{73}$$

To find the three unknown quantities x, $2e \cos E$, and $2e \sin E$, we must have three such equations derived from three angles falling in different parts of the arc,—for example, near 0°, 60°, and 120°. If we have measured a large number of angles, of various magnitudes, we can treat the equations by the method of least squares.

As the index correction is liable to change from one observation to another, we can let γ represent the reading corrected for the index error found at each observation, and then x will be the correction of the zero point for eccentricity.

THE SIMPLE REFLECTING CIRCLE.

106. If the arc of the sextant is extended to a whole circumference, the index arm may be produced and carry a vernier upon each extremity. The mean of the readings of the two verniers may then be taken at every observation, and will be wholly free from the error of eccentricity. This constitutes a simple reflecting circle, the manipulation of which is in every respect the same as that of the sextant. It has not only the advantage of eliminating the eccentricity, but at the same time of diminishing the effect of errors of reading and accidental errors of graduation, since every result is derived from the mean of two readings at two different divisions of the arc. The only objection to the instrument is found in the slight increase of its weight.

The simple reflecting circles of **Troughton** are read by three verniers at distances of 120°; but, as the eccentricity is already fully eliminated by two verniers, the third can increase the accuracy of a result only by diminishing the effect of errors of reading and of graduation. If ε_2 is the probable error of the mean of two readings, that of the mean of three readings will be

$$\varepsilon_3 = \varepsilon_2 \sqrt{\tfrac{2}{3}} = 0.81\, \varepsilon_2$$

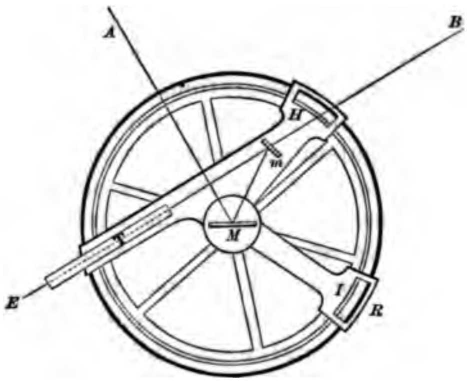
so that if two verniers reduce the error to 5″ the third will only further reduce it to 4″, an increase of accuracy which for a single observation is not worth the additional complication and weight and the trouble of reading. As was to be expected, these instruments, though of very refined and perfect construction, have been but little used.

The prismatic reflecting circles of **Pistor** and **Martins** noticed below have but two verniers, and combine many practical advantages.

THE REPEATING REFLECTING CIRCLE.

107. In the repeating reflecting circle the small mirror, or horizon glass, is not permanently attached to the frame of the instrument, but is attached to an arm which revolves about the centre of the instrument. As the telescope must always be directed through this glass, it is also attached to the same arm and revolves with it. This arm also carries a vernier at its extremity.

Fig. 28.

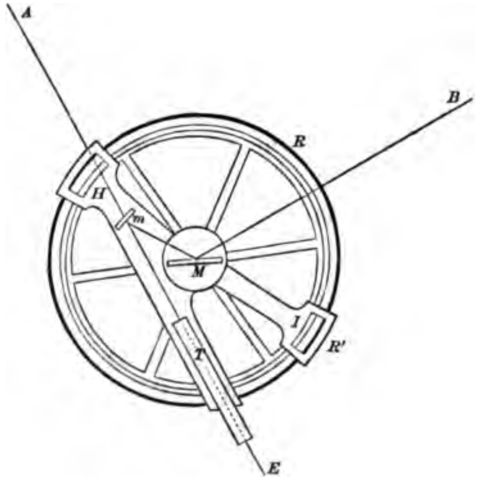


Let *ETH* (Fig. 28) be the revolving arm to which are attached the small mirror *m*, the telescope *T*, and the vernier, or index *H*; *M* the central mirror which is revolved by the arm *MI*, carrying the vernier, or index *I*. In accordance with the nomenclature in nautical works, we shall call *H* the *horizon index*, and *I* the *central index*.

The arc is graduated from 0° to 720° in the direction *HIE*.

Let *A* and *B* be the objects whose angular distance is to be measured. First: let the central index *I* be clamped at any assumed point of the arc. Bring the plane of the instrument to pass through the two objects. Direct the telescope towards the right hand object *B*, and, without touching the central index, move the horizon index *H* (or rather revolve the instrument, keeping the telescope bearing on *B*), until the image of the left hand object *A* is reflected from the central mirror *M* to the horizon glass *m*, and thence to the eye, and thus into coincidence with the object *B* seen directly. This completes the first part of the observation. Now, leaving the horizon index *H* clamped in this position, unclamp the central index *I*; direct the telescope to the left hand object *A*, Fig. 29, and move the index *I* forward (in the direction of the graduations) until the reflected image of the right hand object *B* is brought to coincide with the direct image of *A*. This completes the second part of the observation.

Fig. 29.



Then, *the difference between the readings of the central index in its two positions is twice the angular distance of the objects.* For let R, Fig. 29, be the point of reading of the central index before the first contact, and R' that after the second contact. At each contact the angle of the mirrors is equal to one-half the angle measured (Art. 80); and it is evident that the points R and R' are at equal distances on each side of that point of the arc at which the central index would have stood had we stopped its motion when the mirrors were parallel. Hence the angle RMR' is twice the angle of the mirrors at either contact. Denoting the angle measured by γ, and the readings by R and R', we have, therefore,

$$2\gamma = R' - R$$

The half difference of the two readings is then the mean of *two* measures of the required angle; while with the sextant two observations are necessary to furnish one measure of an angle, since one observation must be made to determine the index correction, which is here dispensed with.

If we now recommence the observations, starting from the last position of the central index, this index will be found after the fourth contact at a reading R'', which differs from R' by twice the angle γ: so that we have

$$2\gamma = R'' - R'$$

and, consequently,

$$4\gamma = R'' - R$$

Continuing this process as long as we please, we shall have, after any even number n of contacts, a reading R_n of the central index, and

$$n\gamma = R_n - R$$

or

$$\gamma = \frac{R_n - R}{n} \tag{74}$$

Hence it is necessary to read off the arc only before the first and after the last observed contact, which is one of the greatest advantages of this instrument for use on board ship in night observations.

108. If the distance of the objects is changing, as in the case of a lunar distance or an altitude, the difference between the

first and last readings will be the *sum* of all the individual measures, and the value of γ found by dividing this sum by the number of observations will be the *mean* of all these measures. The time of each observation having been noted, this value of γ will be the value of the observed angle at the mean of these times, provided the angular distance is changing *uniformly*.

109. We have thus far supposed the telescope to be directed alternately towards each object; but (as in the measurement of a lunar distance, for example) it is expedient to look directly at the fainter object and reflect the brighter one. This can be done by reversing the face of the instrument after each contact; for the relative position of the mirrors will thus be inverted without requiring the line of sight to be shifted from one object to the other.

It is convenient in practice to distinguish the two kinds of observation by the relative positions of the mirrors. For this purpose, let a plane be conceived to be passed through the axis of the telescope at right angles to the plane of the circle; the instrument is thus divided into two portions, of which that which is on the same side of the perpendicular plane as the central mirror will be called the *right*, and that which is on the opposite side, the *left;* these designations, however, having no reference to the right and left of the observer when the instrument is held in various positions.

An *observation to the right* is one in which the object reflected from the central mirror is on the right of the instrument.

An *observation to the left* is one in which the object reflected from the central mirror is on the left of the instrument.

A *cross observation* is one consisting of two observations, one to the right and one to the left.

The observation to the right is precisely like that with the sextant. We may, in fact, use the instrument as a sextant. Clamp the horizon index at any point of the arc; bring the direct and reflected images of the same object into coincidence by moving the central index, and read off this index. Call this reading R; then, making any observation to the right, let the reading be R'; the angle measured is $R' - R$, and $-R$ may be regarded as the index correction, as in the sextant.

110. In observing altitudes with the repeating circle, the tele-

scope is directed to the image in the artificial horizon. The central index is, for convenience, set upon zero, and we commence with an observation to the left, as in Fig. 28, holding the instrument in the left hand. The next observation is to the right, as in Fig. 29, and the instrument is held in the right hand.

111. In order to facilitate the repetition of the observations, the horizon glass and telescope carry with them an inner circular arc, which is called the *finder*. This finder moves under the central index arm alternately backwards and forwards in the successive observations; and, consequently, when the two places of the index arm have been once noted on the finder, it can be brought approximately to these places for the succeeding observations, whereby the images will be already approximately in contact. Two sliding *stops* are usually placed on the finder, and, when once set, serve to indicate the two positions of the central index. The finder is also roughly graduated for the same purpose.

112. The adjustment and verification of the glasses and telescope are in every respect the same as for the sextant. The theory of the errors is also similar, only we have a compensation of some of them which is worthy of notice and will be considered below.

Dark glasses or shades are placed, as in the sextant, behind the horizon glass and between the horizon glass and central mirror, for observations of the sun. In cross observations, the errors of these glasses are eliminated, since their positions with respect to the incident rays are reversed at each alternate contact. In observations to the left, however, Fig. 28, it is evident that when the angular distance between the objects A and B is small, colored glasses midway between M and m would intercept a portion of the direct rays from A on their way to M. In this case, therefore, it becomes necessary to substitute for them a large shade immediately in front of the central mirror. The same shade serves for the observation to the right; but, as the angle of incidence of rays falling upon it is no longer the same as in the observation to the left, the error of the shade is not wholly eliminated. However, as the angle of incidence is small in both positions, the errors produced by a prismatic form of the shade will be small, and the partial compensation of these errors which occurs will leave a residual error mostly inappreciable.

113. *To determine the error produced by a prismatic form of the central mirror in a cross observation with the circle.*—Let us consider the two contacts separately.

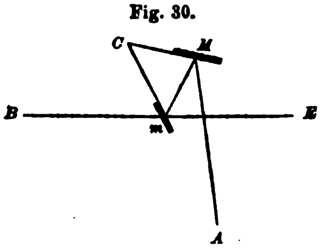
1st. *The observation to the right* is the same as with the sextant, and hence we have, for this observation, by (66),

$$\gamma - \gamma' = 2M\left[\sqrt{1 + q^2 \sec^2(\tfrac{1}{2}\gamma + \beta)} - \sqrt{1 + q^2 \sec^2 \beta}\right] \qquad (75)$$

in which M, q, β, γ, and γ' have the same signification as in Art. 97.

2d. In *the observation to the left*, the central mirror is reversed with respect to the incident ray, and therefore the sign of M must be changed. But the angle of incidence φ is also changed. Let M and m, Fig. 30, be the positions of the mirrors, AM a ray from the left-hand object A reflected from the central mirror to m, and thence to E in coincidence with the direct ray from the object B. Producing the faces of the mirrors, we readily find, from the triangle MCm,

Fig. 30.



$$\varphi = \tfrac{1}{2}\gamma - \beta$$

This value is to be used in the equation (64). The error in the measured angle will be the difference of the values of (64) for $\varphi = \frac{1}{2}\gamma - \beta$ and $\varphi = -\beta$; and we shall therefore obtain for it a formula differing from (75) only in having $-\beta$ instead of $+\beta$ and $-M$ instead of $+M$. Hence the error in an observation to the left is

$$\gamma - \gamma' = -2M\left[\sqrt{1 + q^2 \sec^2(\tfrac{1}{2}\gamma - \beta)} - \sqrt{1 + q^2 \sec^2 \beta}\right] \qquad (76)$$

3d. For the error in *the cross observation* we have, by taking the mean of (75) and (76),

$$\gamma - \gamma' = M\left[\sqrt{1 + q^2 \sec^2(\tfrac{1}{2}\gamma + \beta)} - \sqrt{1 + q^2 \sec^2(\tfrac{1}{2}\gamma - \beta)}\right] \qquad (77)$$

If we suppose, as in Art. 97, $q^2 = 1.4025$, $M = 10''$, $\gamma = 120°$, $\beta = 10°$, we find, by these formulæ, that the error of an observation to the left is $41''$, that of an observation to the right is $11''$, and that of a cross observation is $15''$. The error of the central mirror, though not wholly eliminated, is reduced to about one-third that of a sextant observation.

Borda,* to whom we owe the most important improvements in the reflecting circle, gave the numerical values of the formulæ (75), (76), and (77), in a small table with the argument γ, for a circle in which $\beta = 10°$. Table XXXIV. of Bowditch's Navigator is derived from similar formulæ.

The error produced by the central mirror for a given angle may be found by Art. 97, and then by means of Borda's table we may infer the correction for any other angle, by simple proportion.

114. The errors of reading, of imperfect graduation, and of eccentricity are all nearly eliminated by taking a sufficient number of cross observations. For these errors affect only the first and last readings, and are divided by the number of observations. If the sum of all the measures is very nearly 720° or 1440°, &c., so that the central index has made one or more complete revolutions, the eccentricity is wholly eliminated.

The error resulting from an inclination of the sight line of the telescope is not reduced by repetition, since it makes every measure too great. (Art. 99.)

In theory, therefore, the repeating circle is very nearly a perfect instrument, capable of eliminating its own errors. As, however, we cannot pretend to measure "*what we cannot see*," the refinement of the circle may really be thrown away, so long as the optical power of its telescope is so feeble. In fact, the results obtained with the circle do not appear to have surpassed those obtained with the sextant so much as was expected from its theoretical perfection. This may, however, be due, in a degree, to the mechanical imperfections arising from the centring of two axes one within another.†

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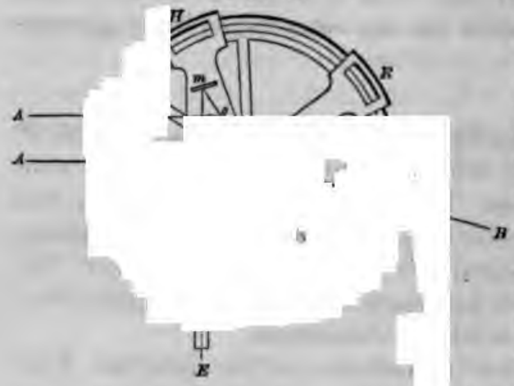
* *Description et usage du Cercle de Réflexion, par* Ch. De Borda, 4^{me} ed. Paris, 1816.

† It seems that the instrument makers have supposed that it was necessary that both the horizon and the central indices should be perfectly centred. In Gambey's circles the axis of the central index turns within that of the horizon index, and any shake of the latter is communicated to the former. But, if we use the instrument as prescribed in the text, *reading off only the central index*, it is quite unimportant whether the horizon index is correctly centred or not. It is only necessary that it should revolve in a plane parallel to the plane of the instrument, and should remain firmly clamped throughout each cross observation; and this will be secured by giving it a broad bearing about the centre. The axis of the central index ought then to pass directly into the solid frame of the instrument, and the horizon index should turn upon a fixed collar, which would entirely separate it from the former. From

. circle, as above described, is capable of measuring no angles greater than about 140°. In this respect, therefore, it does not excel the sextant. A very simple addition proposed by M. DAUSSY obviates this difficulty. On the horizon index arm *EH*, Fig. 31, he places a second small mirror *n*, which is of only one-half the height of the silvered part of the horizon glass *m*. The angle at which it stands is more or less arbitrary, but it is convenient to have it make an angle of about 45° with the mirror *m*. Let *A* be any distant object, and let the instrument be held so that a ray *An*, falling upon *n*, shall be reflected in the line *nm* to *m* and thence to the eye at *E*. Now move the central index until the ray *AC*, from the sam bject, is reflected from the central mirror *MN* in the line *Cm*, passing *over* the small mirror *n* to the horizon glass, and thence to the eye in coincidence with the first ray. (This observation is like the ordinary one of determining the zero point of a sextant or circle, only the line of sight is directed to a point about 90° from the object.) The mirror *MN* and the small mirror *n* are now parallel. Let *R* be the reading of the central index. Now let *B* be a second object which may be even more than 180° from *A* reckoned in the direction *HRR'*. Move the central index until this object is reflected from the central mirror *M'N'* to *m*, and thus into coincidence with the image of *A* reflected from *n*. Let *R'* be the

Fig. 31.



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the fact that such a construction has not been heretofore adopted, I infer that this part of the theory of the instrument has not been well considered.

If this change is made, and the instrument is used on land upon a stand, I cannot see any reason why we should not realize all the theoretical advantages of the instrument, especially if we considerably increase the optical power of the telescope.

The opinion of Sir JOHN HERSCHEL (*Outlines of Astronomy*, Art. 188) that "the abstract beauty and advantage of this principle" (of repetition) "seem to be counterbalanced in practice by some *unknown cause, which probably must be sought for in imperfect clamping*," is hardly sustained by practical experience with instruments having a single central axis.

reading. The angular motion of the mirror *MN* being always equal to one-half the angular distance of the objects, $R' - R$ is the required angle. M. DAUSSY calls this contrivance a *dépressiomètre*, or *dip-measurer*, from its application to the measurement of the dip of the sea horizon, by measuring the angular distance between two diametrically opposite points of the horizon, this angular distance being 180° *plus* or *minus* twice the dip according as we measure through the zenith or through the nadir. It finds, however, another important application in observations with the artificial horizon when the altitude exceeds 65° or 70°, and the double altitude is consequently too great to be measured in the usual manner. The additional mirror is usually furnished with the *Gambey* circles, and is readily applied to any instrument. Since the angle at which it stands is not required to be found, the only adjustment necessary is to make it perpendicular to the plane of the instrument, which is done by the aid of the same test as that which is used in adjusting the horizon glass; we have only to observe that the two images of the same object *A* (which for this purpose may be a bright star) reflected from *MN* and *n* can be brought into coincidence in the middle of the field of the telescope; the mirrors *MN* and *m* having of course been previously adjusted.*

THE PRISMATIC REFLECTING CIRCLE AND SEXTANT.

116. The prismatic reflecting circle, constructed by PISTOR and MARTINS of Berlin, differs from the simple reflecting circle (Art. 106) by the substitution of a glass prism for the horizon glass, and by the position of this prism with respect to the central mirror.

ABC, Fig. 32, represents the circle; *M* the central mirror upon the index arm *ac*, which carries a vernier at each end *a* and *c*; *m* the prism, which is nearer the telescope *T* than the central mirror, and is permanently attached to the frame of the instrument. The prism has two of its faces nearly perpendicular to each other, and the third face acts as the reflector. A ray from the central mirror entering one of the perpendicular faces is totally reflected at the inner face and passes out through the other perpendicular face in the direction of the sight line of the telescope. The height of the prism is only one-half the diameter of the object lens of the telescope, and therefore direct rays from any object passing over the prism enter the telescope and are brought to the same focus as the reflected rays. When the central mirror is parallel to the longer side of the prism, as in Fig. 32, two images of the same object are in coincidence, and the index correction is determined as in the sextant, except that every reading is here the mean of the readings of the two verniers.

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* Special instruments for measuring the dip of the sea horizon have been contrived. For an account of TROUGHTON'S *Dip-Sector*, see SIMMS'S *Treatise on Mathematical Instruments*.

Now revolving the index into the position, Fig. 33, an object to the right will be reflected into coincidence with the direct object, and the angular distance of the two objects is given by the reading corrected for the index error. When the central mirror becomes nearly perpendicular to the line *Mm*, the prism intercepts the rays from the right hand object. This occurs when the angular distance of the two objects is about 130°. Beyond this point the head of the observer also intercepts the rays, until we come to the position of Fig. 34.

Fig. 32.

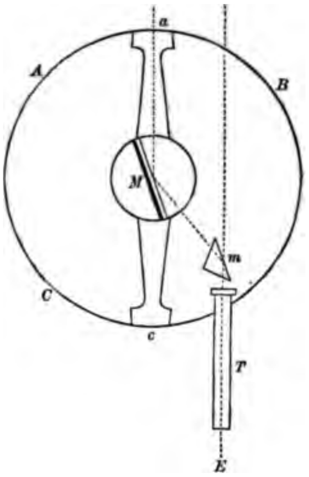
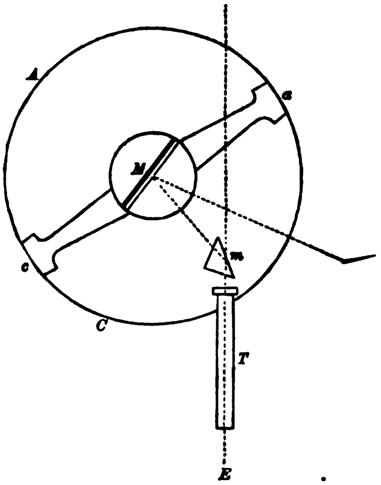


Fig. 33.



In this position two objects 180° apart can be brought into optical coincidence. But, although the prism does not interfere with the rays from the second object, the head of the observer may; and this is obviated by placing a small prism *D* at the eye end of the telescope, to reflect the two images which are in coincidence, to the eye in the direction *DE*.

Continuing the motion of the index, we see, by Fig. 35, that angles greater than 180° can now be obtained until the index arm comes against the prism, which occurs when the angle is about 280°. The angles thus measured may be reckoned either as between 280° and 180° or between 80° and 180°. Of these, the angles falling between 80° and 180° may be observed in two reversed positions of the instrument, constituting a cross observation, as with the repeating circle, whereby the index correction becomes unnecessary, and the errors arising from a prismatic form of the central mirror are partially eliminated.

Fig. 34.

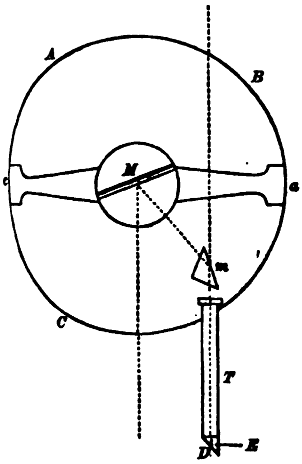
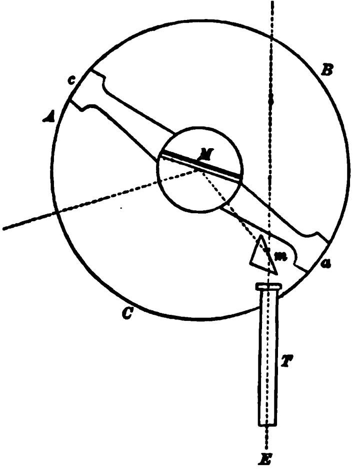


Fig. 35.



When the index is on zero, Fig. 32, the rays incident upon the central mirror make an angle with it of 20°, and in this position we obtain the feeblest reflected images. When the index is at 130°, the incident rays make an angle with the mirror of 85°, and we obtain the brightest reflected images. In the common sextant, the reverse takes place; the feeblest images occur for the angle 130° when the incident rays make an angle of only 10° with the central mirror; and the brightest images when the index is on zero and the rays make an angle of 75° with the mirror. The angles of incidence in the prismatic instruments are, therefore, more favorable for the production of distinct images than in the common sextant, since even the smallest angle which the incident rays make with the mirror in the former is double the corresponding angle in the latter.

The adjustments of the prism and central mirror are similar to those of the horizon and index glasses of the sextant.

The theory of the errors is also similar to that above given for the sextant and circle.

117. The advantages of these instruments over the common sextants are: 1st. Angles of all magnitudes can be measured; 2d, the eccentricity is completely eliminated by always employing the mean of the readings of the two verniers; 3d, the reflected images are brighter than in other reflecting instruments, both because the angles of incidence upon the central mirror are more favorable, and because the inner face of a glass prism is a much better reflector than a silvered glass; 4th, the errors arising from a prismatic form of the central mirror are much less than in the sextant. The instruments, as made by Pistor and Martins combine also other improvements which might be introduced into the common sextant. Thus, the shade glasses admit of reversal, by which their errors are wholly eliminated; a revolving disc, containing small colored glasses or shades, is adapted to the eye piece of the telescope, for use in taking altitudes of the sun with the artificial horizon; all lost motion is avoided in the tangent screw, by causing it to act against a spring; the arc is read off at night by the aid of a lantern which is placed over the centre of the instrument and the light of which is concentrated upon the arc by a lens.

The prismatic sextant differs from the circle only in dispensing with the second vernier (the vernier a in the above figures), and that portion of the arc upon which it reads. The same angles can be measured with this instrument as with the circle, but without the advantage of eliminating the eccentricity.

For an extensive series of observations, illustrating the capabilities of the sextant in the hands of a good observer, and especially demonstrating the excellence of the prismatic sextants, see an article of Schumacher, in the *Astron. Nach.*, Vol. XXIII. p. 321.

CHAPTER V.

THE TRANSIT INSTRUMENT.

118. The *transit instrument* is an instrument for determining the instant of a star's passage through any given vertical plane; or (which is the same thing) the time of a star's *transit* over any given vertical circle. For this purpose, it is necessary that the motion of the telescope be confined to the vertical plane; and this is effected by attaching the tube to a horizontal axis and perpendicular to it, so that by revolving the instrument upon this axis the principal sight-line of the telescope describes a plane passing through the zenith. The common theodolite may therefore be used as a transit instrument when its telescope admits of a complete revolution upon its horizontal axis.

The time of transit over the assumed vertical circle is deduced from the time when a star passes a given thread placed in the focus of the objective.

The instrument may be mounted in any vertical plane, but is chiefly used either in the meridian or in the prime vertical: in the first position, for finding either the true local time or the right ascensions of stars; in the second, for finding either the latitude of the place of observation or the declinations of stars. When spoken of simply as "the transit instrument," however, it is usually understood to be in the meridian.

It admits of some variety of form. In the old and still most common form, the telescope and horizontal axis bisect each other,* and the two ends of the axis are supported on pillars between which the telescope revolves.

A second form is that in which, starting from the first form, one-half the telescope tube is dispensed with, that half which contains the object glass being retained, while the horizontal axis is made to perform the part of the other half. At the intersection of the tube with the axis is a glass prism which bends the rays from the object glass at right angles, and transmits them through the hollow axis to the eye piece which is placed at the end of this axis. The chief advantage of this construction is that the observer does not have to change his position to observe all the stars which cross the plane of the telescope. It has also the advantage, for a portable instrument, of diminished weight and a more compact form.

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* In Halley's transit instrument (still preserved as a relic in the Greenwich Observatory) the pivots of the axis are at unequal distances from the telescope.

In a third form, proposed by STEINHEIL* of Munich the telescope tube is dispensed with entirely, or rather the horizontal axis is converted into a telescope, by starting from the second form just described and shortening the tube until the object glass is brought next to the prism, so that the rays are bent immediately after entering the instrument. This is therefore, practically, an instrument of the second form with the telescope tube reduced to its minimum length; but, to gain sufficient focal length, the object glass and prism (which are connected together) are placed near one end of the axis. This form evidently offers the greatest advantages for a portable instrument; its want of symmetry, and the loss of light incurred by the introduction of the prism, seem to prevent its adoption for the larger instruments intended for the more refined purposes of the observatory.

The principles governing the use of such instruments being essentially the same as those which apply to the transit instrument of the common form, I shall here treat exclusively of the latter.

119. Plate IV. represents the meridian transit instrument of the Washington Observatory, made by ERTEL AND SONS, Munich. It has a focal length of 85 inches, with a clear aperture of 5.3 inches. The dimensions of all the parts may be found from the drawing. The portions of the telescope tube *TT*, which are made conical to prevent flexure, are screwed to the hollow cube *M*. The conical portions of the horizontal or *rotation* axis *NN* are also screwed to this cube; this axis is hollow, and terminates in two steel cylindrical pivots which rest in Vs at *VV*. It is highly important that these pivots be perfect cylinders and of precisely equal diameters.

If the whole weight of an instrument of this size were permitted to rest upon the Vs, the friction would soon destroy the perfect form of the pivots, and hence a *portion* of this weight is counterpoised by the weights *WW*, which, by means of levers, act at *XX*, where there are friction rollers upon which the axis turns. By this arrangement, only so much of the weight of the instrument is allowed to rest upon the Vs as is necessary to insure a perfect contact of the pivots with the Vs. This not only saves the pivots, but gives the greatest possible freedom of motion to the telescope, the lightest touch of the finger being now sufficient to rotate the instrument upon the axis.

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* *Astron. Nach.*, Vol. XXIX. p. 177.

The counterpoises may be made to perform another important service in diminishing the *flexure* of the horizontal axis, which they will evidently do if they are applied nearer to the cube than in this instrument. With cones, such as *NN*, of very broad base, the amount of flexure must be extremely small; still, with counterpoises properly placed, the necessity of making the cones so large and heavy would be obviated. (See the arrangement of the counterpoises in the meridian circle, Plate VII.)

In the principal focus of the objective, at *m*, is the *reticule*, consisting of seven parallel transit threads; these are parallel to the vertical plane of the telescope and perpendicular to its optical axis (Art. 5). These threads and the images of stars in their plane are observed with the eye piece *E*. Eye pieces, or oculars, of various magnifying powers are usually supplied, to be used according to the nature of the object observed and the state of the atmosphere, the highest powers being available only in the most favorable circumstances. One of these eye pieces (and usually one of the lowest powers) is fitted with a mirror to throw light down the tube in observations for collimation, as will be fully explained hereafter. This constitutes what is called the *collimating eye piece;* but the plan of placing a small piece of mica *outside* the eye piece (Art. 47) converts any one of the eye pieces into a collimating eye piece.

There is also a micrometer thread which moves so nearly in the plane of the transit threads as to be sensibly in the same focus. This thread may be either parallel or at right angles to the transit threads according to the application of it intended; but in the simple transit instrument its use will be chiefly to determine the collimation with the mercury collimator, and then it will be most convenient to make it parallel to the transit threads. For this purpose, it will be still better to substitute for the single movable thread a *cross-thread* or two very close parallel threads.

The transit threads are rendered visible at night by light thrown into the interior of the telescope through the hollow rotation axis from a lamp on either side. The light is reflected down the telescope tube by a small silver mirror in the cube *M*, or by an open metallic ring, which does not interfere with rays from the object glass. The amount of light can easily be regulated by a contrivance which it is not necessary to describe. The color of the light may be varied by passing it through glass of the desired shade.

The light thus thrown down the tube illuminates the *field*, and the transit threads appear as black lines upon a bright ground. For very faint stars it may be necessary to reduce this field illumination to such an extent that the threads cease to be distinctly visible, and then the *direct* illumination of the threads is to be resorted to. This direct illumination of the threads is effected, in the instrument here represented, by two small lamps (omitted in the drawing) suspended upon the telescope near the eye piece, which throw their light obliquely upon the threads without illuminating the field. The lamps are so suspended that their flames occupy the same position relatively to the threads for all positions of the telescope. The threads are thus made to appear as bright lines on a dark ground. Two lamps, one on each side, are used in order to produce symmetrical illumination of the threads. The threads may also be illuminated by light admitted through the axis, but so brought down the tube (by the aid of a small lens) as not to illuminate the field; this light being finally received by small reflectors near the eye piece, and by them thrown upon the threads in such a manner as to produce the required symmetrical illumination.

At *F* and *F* are two small *finding circles*, also called *finding levels*, or simply *finders*, which serve in setting the telescope at any given elevation or zenith distance. They will be more fully explained in connection with the portable transit instrument in the next article.

The handles, *A* and *B*, which are within reach of the observer's hand, act upon a clamp and fine motion screw by which the telescope is fixed and accurately set at any zenith distance.

The inclination of the rotation axis to the horizon is measured with the striding level L (Art. 51), which is applied to the pivots VV. The feet of the level have also the form of Vs.

The piers are so nearly adjusted in the first place that the Vs are nearly in a true east and west line, but a small final correction is still possible by means of screws which act horizontally upon one of the Vs. In the same manner, the inclination of the axis to the horizon is made as small as we please by screws acting vertically upon the other V. These screws are not shown in the drawing.

In order to eliminate errors of the instrument, it is necessary to reverse the rotation axis from time to time, that is, to make the east and west ends of the axis change places. The *reversing apparatus* or *car* for this purpose is shown at R. It runs upon grooved wheels which roll upon two rails laid in the observatory floor between the piers PP, and is thus brought directly beneath the axis. By the crank h acting upon the beveled wheels e and f, two forked arms aa are lifted and brought into contact with the axis at NN; then, continuing the motion, the telescope is lifted just sufficiently to clear the Vs, and the friction rollers at XX; the car is then rolled out from between the piers, bearing the telescope upon its arms; a semi-revolution is given to the arms, the exact semi-revolution being determined by a stop d, the car is rolled back between the piers, and the telescope lowered into the Vs. It is hardly necessary to observe that the telescope is placed in a horizontal position during this operation.

An *observing couch* C runs on the rails between the piers. It is so arranged that the observer reclining upon it may give his head any required elevation, and thus be able to observe stars at high altitudes without the discomfort which would destroy the accuracy of his observations.

The piers PP are of granite, and rest upon a foundation of stone sunk ten feet below the surface of the ground. They are wholly insulated from the walls and floor of the building.

Between the piers, a granite slab about a foot broad and ten feet long is placed on a level with the floor. This rests firmly upon the foundation which supports the instrument, and, like the piers, is insulated from the floor. On this slab may be placed a basin of mercury at various distances from the instrument, for observing stars by reflexion.

I do not propose to enter into the details of constructing the observatory itself, as many of these details will vary according to the taste and means of the builder; but it is essential to remark that the opening in the roof and sides of the building through which the observations are to be made should be much wider than the mere aperture of the telescope; for there are always currents of air of various temperatures near the edges of the openings, which produce unsteadiness in the images of stars. A width of two feet at least should be allowed.

It is also well to observe that the observing room should be large and high, that the radiation from the walls may not have too much effect upon the instrument. No artificial heat should be permitted in it or near it. Its temperature at the time of an observation, and that of the whole instrument, should be as nearly as possible the same as the temperature of the atmosphere outside the observatory.

The indispensable companion of the transit instrument in the observatory is the sidereal clock, which is to be secured to a stone pier, resting upon a foundation which is insulated from the floor, and so placed that its dial may be seen by the observer from any position he may occupy at the telescope. If, however, the transits are recorded by the chronograph (Arts. 71–77) the clock may be in any part of the observatory, and a single clock may be used *for all the observations* with all the instruments. It will only be necessary that each instrument should have its own chronographic register, which is graduated into seconds by the one standard clock. However, a clock in the room with the instrument is still necessary to enable the observer to prepare for his observations at the proper time; but this may then be regarded as a sort of *finder* merely, and it will be necessary to regulate it only approximately.

120. Plate V. represents a portable transit instrument as constructed by Mr. W. Würdemann (Washington, D. C.). The focal length of such an instrument is usually from 24 to 36 inches.

The letters common to Plate V. and Plate IV. represent the same parts. The peculiar feature is the portable frame *PP*, which here takes the place of the piers. It is made of iron, and is made as light as possible without the sacrifice of strength and stability. The screws *tt* being removed, the inclined supports *pp* fold in against the upright ones, and then the latter fold down upon the horizontal frame; and the whole frame can be placed in a box. This box is deep enough to receive the telescope also. The instrument can thus be conveniently transported and set up in a few minutes upon any temporary pillar Q. In the field it will often be convenient to mount the instrument upon the trunk of a tree cut off to the required height. The frame is quickly levelled approximately by the foot screws S, S, S.

A *diagonal eye piece* E (Art. 12) is necessary for observing stars at considerable altitudes.

The eye tube of the telescope is moved out and in by a rack and pinion r, to bring the threads precisely into the focus of the object glass. The rack and pinion k carry the eye piece to the right and left so as to bring it opposite each thread in succession as a star crosses it.

The finder F consists, 1st, of a small graduated circle which is permanently attached to the telescope; 2d, of a spirit level g attached to an arm which revolves about the centre of the circle. This arm carries a vernier, and has a clamp and fine motion screw at f. When the vernier reads 0°, the axis of the level is parallel to the optical axis of the telescope; consequently, if we set the vernier to this reading, 0°, and then revolve the telescope until the bubble stands in the middle of the tube, the optical axis will be horizontal. If then we set the vernier at any other given reading R, and revolve the telescope until the bubble stands in the middle of the tube, the inclination of the telescope to the horizon will be $= R$. The altitude of a star whose transit is to be observed is known from its declination and the latitude of the place of observation, and it is usually necessary to prepare for the observation by setting the telescope at the proper altitude by means of the finder.

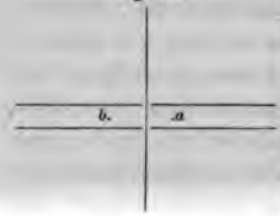
A rack and pinion (not shown in the drawing) serve to revolve the eye piece and micrometer so as to make the threads vertical, or rather parallel to the vertical plane of the telescope.

The illuminating lamps are shown in their position. Their light is thrown into the axis in nearly parallel lines by means of a lens in the lantern opposite the middle point of the flame, the flame being nearly in the focus of the lens.

120*. A small altitude and azimuth instrument so constructed that it may be used also as a transit instrument is called a *universal instrument.* The horizontal graduated circle renders such an instrument very convenient for observations out of the meridian. See Chapter VII.

121. *Method of observation.*—In all cases, the celestial observation made with the transit instrument consists only in noting, by a clock or chronometer, the several instants when a star or other object crosses the threads. The method of doing this with precision is as follows. The instrument remaining stationary, the diurnal motion causes the star to pass across the field of the telescope. As it approaches a thread, the observer looks at the clock and begins to count its beats; and, keeping the count in his head by the aid of the audible beats of the clock, he then turns his eye to the telescope and notes the beat when the star appears *on* the thread. The transit over the thread may, however, fall between two beats; and then the fraction of a beat is to be estimated. This estimate is made rather by the eye than the ear. Suppose the clock beats seconds. Let *a*, Fig. 36, be the position of the star at the last beat before the star comes to the thread, and *b* its position at the next following beat. The observer compares the distance from *a* to the thread with the distance from *a* to *b*, and estimates the fraction which expresses the ratio of the former to the latter in tenths; and these tenths are then to be added to the whole number of seconds counted at *a*, to express the instant of transit. Thus, if he counts 20 seconds by the clock at *a*, and estimates that from *a* to the thread is $\frac{4}{10}$ of *ab*, the instant of transit is $20^s.4$, which he records, together with the minute and hour by the clock.

Fig. 36.



In the transit of the sun, the moon, or a planet, the instant when the limb is a tangent to the thread is noted. The mode of inferring the time of transit of the centre from that of the limb will be explained hereafter.

The most accurate method of observing transits is by the aid of the chronograph. At the precise instant when the star is *on* the thread, the observer presses the signal key and makes a record on the register, which is read off at his leisure, according to the methods explained in Arts. 71–77. The record of several transits of stars over the five threads of the Cambridge telescope is shown in Plate I. Fig. 6. . Each transit is preceded by an irregular signal, produced by a rapid succession of taps on the signal key, by means of which the place of the observation on the register is afterwards readily found. As the observer is relieved by the chronograph from the necessity of counting the seconds and estimating the fractions, the transit threads may be placed much closer to each other and their number greatly increased. In the transit instruments used in the United States Coast Survey for the telegraphic determination of differences of longitude (see Vol. I. Art. 227), the diaphragms contain twenty-five threads, arranged in groups, or "tallies," of five, as in Plate I Fig. 1.

GENERAL FORMULÆ OF THE TRANSIT INSTRUMENT.

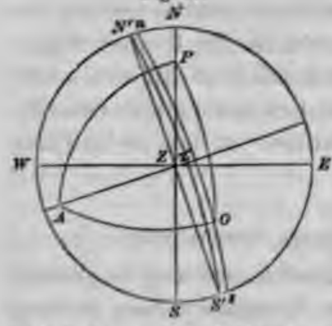
122. In whatever position the transit instrument may be placed, we may consider its *rotation axis* as an imaginary line, passing through the central points of the pivots, which, produced to the celestial sphere, becomes a diameter of the sphere; and the *axis of collimation* as an imaginary line, drawn from the optical centre of the object glass perpendicular to the rotation axis, and describing a great circle of the sphere as the telescope revolves. The position of this great circle in the heavens is fully determined when we have given the position of the rotation axis; and the position of the rotation axis is given when we know the altitude and azimuth of either of the points in which it meets the celestial sphere.

The *sight-line* marked by a thread in any part of the field is a line drawn from the thread through the optical centre of the object glass. The angle which this line makes with the axis of collimation does not change as the telescope revolves: so that, while the axis of collimation describes a great circle, the sight-line describes a small circle parallel to it whose distance from it is everywhere the constant measure of the inclination of the sight-line. If then a star is observed on the thread, the position of the star with respect to the great circle of the instrument becomes known when we know the inclination of the sight-line or the angular distance of the thread from the axis.

The general problem to which the use of the transit instrument gives rise is the following:

123. *To find the hour angle of a star observed on a given thread of the transit instrument in a given position of the rotation axis.*—Let Fig. 37 represent the sphere stereographically projected upon the plane of the horizon, *NS* the meridian, *WE* the prime vertical. Suppose the axis of the instrument lies in the vertical

Fig. 37.



plane ZA, and that A is the point in which this axis produced towards the west meets the celestial sphere. Let $N'Z'S'$ be the great circle described by the axis of collimation; A is the pole of this circle. Let nOs be the small circle described by the sight-line drawn through a thread whose constant angular distance from the collimation axis is given $= c$. Let b denote the altitude, $90° + a$ the azimuth, $90° - m$ the hour angle, n the declination of the point A; φ the latitude of the observer; δ the declination of a star observed at O on the given thread. Join PA, PO, AO. We have

$$\begin{aligned} NZA &= 90° + a, & ZPA &= 90° - m \\ ZA &= 90° - b, & PA &= 90° - n \\ AO &= 90° + c, & PZ &= 90° - \varphi \\ & & PO &= 90° - \delta \end{aligned}$$

and the triangle PZA gives the equations [Sph. Trig. (6), (3), (4)]

$$\left.\begin{aligned} \cos n \sin m &= \sin b \cos \varphi + \cos b \sin a \sin \varphi \\ \cos n \cos m &= \cos b \cos a \\ \sin n &= \sin b \sin \varphi - \cos b \sin a \cos \varphi \end{aligned}\right\} \quad (78)$$

which determine m and n when a and b are given. Now let

$$\tau = \text{the hour angle of } O \textit{ east} \text{ of the meridian;}$$

then the angle $APO = 90° - m + \tau = 90° + (\tau - m)$, and the triangle APO gives

$$-\sin c = \sin n \sin \delta - \cos n \cos \delta \sin (\tau - m)$$

whence

$$\sin (\tau - m) = \tan n \tan \delta + \sin c \sec n \sec \delta \quad (79)$$

which determines $\tau - m$, whence also τ.

These general formulæ admit of simplification when the instrument is either near the meridian or near the prime vertical.

THE TRANSIT INSTRUMENT IN THE MERIDIAN.

124. The instrument is said to be in the meridian when the great circle described by the axis of collimation is the meridian.

The axis of rotation is then perpendicular to the plane of the meridian, and, consequently, lies in the intersection of the prime vertical and the horizon. If, further, the thread on which the star is observed is in the axis of collimation, the time of observation is that of the star's transit over the meridian; and, since at that instant the sidereal time is equal to the star's right ascension, the error of the clock on sidereal time is obtained at once by taking the difference between that right ascension and the observed clock time of transit. (Vol. I. Art. 138.)

Practically, however, we rarely fulfil these conditions exactly, but must correct the time of observation for the small deviations expressed by a, b, and c, of which a is the excess of the azimuth of the west end of the axis above 90° (reckoned from the north point), and is called the *azimuth constant;* b is the elevation of the west end of the axis, and is called the *level constant*; and c is the inclination of the sight-line to the collimation axis, and is called the *collimation constant.*

We must first show how to adjust the instrument approximately, or to reduce a, b, and c to small quantities.

125. *Approximate adjustment in the meridian.*—1st. The middle thread of the diaphragm should coincide as nearly as possible with the collimation axis. This adjustment can be approximately made before putting the instrument in the meridian, by moving the thread plate laterally until the middle thread cuts a well defined distant point in both positions of the rotation axis in the Vs.

2d. The middle thread (and, consequently, all the transit threads) should be vertical when the rotation axis is horizontal; that is, it should be perpendicular to the rotation axis. This can be verified while adjusting the sight-line, by observing whether the distant point continues to appear on the thread as the telescope is slightly elevated or depressed. After the instrument has been placed in the meridian and the axis levelled, the verticality of the threads may also be proved by an equatorial star running along the horizontal thread, which is at right angles to the transit threads.

The axis, being placed nearly east and west (at first by estimation), is levelled by means of the striding level. Thus c and b are easily reduced to small quantities.

3d. To reduce a to a small quantity, or to place the instrument very near to the meridian, we must have recourse to the observation of stars. The following process will be found as simple as any other with a portable instrument.

Compute the mean time of transit of a slow moving star (one near the pole), and bring the telescope upon it at that time. For the first approximation, the time may be given by a common watch, and the telescope may be brought upon the star by moving the frame of the instrument horizontally. Then level the axis, and note the time by the clock of the transit of a star near the zenith over the middle thread. It is evident that the time of transit of a star near the zenith will not be much affected by a deviation of the instrument in azimuth, and therefore the difference between the star's right ascension and the clock time will be the approximate error of the clock on sidereal time. With this error, we are prepared to repeat the process with another slow moving star, this time employing the clock and causing the middle thread to follow the star by moving only the azimuth V. When the clock correction has been previously found by other means (as with the sextant), the first approximation will usually be found sufficient. The instrument is now sufficiently near to the meridian, and the outstanding small deviations can be found and allowed for as explained below.

In mounting a large transit instrument in an observatory, it will be convenient first to establish the approximate direction of the meridian with a theodolite, and to set up a distinct mark at a sufficient distance to be visible in the large telescope without a change of the stellar focus. The middle thread of the instrument can then be brought upon this mark before proceeding to the observation of stars.

4th. Finally, it is necessary to adjust the *finder* whereby the telescope is to be directed to that point of the meridian through which a given object will pass. If the finder is intended to give the zenith distance (ζ), we take

$$\zeta = \varphi - \delta - r + p \text{ for an object south of the zenith,}$$
$$\zeta = \delta - \varphi - r + p \quad \text{“} \quad \text{“} \quad \text{north} \quad \text{“} \quad \text{“}$$

in which r is the refraction, and p the parallax of the object for the zenith distance ζ. But, for the purpose of *finding* an object merely, we may neglect r, except for very low altitudes, and p may be neglected for all bodies except the moon.

To adjust the finder, we have only to clamp the telescope when some known star is on the horizontal thread, and in that position cause the finding circle to read correctly for that star, by means of the proper adjusting screws. It will then read correctly for all other stars. In large instruments the finder is sometimes graduated from 0° to 360°.

With respect to the *time* when a star is to be expected on the meridian, the sidereal clock or chronometer answers as a finder, since (after allowing for its error) it shows the right ascensions of the stars that are on the meridian.

126. *Equations of the transit instrument in the meridian.*—By the preceding process we can always easily reduce a, b, and c to quantities so small that their squares will be altogether insensible, or, which is the same thing, we can substitute them for their sines, and put their cosines equal to unity. And, since m, n, and τ will be quantities of the same order as a, b, and c, the general formulæ (78) will become

$$\left.\begin{aligned} m &= b\cos\varphi + a\sin\varphi \\ n &= b\sin\varphi - a\cos\varphi \end{aligned}\right\} \quad (80)$$

and (79) gives

$$\tau = m + n\tan\delta + c\sec\delta \qquad (81)$$

which is Bessel's formula for computing the correction to be added to the observed sidereal clock time of transit of a star over the middle thread to obtain the clock time of the star's transit over the meridian. It is hardly necessary to observe that the unit of all the quantities a, b, c, m, n, τ should be the second of time.

If now we put

T = the observed clock time of the star's transit over the middle thread,
ΔT = the correction of the clock,
α = the star's apparent right ascension,

the true sidereal time of transit will be $T + \tau + \Delta T$, and this quantity must be equal to α. Hence we have

$$\left.\begin{aligned} \alpha &= T + \Delta T + \tau \\ \text{or}\quad \alpha &= T + \Delta T + m + n\tan\delta + c\sec\delta \end{aligned}\right\} \quad (82)$$

by which formula the right ascension of an unknown star can be found when ΔT and the constants of the instrument are known From the transits of known stars, on the other hand, this equation enables us to find ΔT, when the constants of the instrumen are given.

The apparent right ascension in this equation should be affected by the diurnal aberration, which, by Vol. I. Art. 393, is $0''.311 \cos\varphi \sec\delta = 0^s.021 \cos\varphi \sec\delta$ when the star is on the meridian. If then α denotes the right ascension as given in the Ephemeris, the first member of (82) ought to be $\alpha + 0''.311 \cos\varphi \sec\delta$, so that the equation becomes

$$\alpha = T + \Delta T + m + n \tan\delta + (c - 0^s.021 \cos\varphi) \sec\delta \tag{83}$$

Hence, if instead of c we take

$$c' = c - 0^s.021 \cos\varphi$$

we may use (82) without further modification, and the diurnal aberration will be fully allowed for. Since, for each place of observation, the quantity $0^s.021 \cos\varphi$ is constant, there is no reason for omitting to apply this correction, although its influence is scarcely appreciable except with the larger instruments of the observatory.

127. Bessel's form for the correction τ is usually the most convenient; but other forms have their advantages in certain applications. From (80) we deduce

$$\left.\begin{aligned} a &= m \sin\varphi - n \cos\varphi \\ b &= m \cos\varphi + n \sin\varphi \end{aligned}\right\} \quad (84$$

and from the second of these we have

$$m = b \sec\varphi - n \tan\varphi \quad (85$$

which substituted in (81) gives Hansen's formula,

$$\tau = b \sec\varphi + n(\tan\delta - \tan\varphi) + c \sec\delta \quad (86$$

This is convenient in reducing observations of stars near th zenith, where the coefficient $\tan\delta - \tan\varphi$ becomes small. I shows that for a star in the zenith the correction depends onl on b and c, and that in general the best stars for determinin the clock correction are those which pass nearest to the zenith.

If we substitute the values of m and n from (80) in (81), w readily bring it to the form

$$\tau = a \cdot \frac{\sin(\varphi - \delta)}{\cos \delta} + b \cdot \frac{\cos(\varphi - \delta)}{\cos \delta} + \frac{c}{\cos \delta} \qquad (87)$$

which is known as MAYER's formula. This is the oldest form; but where many stars are to be reduced for the same values of the constants, it is much less convenient than the preceding. It has its advantages, however, in cases where the constant a is directly given, or in discussions in which this constant is directly sought.

128. These formulæ apply directly to the case of a star at its upper culmination. To adapt them to lower culminations (that is, of circumpolar stars at their transits below the pole), we observe that in the general investigation Art. 123, δ represents the distance of the star from the equator reckoned towards the zenith of the place of observation, and, consequently, the formula will be applicable to lower culminations if we still represent by δ the distance of the star from the equator through the zenith and *over the pole;* that is, if we take for δ the supplement of the declination. This being understood, we shall be saved the necessity of duplicating our formulæ.

Again, the time of the lower culmination differs by 12^h of sidereal time from that of the upper culmination of the same star. Hence, to apply the formulæ to the case of a lower culmination, it is also necessary to suppose that α represents the star's right ascension increased by 12^h.

In short, for lower culminations, we must substitute $12^h + \alpha$ and $180° - \delta$ for α and δ.

129. Since the instrument may be used in two positions of the rotation axis, it is necessary to distinguish these positions. We shall suppose that the *clamp* is at one end of the axis, and shall distinguish the two positions by "clamp west" and "clamp east." If the value of c has been found for clamp west, its value for clamp east will be numerically the same, but will have a different sign; for, since in reversing the collimation axis remains in the same plane,* any thread will be at the same absolute distance from this axis, but on opposite sides of it in the two positions.

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* Except when the pivots are unequal, the correction for which will be considered hereafter.

For example, if we have found for clamp west $c = -0^s.292$, we must take for clamp east $c = +0^s.292$.

If, however, we take the diurnal aberration into account, we must observe that c' is not numerically the same in the two positions of the axis. For example, if $\varphi = 38° 59'$, the correction $0^s.021 \cos\varphi$ is $0^s.016$; and if for clamp west we have $c = -0^s.292$, we shall have for this position $c' = -0^s.292 - 0^s.016 = -0^s.308$, but for clamp east $c' = +0^s.292 - 0^s.016 = +0^s.276$.

130. In the above, we have assumed that the star has been observed on a single thread whose distance from the collimation axis is known. The same method may be applied to each thread; but when the intervals between the threads are known, each observation may be reduced to the middle thread or to a point corresponding to the "mean of the threads," and the correction τ will then be computed only for this middle thread or this mean point. I proceed to show how these intervals are to be determined and applied.

THREAD INTERVALS.

131. An odd number of threads is always used, and they are placed as nearly equidistant as possible, or, at least, they are symmetrically placed with respect to the middle one, and this middle thread is adjusted as nearly as possible in the collimation axis. If the threads were exactly equidistant, the mean of the observed times of transit over all of them could be taken as the time of transit over the middle one, and this with the greater degree of accuracy (theoretically) the greater the number of threads.* But since it rarely happens that the threads are perfectly equidistant or symmetrical, it becomes necessary to determine their distances; and this is usually the first business of the observer after he has mounted his instrument and brought it approximately into the meridian.

Let i denote the angular interval of any thread from the middle thread; I the time required by a star whose declination is δ to pass over this interval. Then i, being expressed in seconds of time, will also denote the interval of sidereal time required by a star in the equator to describe the space between the threads; for this is the case in which the apparent path of the star is a great circle. Our notation, therefore, may be expressed by putting

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* The practical limits to the number of threads will be considered in another place.

i = the equatorial interval of a thread from the middle thread,
I = the interval for the declination δ.

If now c denotes the collimation constant for the middle thread, the distance of the side thread from the collimation axis is $i + c$; and if τ is the hour angle of a star when on the middle thread, $I + \tau$ is its hour angle when on the side thread. Hence, by our rigorous formula (79), applied to each thread, we have

$$\begin{aligned}\sin(I + \tau - m) &= \tan n \tan \delta + \sin(i + c) \sec n \sec \delta \\ \sin(\tau - m) &= \tan n \tan \delta + \sin c \sec n \sec \delta\end{aligned}$$

the difference of which is

$$2 \cos(\tfrac{1}{2} I + \tau - m) \sin \tfrac{1}{2} I = 2 \cos(\tfrac{1}{2} i + c) \sin \tfrac{1}{2} i \sec n \sec \delta$$

for which, since $\tau - m$, c, and n are here very small quantities, we may write, without sensible error,

$$\left.\begin{aligned}2 \cos \tfrac{1}{2} I \sin \tfrac{1}{2} I &= 2 \cos \tfrac{1}{2} i \sin \tfrac{1}{2} i \sec \delta \\ \text{or} \qquad \sin I &= \sin i \sec \delta\end{aligned}\right\} \qquad (88)$$

From this, I can be found when i is given. On the other hand, if I is observed in the case of a star of known declination, we deduce i by the formula

$$\sin i = \sin I \cos \delta \qquad (89)$$

If the star is not within 10° of the pole, it is quite accurate to take for these the more simple forms

$$I = i \sec \delta \qquad\qquad i = I \cos \delta \qquad (90)$$

These formulæ show that the observed interval will be the greater the nearer the star is to the pole. Hence, for finding i from observed values of I, it is expedient to take stars near the pole, since errors in the observed times will be reduced in the ratio $1 : \cos \delta$.

When the star is so near to the pole that either (88) or (89) is to be used, it will be found convenient to substitute for them the following:

$$I = i \sec \delta . k \qquad\qquad i = \frac{I \cos \delta}{k} \qquad (91)$$

in which $k = \frac{I \sin 15''}{\sin I}$, and its logarithm may be readily taken from the following table:

I	log $i \sec \delta$	log k
1^m	1.778	0.00000
2	2.079	.00001
3	2.255	.00001
4	2.380	.00002
5	2.477	.00003
6	2.556	.00005
7	2.623	.00007
8	2.681	.00009
9	2.732	.00011
10	2.778	.00014
11	2.819	.00017
12	2.857	.00020
13	2.892	.00023
14	2.924	.00027
15	2.954	.00031

I	log $i \sec \delta$	log k
15^m	2.954	0.00031
16	2.982	.00035
17	3.008	.00040
18	3.033	.00045
19	3.056	.00050
20	3.079	.00055
21	3.100	.00061
22	3.120	.00067
23	3.139	.00073
24	3.158	.00080
25	3.175	.00086
26	3.192	.00093
27	3.209	.00101
28	3.224	.00108
29	3.239	.00116
30	3.254	.00124

EXAMPLE 1.—If for a star whose declination is $\delta = 88° 33'$ we have observed the interval between a side thread and the middle thread to be $I = 25^m\ 17^s.6$, required the value of i.

We have

	log I	3.18116
	log cos δ	8.40320
	ar. co. log k	9.99912
$i = 38^s.325$	log i	1.58348

EXAMPLE 2.—Given $i = 38^s.325$, find I for $\delta = 87° 15'$.

We have

	log i	1.58348
	log sec δ	1.31896
	log i sec δ	2.90244
	(Argument 2.902) log k	0.00024
$I = 799^s.25$	log I	2.90268

132. The thread intervals may also be found by GAUSS's method, with a theodolite, precisely as in the method of determining the value of a micrometer screw in Art. 46.

If the instrument is furnished with a micrometer, the value of the screw may be determined by the transits of circumpolar stars over the micrometer thread, and then it may be employed to measure the thread intervals.

REDUCTION TO THE MIDDLE THREAD.

133. Suppose that the reticule contains five transit threads, and that they are numbered consecutively from the side next to the clamp: so that for "clamp west" stars at their upper culminations cross the threads in the order of their numbers. Then, if we denote the observed clock times of a transit over them by t_1, t_2, t_3, t_4, t_5, and the equatorial intervals of the side threads from the middle thread by i_1, i_2, i_4, i_5 (observing that i_4 and i_5 will be essentially negative), the time of passing the middle thread according to the five observations is either $t_1 + i_1 \sec \delta$, $t_2 + i_2 \sec \delta$, t_3, $t_4 + i_4 \sec \delta$, or $t_5 + i_5 \sec \delta$, which, if the observations were perfect, would be equal to each other. Taking their mean, which we shall denote by T, we have

$$T = \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} + \frac{i_1 + i_2 + i_4 + i_5}{5} \sec \delta$$

If we put

$$\Delta i = \frac{i_1 + i_2 + i_4 + i_5}{5}$$

and denote the mean of the observed times by T_0, we have

$$T = T_0 + \Delta i \sec \delta \quad \text{for clamp west,}$$
$$T = T_0 - \Delta i \sec \delta \quad \text{for clamp east}$$

If the threads are equidistant, Δi vanishes; otherwise $\Delta i \sec \delta$ is the correction to be applied to what is called the *mean of the threads*, to obtain the time of passage over the middle thread.

If there are seven threads,

$$\Delta i = \frac{(i_1 + i_2 + i_3) + (i_5 + i_6 + i_7)}{7} \qquad (92)$$

and so on for any number of threads.

At the lower culmination, a star crosses the threads in the reverse order, and, consequently, the sign of the correction $\Delta i \sec \delta$ must be changed; but this change of sign is effected by taking for δ the supplement of the declination, according to the method pointed out in Art. 128. We shall, therefore, regard the above formulæ as entirely general.

A *broken transit* (one in which the transits over some of the threads have not been observed) is reduced in the same manner; that is, we take the mean of the observed times and apply to it a correction which is the mean of the equatorial intervals of the observed threads multiplied by sec δ. Thus, if only the 1st, 3d, and 4th of five threads have been observed, we have for T the several values $t_1 + i_1 \sec \delta$, t_3, $t_4 + i_4 \sec \delta$, the corresponding thread intervals being i_1, 0, i_4: so that we have

$$T = \frac{t_1 + t_3 + t_4}{3} + \frac{i_1 + i_4}{3} \sec \delta .$$

In general, if we put

$M =$ the mean of the observed times on any number of threads,
$f =$ the mean of the equatorial intervals of these threads,

the time T of transit over the middle thread will be

$$T = M + f \sec \delta \tag{93}$$

If the clock rate is considerable, the reduction of M to T must be corrected accordingly. Thus, if

$$\Delta T = \text{the clock rate per hour},$$

the reduction $f \sec \delta$ becomes $f \sec \delta \left(1 - \frac{\Delta T}{3600}\right)$; or, putting

$$\left.\begin{aligned} \rho &= \text{the factor for rate} = 1 - \frac{\Delta T}{3600} \\ T &= M + \rho f \sec \delta \end{aligned}\right\} \tag{94}$$

For a sidereal clock which *gains* 1^s per day, we have $\Delta T = -\frac{1}{24}$, whence $\log \rho = 0.000005$, and for a gain of x seconds daily $\log \rho = 0.000005\,x$.

For a mean time clock which has no rate on mean time, and, consequently, loses $9^s.83$ per hour on sidereal time, we find $\log \rho = 9.99881$; and, if it *gains* x seconds per day, $\log \rho = 9.99881 + 0.000005\,x$.

If the star is very near the pole, each thread should be separately reduced, the reduction to the middle thread being computed by the formula $I = i \sec \delta . k\rho$, $\log k$ being taken from the table in Art. 131.

REDUCTION TO THE MEAN OF THE THREADS.

134. Another mode of reducing transits is commonly used in the observatory. We may suppose an imaginary thread so placed in the field that the time of transit over it will be the same as the mean of the times on all the threads, and for brevity this imaginary thread is called the *mean of the threads*, or the *mean thread*. Then all observations are reduced to this imaginary thread, and the constant c as well as the intervals of the several threads are referred to it, precisely as if it were a real thread. It is evident that, where many complete transits are to be reduced, this method saves labor, as the correction $\Delta i \sec \delta$ is avoided.

135. EXAMPLE 1.—The upper transit of *Polaris* was observed with the meridian instrument of the Naval Academy on Jan. 26, 1859, as in the second column of the following table:

Clamp East. $\delta = 88° 38' 54''.3$

Thread.	Sid. clock.	I	log I	log k	log i	i
VII	$0^h\ 44^m\ 55^s$	$-28^m\ 49^s$	$n3.15508$	0.00079	$n1.55290$	$-35^s.720$
VI	52 56	$-15\ 48$	$n2.97681$	84	$n1.87518$	-28.721
V	1 0 54	$-7\ 50$	$n2.67210$	09	$n1.07067$	-11.767
IV	8 44					
III	16 32	$+7\ 48$	2.67025	09	1.06882	$+11.717$
II	24 31	$+15\ 47$	2.97635	84	1.87467	$+28.696$
I	32 30	$+23\ 46$	3.15412	78	1.55200	$+35.645$

The table exhibits the computation of the equatorial intervals of the side threads from the middle thread. The values of log k are taken from the table in Art. 131, and each value of log i is found by the formula $\log i = \log I + \log \cos \delta - \log k$. The signs of I and i are given for *clamp west.*

The values of the intervals must be found from a number of observations of this kind, and the mean of all the determinations should be finally adopted.

According to this single observation, the value of Δi for this instrument will be

$$\Delta i = -0^s.021$$

If the reductions are to be made to the mean of the threads, we find the values of I by taking the difference between the mean of all the observed times and the time on each thread, and compute i as before. The values of i that would result in the above example may be immediately inferred, since they will be equal to those above found diminished by Δi. Thus, arranging the values in their order for clamp west, we have—

Thread.	Intervals to middle thread.	Intervals to mean thread.
I	$+ 35^s.645$	$+ 35^s.666$
II	$+ 23.696$	$+ 23.717$
III	$+ 11.717$	$+ 11.738$
IV	$0.$	$+ 0.021$
V	$- 11.767$	$- 11.746$
VI	$- 23.721$	$- 23.700$
VII	$- 35.720$	$- 35.699$

EXAMPLE 2.—With the same instrument on the same date, the transit of *α Arietis* was observed as follows (*clamp east*):

Thread.	Clock.
VII	$1^h\ 58^m\ 58^s.2$
VI	lost
V	1 59 24.1
IV	36.9
III	49.8
II	2 0 2.8
I	15.9
Mean	= 1 59 41.28

$\delta = + 22° 47' 49''$.

The algebraic sum of the intervals to the middle thread for the threads here observed, taken from the table in the preceding example, is $+ 23^s.571$, or for *clamp east* $- 23^s.571$; and therefore the time of transit over the middle thread is

$$T = 1^h\ 59^m\ 41^s.28 - \frac{23^s.571}{6} \sec\delta = 1^h\ 59^m\ 37^s\ 02$$

To reduce this observation to the mean of the threads, the shortest method is to take one-sixth of the interval corresponding to the missing thread,—thus:

$$T_s = 1^h\,59^m\,41^s.28 - \frac{23^s.700}{6}\sec\delta = 1^h\,59^m\,37^s.00$$

136. Having shown how the quantity T in (82) or (83) is found, I now proceed to show how to determine the constants m, n, and c. Since m and n both involve b, let us begin with the investigation of this quantity.

THE LEVEL CONSTANT.

137. The inclination of the rotation axis to the horizon is usually found by applying the spirit level as explained in Art. 52; and this inclination expressed in seconds of time is the value of the level constant b, positive when the west end of the axis is too high.

But the spirit level applied to the outer surface of the cylinders which form the pivots does not directly determine the inclination of the rotation axis which is the common axis of these cylinders, unless the pivots are of equal diameters.

To find the *correction for inequality of the pivots*, let C, Fig. 38, be the centre of a cross section of a pivot, A the vertex of the V in which the pivot rests, B the vertex of the V of the spirit level applied to it. Put

Fig. 38.

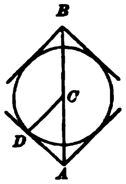
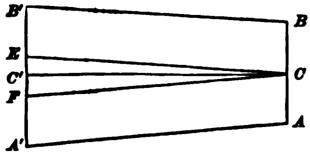


Fig. 39.



$2i$ = the angle of the V of the level,
$2i_1$ = " " " V " transit inst.,
r = the radius of the pivot,
d = the vertical distance of B above C,
d_1 = " " " C " A,

we have

$$d = \frac{r}{\sin i} \qquad d_1 = \frac{r}{\sin i_1}$$

If now, in Fig. 39, CC' is the rotation axis, A and B the vertices of the transit and level Vs at the end next the clamp, A' and B' the vertices of the Vs at the other end of the axis,

r' the radius of the pivot at that end, then we have for the distances $B'C'$, $A'C'$,

$$d' = \frac{r'}{\sin i} \qquad d_1' = \frac{r'}{\sin i_1}$$

The level gives the inclination of the line BB' to the horizon, and we wish to find that of CC'. Let us suppose the clamp at first is west, and afterwards east, and that in both positions of the axis the inclination given by the level is observed. Let

B, B' = the inclinations given by the level for clamp west and clamp east, respectively,
b, b' = the true inclinations of the rotation axis for clamp west and clamp east,
β = the constant inclination of the line AA'.

Also draw CE and CF parallel to BB' and AA', and put

$$p = ECC' \qquad p_1 = FCC'$$

then, L being the length of the level, we have

$$\sin p = \frac{d' - d}{L} = \frac{r' - r}{L \sin i}$$

$$\sin p_1 = \frac{d_1' - d_1}{L} = \frac{r' - r}{L \sin i_1}$$

for which we may take

$$p = \frac{r' - r}{L \sin i \sin 15''} \qquad p_1 = \frac{r' - r}{L \sin i_1 \sin 15''}$$

in which p and p_1 are in seconds of time. Now, we have evidently for clamp west (b denoting the elevation of the *west* end)

$$b = B + p \qquad b = \beta - p_1$$

and for clamp east,

$$b' = B' - p \qquad b' = \beta + p_1$$

whence

$$b' - b = B' - B - 2p = 2p_1$$

$$\frac{B' - B}{2} = p + p_1 = p + p \cdot \frac{\sin i}{\sin i_1} = p\left(\frac{\sin i + \sin i_1}{\sin i_1}\right)$$

and, consequently,

$$p = \frac{B' - B}{2}\left(\frac{\sin i_1}{\sin i + \sin i_1}\right) \tag{95}$$

By this formula, when i and i_1 are known, we can directly compute the value of p from the level indications B and B', observed in the two positions of the axis.

If the angles of both the transit and the level Vs are equal to each other, which is usually the case, we have $\sin i = \sin i_1$; and then we have

$$p = \frac{B' - B}{4} \tag{96}$$

The value of p thus found is called the correction for inequality of pivots. It is to be carefully found by taking the mean of a great number of level determinations in the two positions of the axis. By determining it according to the above formula, it is a correction algebraically additive to the level indication for clamp west: so that the true level constant in any case is found by the formulæ

$$\left.\begin{aligned} b &= B + p \quad \text{for clamp west,} \\ b' &= B' - p \quad \text{for clamp east.} \end{aligned}\right\} \tag{97}$$

138. The inequality of the pivots may also be found without reversing the axis, by using successively two spirit levels, the angles of whose Vs are quite different. Let $2i$ and $2i'$ be their angles, and B and B' the apparent inclination of the axis given by the two levels respectively. If then b is the true inclination, and we put

$$q = \frac{r' - r}{L \sin 15''}$$

we have, by the preceding article,

$$b = B + \frac{q}{\sin i}$$

$$b = B' + \frac{q}{\sin i'}$$

whence

$$q = (B - B')\cdot\frac{\sin i \sin i'}{\sin i - \sin i'} \tag{98}$$

and the correction of inclinations found with the level the angle of whose Vs is $2i$ will be

$$p = \frac{q}{\sin i} = (B - B')\,\frac{\sin i'}{\sin i - \sin i'} \qquad (99)$$

If we construct the levels so that their angles are supplements of each other, that is, make $2i' = 180° - 2i$, the formula becomes

$$p = \frac{B - B'}{\tan i - 1}$$

For example, if $2i = 157° \; 23'$ and $2i' = 22° \; 37'$, we have $p = \frac{1}{4}(B - B')$: so that as accurate a determination of p may be found in this way as by reversing and employing the formula (96).

139. Example 1.—The following example of a case in which the angle of the level V differed from that of the transit V is given by Sawitsch. A portable instrument was mounted in the meridian, and three sets of observations were made consecutively for the determination of p, as in the following table:

No. of determination.	Clamp.	Level readings.		B and B'	$B' - B$
		West.	East.		
1	W.	B. 13.2	13.1	div. $B = + 0.42$	div. $+ 4.50$
		A. 14.0	12.4		
	E.	A. 18.4	8.4	$B' = + 4.92$	
		B. 17.9	8.2		
2	E.	B. 18.8	8.3	$B' = + 5.60$	$+ 5.15$
		A. 19.1	7.2		
	W.	A. 13.6	12.0	$B = + 0.45$	
		B. 13.2	13.0		
3	W.	B. 13.6	13.0	$B = + 0.52$	$+ 4.53$
		A. 14.0	12.5		
	E.	A. 18.2	8.3	$B' = + 5.05$	
		B. 18.3	8.0		

The letters A and B in the first column of level readings refer to the position of the level on the axis.

The value of one division of the level was $1''.68$, or, in time, $0^s.112$.

The angle of the level Vs was $85° = 2i$: that of the transit Vs was $91° = 2i_1$.

We find, by taking the mean,

$$B' - B = + 4.73 \text{ div.} = + 0^s.53$$

and hence, by (95),

$$p = +\,0''.14$$

If we had assumed $i = i_1$, we should have found, by (96), $p = +\,0''.13$, very nearly the same as by the complete formula, although there is a considerable difference between i and i_1.

To find the true inclination of the axis during these observations, we have, by taking the mean of the values of B and B',

$$\begin{aligned} B &= +\,\overset{\text{div.}}{0.46} = +\,0''.05 \\ B' &= +\,5.19 = +\,0\;.58 \end{aligned}$$

whence

$$\begin{aligned} b &= +\,0''.05 + 0''.14 = +\,0''.19 \\ b' &= +\,0\;.58 - 0\;.14 = +\,0\;.46 \end{aligned}$$

Example 2.—In October, 1852, the pivots of the Repsold meridian circle of the U.S. Naval Academy were examined by twenty-four determinations of the inclination of the axis, twelve in each position, and the means were

$$\begin{aligned} \text{Clamp west,}\quad B &= +\,\overset{\text{div.}}{0.68} \\ \text{" east,}\quad B' &= +\,0.74 \end{aligned}$$

One division of the level was equal to $0''.079$; and hence

$$p = +\,\overset{\text{div.}}{0.015} = +\,0''.0012$$

which was neglected, as of no practical importance. Indeed, it is hardly to be presumed that the level readings were sufficient to determine so small a quantity with certainty; nevertheless they suffice to prove the same excellence of workmanship in these pivots as in those of other instruments of Repsold's. In the meridian circle of Pulkowa, made by the same distinguished artist, Struve found an inequality of pivots of only $0''.0025$.

140. The linear difference of the radii of the pivots may also be found; for, by the above formulæ, we have

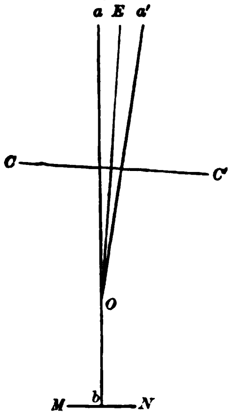
$$r' - r = pL \sin i \sin 15'' = \frac{(B' - B)\,L \sin 15'' \sin i \sin i_1}{2\,(\sin i + \sin i_1)} \tag{100}$$

The value of L in the Example 1 of the preceding article was

10.85 inches, and hence $r' - r = 0.000075$ inch. Small as this difference appears, it is satisfactorily determined by the level.

141. The level constant may also be found by the aid of the mercury collimator (Art. 47) and the micrometer. For large instruments, it is convenient to have the mercury basin permanently placed immediately under the instrument, a little below the level of the floor, and covered only by a small movable trap-door in the floor.

Fig. 40.



Let CC', Fig. 40, be the rotation axis of the instrument; EO the collimation axis, perpendicular to CC'; MN the surface of mercury. There will be formed in the field of the telescope a reflected image of each thread of the reticule; but we shall here use only the movable micrometer thread (which will be assumed to be parallel to the transit threads). Let this micrometer thread be brought into coincidence with its own reflected image, which occurs when it is at that point a of the field which lies in the line bO drawn through the optical centre of the objective, perpendicular to the horizontal surface of the mercury; and hence it follows that, in this position, the angle aOE is equal to the inclination of the rotation axis CC' to the surface MN, or that aOE is equal to the required level constant. Now, let the rotation axis be reversed; the directions CC' and EO remain unchanged (provided the pivots are equal), and the micrometer thread is now at a', at the same distance as before from the collimation axis; if then the thread is again brought into coincidence with its image, it must be moved over a distance $a'a$ = twice the required level constant. If then we put

M = the micrometer interval (expressed in seconds of time), positive or negative according as the micrometer thread is east or west of its image after reversal,

we shall have

$$b = \frac{M}{2} \qquad (101)$$

and b will thus be positive when the west end is elevated.

If the pivots are unequal, b and b' being the true inclinations of the axis for clamp west and clamp east respectively, we shall have, after reversal, $EOa' = b$, and after making a coincidence again, $EOa = b'$; and hence

$$b' + b = M$$

and, from (96) and (97),

$$b' - b = 2p$$

whence

$$b = \frac{M}{2} - p \qquad b' = \frac{M}{2} + p \qquad (102)$$

It appears, then, that the mercury collimator alone is not adequate to the determination of the level constant when the pivots are unequal, since the quantity p must be otherwise determined. The only independent method of finding p is by the spirit level; but we shall see hereafter how the level may be dispensed with (or its indications verified) by means of the mercury collimator *in combination* with collimating telescopes.

142. The pivots may be not only unequal, but also of irregular figures. To determine the existence of irregularities of form, the level should be read off with the telescope placed successively at every 10° of zenith distance on each side of the zenith. The mean of all the inclinations found being called B_0, and B' being that found at a given zenith distance z, $B_0 - B'$ is the correction to be applied to any level reading afterwards taken in the same position of the rotation axis and at the same zenith distance. The level readings are thus freed from the *irregularities* of the pivots, but we still have to apply the correction for *inequality* of the two pivots; and this inequality will be determined by taking one-fourth of the difference of the mean values of B_0 (found as just explained) in the two positions of the rotation axis.

For the examination of the form of the pivots of the great Transit Circle of Greenwich, "each is perforated, and within the hollow of the eastern pivot is fixed a plate of metal perforated with a very small hole, behind which a light can be placed for illumination; and in the hollow of the western pivot there is fixed an object glass at a distance from the perforated plate equal to its focal length. This combination forms a collimator revolving with the instrument. It is viewed by a telescope of 7 feet focal length, which, when required, is placed on Vs, one of them planted in the opening of the western pier, and the other in a hole made for that purpose in the western wall of the room. By a series of most careful observations in 1850, '51, and '52, no appreciable error could be discovered in the form of the pivots."* These pivots are six inches in diameter.

THE COLLIMATION CONSTANT.

143. The constant c may express the distance from the collimation axis either of the middle thread or of the fictitious thread denoted by the "mean of the threads;" the former, when T in (82) is the time of transit over the middle thread, and the latter when T is the time of transit over the mean of the threads

Let us first determine c for the middle thread; its value for the mean of the threads can afterwards be found by adding the quantity Δi (Art. 133); thus, denoting the latter by c_0, we shall have

$$c_0 = c + \Delta i \tag{103}$$

144. *First Method.*—Place the telescope in a horizontal position, and select any terrestrial object that presents some well defined point, and so remote that the stellar focus of the telescope need not be changed to obtain a good definition of the point.† Measure with the micrometer the distance of the point from the middle thread. Reverse the rotation axis, and again measure this distance. If it is the same as before, the thread is in the collimation axis, and $c = 0$; otherwise c is one-half the difference of the micrometer measures. To obtain a simple practical rule which will fix the sign of c for clamp west, put

M, M' = the micrometer distances of the middle thread from the point, positive when the thread appears in the field to be nearer to the clamp than the point;

then, for clamp west,

$$c = \tfrac{1}{2}(M + M') \tag{104}$$

This gives c with the positive sign when the thread is nearer to the clamp than the collimation axis, in which case stars at heir upper culminations arrive at the thread before they reach he axis, and the correction $c \sec \delta$ must be additive.

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* Greenwich Obs. for 1852. Introd. p. iv.

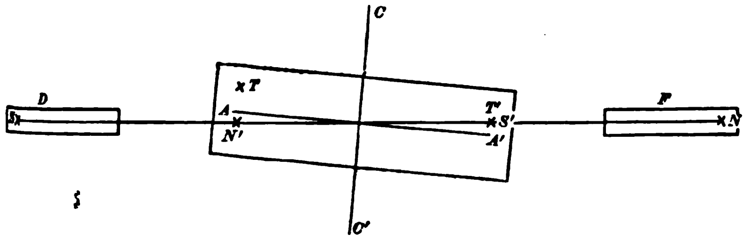
† The meridian mark, if one has been established, will, of course, be used for this point. See Art. 159.

By this method, no correction for the inequality of the pivots is required, since the telescope is horizontal.

Instead of a distant terrestrial point, we may substitute the intersection of two threads in the focus of a horizontal collimating telescope, placed north or south of the instrument. To avoid reversing the axis, two such collimators are used, as in the following method.

145. *Second Method.*—Let two horizontal collimating telescopes D and F, Fig. 41, be mounted on piers in the transit room, one

Fig. 41.



north and the other south of the transit instrument, in the same plane with its rotation axis, their objectives turned towards this axis, and, consequently, towards each other. Suppose, for simplicity, that the collimators have each a single vertical thread N or S in the principal focus. The transit instrument being at rst removed so as not to obstruct the view of one collimator om the other, an image of the thread of either collimator will e formed at the focus of the other, and either thread may be justed so as to coincide exactly with the image of the other. Then the two sight lines of the collimators are in the same ne, or at least are parallel to each other, and their threads hen viewed by the transit telescope represent two infinitely stant objects whose difference of azimuth is precisely 180°. eplacing the transit instrument, direct it first towards the rth collimator. Let CC' be its rotation axis, AA' perpendi-lar to CC' its collimation axis, T the middle thread of the aphragm at the distance $AT = c$ west of the axis. An image N will be formed at N' at a distance AN' from the collima-n axis, which is the measure of the *difference of directions* of the common sight line of the collimators and the axis AA'. Measure with the transit micrometer the distance ($= M$) of T from N'. Next revolve the telescope upon its rotation axis and direct it towards the south collimator. The axis CC' is unchanged, and the point A of the focus which represents the collimation axis is now found at A'. The image of S is formed at S' at a distance $A'S'$ from the collimation axis, which is again the measure of the difference of directions of the common sight line of the collimators and the axis AA': so that we have $AN' = A'S'$; but the points S' and N' are on opposite sides of the axis. The middle transit thread is now at T' on the same side of the collimation axis and at the same distance from it as before: so that we have also $A'T' = c$. Hence, remembering that

M, M' = the micrometer distances of the middle thread *west* of the north and south collimator threads, respectively,

we evidently have

$$c = \tfrac{1}{2}(M + M')$$

To give this method the greatest degree of precision, it will not suffice to use single vertical threads in the collimators, on account of the difficulty of estimating the coincidence of two superposed threads. It is also clear that the sight lines of the two collimators must not be marked by two entirely similar and equal systems of threads, since to bring the sight lines into coincidence we should still have to superpose one system upon the other. A simple method is to substitute for the single thread in the north collimator two very close parallel vertical threads, and in the south collimator two threads intersecting at an acute angle and making equal angles with the vertical. Then the middle point between the close parallel threads marks the sight line of the north collimator, and the coincidence of the intersection of the cross threads of the south collimator with this point can be judged of by the eye with great delicacy. It will assist the eye somewhat if the collimators have also two parallel horizontal threads equidistant from the middle of the field, but not at the same distance from each other in both telescopes.

In the large transit-circle of the Greenwich Observatory the whole system of transit threads is moved by the micrometer screw. In this case let M and M' be the micrometer *readings* when the middle thread is in coincidence with the two collimators respectively; then $M_0 = \frac{1}{2}(M + M')$ is the reading when the middle thread is in the axis of collimation, and $c = 0$; and if during any subsequent observations the micrometer is placed at a different reading m, we must take for the reduction of such observations $c = M_0 - m$.

EXAMPLE.—On Feb. 7, 1853, the collimators of the Greenwich transit-circle having been brought into coincidence, the middle transit thread was brought successively upon each collimator, and the reading of the micrometer for the north collimator was $31^r.300$, and for the south collimator $31^r.521$. Hence, the micrometer being set at the mean $31^r.411$, the middle thread would be in the collimation axis, and then $c = 0$. But if the transit of a star was observed on that date with the micrometer set at $31^r.5$, we should have $c = 31^r.411 - 31^r.5 = - 0^r.089$, or, since $1^r = 0^s.985$, $c = - 0^s.088$.

146. For merely determining the collimation constant, it is not necessary, as has been above supposed, that the collimators be in the same horizontal plane with the axis of the transit instrument. They may be in a plane so far above (or below) that of the transit instrument that the telescope of the latter when horizontal will not intercept the view from one to the other. If then each collimator is mounted as a transit instrument and its rotation axis is level, it can be depressed (or elevated) until its threads can be viewed by the transit telescope. If the inclination of each collimator to the horizon is the same, and the measures of the distances of the middle transit threads from the two collimating threads are as before M and M', we still have $c = \frac{1}{2}(M + M')$. The objection to this arrangement is that the sight lines of the collimators must be made perpendicular to their rotation axes, and these axes must be levelled, adjustments which are unnecessary when they are in the same or very nearly the same horizontal plane as the axis of the principal instrument.

To avoid the necessity of raising the transit instrument out of the Vs (when the three instruments are in the same horizontal plane), two apertures may be made in the cube of the telescope, through which, when the telescope is vertical, the horizontal rays from the collimators may pass.

147. *Third Method.*—Direct the instrument vertically towards the mercury collimator, and measure with the micrometer the distance of the middle thread from its image; put

M = the micrometer distance of the thread from its image, *positive* when the thread is *west* of its image;

then it is evident that, if the rotation axis is horizontal, we shall have $M = 2c$; but, if the west end is elevated by the quantity b, the apparent distance of the thread and its image will be diminished by $2b$: so that we shall then have $M = 2c - 2b$, whence

$$c = \tfrac{1}{2}M + b \tag{105}$$

which gives c with its proper sign for the actual position of the rotation axis.

If we wish to determine the level constant at the same time, we reverse the axis, and again measure the distance of the middle thread from its image. Then, putting

M, M' = the distances of the thread west of its image for clamp west and clamp east, respectively,
b, b' = the level constants in the two positions,

we have, for clamp west,

$$c = \tfrac{1}{2}M + b$$

and (since the sign of c is changed by the reversal), for clamp east,

$$-c = \tfrac{1}{2}M' + b'$$

whence

$$c = \tfrac{1}{4}(M - M') - \tfrac{1}{2}(b' - b)$$

or, since $b' - b = 2p$,

$$\left.\begin{aligned} c &= \tfrac{1}{4}(M - M') - p \quad \text{clamp west,} \\ \text{and} \qquad c &= -\tfrac{1}{4}(M - M') + p \quad \text{“ \ \ east,} \end{aligned}\right\} \tag{106}$$

We have also

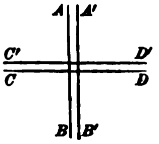
$$\begin{aligned} b' + b &= -\tfrac{1}{2}(M + M') \\ b' - b &= \quad 2p \end{aligned}$$

whence

$$\left.\begin{aligned} b &= -\tfrac{1}{4}(M + M') - p \quad \text{clamp west,} \\ b' &= -\tfrac{1}{4}(M + M') + p \quad \text{“ \ \ east,} \end{aligned}\right\} \tag{107}$$

When the micrometer thread is at right angles to the meridian, and, consequently, moves only in declination, it can nevertheless

Fig. 42.



be used for determining the small quantities c and b according to the above method, as follows. Let AB, Fig. 42, be the middle transit thread, $A'B'$ its reflected image in the collimator, CD the micrometer thread. Move the micrometer thread CD until the distance between it and its image $C'D'$, estimated by the eye, is equal to the distance between the transit thread AB and its image, that is, until the two threads and their images form, to the eye, a perfect square. This square is always very small in a tolerably well adjusted instrument, and can be very accurately formed by estimation. We have then only to measure the distance of CD and $C'D'$ to obtain the required distance. Now, if we move CD we also cause the image $C'D'$ to move; but it is evident that (the telescope not being disturbed) if CD is moved to $C'D'$, the image will be seen at CD, and, in passing from one position to the other, the thread and its image will be in coincidence at the point midway between the two positions. If this coincidence could be observed with perfect accuracy, we might read the micrometer head first when the square was formed, and secondly when the coincidence occurred and the difference of the readings would be one-half the required measure of the side of the square. But, as the threads have sensible thickness, it is difficult to estimate the coincidence of the middle of the thread with the middle of its image, and therefore it will be better to read the micrometer, first when the square is formed by the thread at CD and its image at $C'D'$, and secondly when the square is again formed by the thread at $C'D'$ and its image at CD. The difference of the readings will then be the required measure of the side of the square or of the quantity above denoted by M.

Example 1.—In 1857, June 28, at the Naval Academy, to find the collimation constant of the meridian circle, the distance of the image of the middle thread from its image in the mercury collimator was measured, by forming a square, as above explained, with the declination micrometer thread, alternately north and south of its own image. The readings of the micrometer were 53.5 div. and 59.5 div. The middle thread was west of its image. The value of one division of the micrometer was $0^s.0618$. The level constant found by the spirit level was $b = -0^s.247$. Clamp West.

We find

$$\begin{aligned} M &= +\overset{\text{div.}}{6.0} = +0^{s}.371 \\ c &= \tfrac{1}{2}M + b = +0^{s}.186 - 0^{s}.247 = -0^{s}.061 \end{aligned}$$

EXAMPLE 2.—In 1855, May 11, with the same instrument, a similar observation was made, both with clamp west and clamp east, and there were found

$$\begin{array}{lll} \text{Clamp W.,} & M = -\overset{\text{div.}}{5.4} & \text{(Thread east of its image)} \\ \text{" E.,} & M' = -2.7 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \end{array}$$

Hence, since for this instrument $p = 0$, we find

$$\begin{aligned} c &= \tfrac{1}{4}(M - M') = -0^{s}.042 \text{ for clamp W,} \\ b &= -\tfrac{1}{4}(M + M') = +0.125 \end{aligned}$$

148. By combining the collimating telescopes with the mercury collimator, we can deduce both the collimation and level constants without reversing the rotation axis and without involving the inequality of the pivots. For, by the collimating telescopes, we deduce the value of c, and by the mercury collimator in the same position of the axis, the value of $b = c - \frac{1}{2}M$. This is the method now employed at the Greenwich Observatory, where the transit circle is never reversed; but it is better also to reverse, and thus obtain two independent determinations of our constants for verification.

If we reverse the instrument and determine the level constant by this method in both positions, we can find the inequality of the pivots; for we shall have $p = \frac{1}{2}(b' - b)$.

149. *Fourth Method.*—The preceding methods are very precise and convenient, but are practicable only with instruments provided with collimators. The following method is independent of these auxiliaries, and is practicable with all instruments which admit of reversal; and, being quite accurate, it may be used also with the larger instruments in connection with the other methods, as a check upon them.

Direct the telescope upon a star near the pole, and observe its transits over one or more of the side threads (and also over the middle thread, if the instrument can be reversed in the interval between two threads). Then immediately reverse the rotation axis and observe the transits of the star over one or more of the same side threads again. Let T and T' be the mean of the clock times of transit over the middle thread, deduced from the several observations for clamp west and clamp east respectively (Art. 133); b and b' the level constants in the two positions (the pivots being supposed unequal); then, by (82), (83), and (87), we have, for clamp west,

$$\alpha = T + \Delta T + a\frac{\sin(\varphi - \delta)}{\cos\delta} + b\frac{\cos(\varphi - \delta)}{\cos\delta} + \frac{c}{\cos\delta} - \frac{0^s.021\cos\varphi}{\cos\delta}$$

and, for clamp east,

$$\alpha = T' + \Delta T + a\frac{\sin(\varphi - \delta)}{\cos\delta} + b'\frac{\cos(\varphi - \delta)}{\cos\delta} - \frac{c}{\cos\delta} - \frac{0^s.021\cos\varphi}{\cos\delta}$$

From the difference of these equations we deduce

$$c = \tfrac{1}{2}(T' - T)\cos\delta + p\cos(\varphi - \delta) \qquad (108)$$

in which we have substituted p for $\tfrac{1}{2}(b' - b)$. If the pivots are equal, the term $p\cos(\varphi - \delta)$ will disappear.

If T and T' are the times of passing the *mean thread* (Art. 134), then c is the collimation of this fictitious thread.

150. If the equatorial intervals have not been previously well determined, the mean of the transits over the same thread in the two positions must be compared with the transit over the middle thread. Thus, if T_1 and T_1' are the clock times on the same thread for clamp west and clamp east, we have, for this thread, i_1 being its equatorial interval (omitting the diurnal aberration, which would be eliminated),

$$\alpha = T_1 + i_1\sec\delta + \Delta T + a\frac{\sin(\varphi - \delta)}{\cos\delta} + b\frac{\cos(\varphi - \delta)}{\cos\delta} + \frac{c}{\cos\delta}$$

$$\alpha = T_1' - i_1\sec\delta + \Delta T + a\frac{\sin(\varphi - \delta)}{\cos\delta} + b'\frac{\cos(\varphi - \delta)}{\cos\delta} - \frac{c}{\cos\delta}$$

and, for the middle thread, supposed to be observed with clamp west,

$$\alpha = T + \Delta T + a\frac{\sin(\varphi - \delta)}{\cos\delta} + b\frac{\cos(\varphi - \delta)}{\cos\delta} + \frac{c}{\cos\delta}$$

The difference between the last equation and the mean of the first two gives

$$c = \left(\frac{T_1 + T_1'}{2} - T\right)\cos\delta + p\cos(\varphi - \delta) \qquad (109)$$

but, since the error of observation in T will appear in all the values of c thus found from the several threads, their mean will also involve this error, so that but a slight increase of accuracy will be gained by observing more than one side thread. Hence, for the greatest precision, it is indispensable that the thread intervals should be previously well determined, and that several threads should be used as prescribed in the preceding article.

These formulæ apply without modification to the case of a lower transit, if for δ we use the supplement of the star's declination (Art. 128).

Example.—On Sept. 30, 1858, the lower transit of *Polaris* was observed with the meridian circle of the Naval Academy on the three side threads and the middle thread with clamp east, and on the same side threads with clamp west, as below:

Polaris (lower culm.) $\delta = 91^\circ\ 26'\ 34''$.

	Thread.	Clock.	Reduction to middle thread.	Clock time on middle thread.
Cl. E.	I	$12^h\ 44^m\ 45^s.$	$+\ 23^m\ 39^s.2$	$13^h\ 8^m\ 24^s.2$
	II	12 52 41	+ 15 44.8	25.8
	III	13 0 39	+ 7 47.5	26.5
	IV	13 8 24.5		24.5
			Mean $T' =$	13 8 25.25
Cl. W.	III	13 16 21.	− 7 47.5	13 8 33.5
	II	24 20.	− 15 44.8	35.2
	I	32 13.	− 23 39.2	33.8
			Mean $T =$	13 8 33.17

The adopted intervals for these threads were $i_1 = +\,35^s.67$, $i_2 = +\,23^s.77$, $i_3 = +\,11^s.77$, with which the reductions to the middle thread were computed as in the table. As a test of the accuracy of the observation, each thread is here reduced separately. We have then, taking only the seconds of T and T', and putting $p = 0$, by (108),

$$c = \frac{25^s.25 - 33^s.17}{2} \cos 91^\circ\ 26'\ 34'' = +\,0^s.100\ (\text{Cl. W.})$$

On the same day the distance of the middle thread west of its image in the mercury collimator was found with clamp east to be -19.9 div. $= -1^s.230$, and by the spirit level there was found $b = +0^s.521$, whence $c = -0^s.615 + 0^s.521 = -0^s.094$ (Cl. E.), agreeing almost exactly with the value found by *Polaris*.

THE AZIMUTH CONSTANT.

151. To find the azimuth constant, we must have recourse to the observations of stars, since it is only by a reference to the heavens that the direction of the meridian can be determined. We can either find a directly, or first find n and m, from which a can be deduced.

To find a *directly.*—Observe the transits of two stars of different declinations δ and δ'. Let T and T' be the clock times of transit reduced to the middle thread (or the mean thread), b the level constant, c the collimation constant for the middle thread (or the mean thread), and put $c' = c - 0^s.021 \cos\varphi$ (Art. 126). Let ΔT_0 be the clock correction at any assumed time T_0, δT the hourly rate; then the clock corrections at the times of observation are

$$\begin{aligned} \Delta T &= \Delta T_0 + \delta T(T - T_0) \\ \Delta T' &= \Delta T_0 + \delta T(T' - T_0) \end{aligned}$$

Then, if α and α' are the apparent right ascensions of the stars at the time of the observation, as found from the Ephemeris, we have, by (82) and (87),

$$\begin{aligned} \alpha &= T + \Delta T + a \sin(\varphi - \delta) \sec\delta + b \cos(\varphi - \delta) \sec\delta + c' \sec\delta \\ \alpha' &= T' + \Delta T' + a \sin(\varphi - \delta') \sec\delta' + b \cos(\varphi - \delta') \sec\delta' + c' \sec\delta' \end{aligned}$$

If in these we substitute the above values of ΔT and $\Delta T'$, and suppose the *rate* of the clock to be given, every thing in the equations will be known except ΔT_0 and a. To abbreviate, put

$$\left.\begin{aligned} t &= T + \delta T(T - T_0) + b \cos(\varphi - \delta) \sec\delta + c' \sec\delta \\ t' &= T' + \delta T(T' - T_0) + b \cos(\varphi - \delta') \sec\delta' + c' \sec\delta' \end{aligned}\right\} \quad (110)$$

that is, let t and t' denote the observed clock times reduced to the assumed epoch T_0 and corrected for level and collimation; then we have

$$\begin{aligned} \alpha &= t + \Delta T_0 + a \sin(\varphi - \delta) \sec\delta \\ \alpha' &= t' + \Delta T_0 + a \sin(\varphi - \delta') \sec\delta' \end{aligned}$$

which give

$$\alpha' - \alpha = t' - t + a\left[\frac{\sin(\varphi - \delta')}{\cos\delta'} - \frac{\sin(\varphi - \delta)}{\cos\delta}\right]$$

$$= t' - t + a\,\frac{\cos\varphi\sin(\delta - \delta')}{\cos\delta\cos\delta'}$$

whence

$$a = [(\alpha' - \alpha) - (t' - t)]\,\frac{\cos\delta\cos\delta'}{\cos\varphi\sin(\delta - \delta')}$$

or

$$a = \frac{(\alpha' - \alpha) - (t' - t)}{\cos\varphi\,(\tan\delta - \tan\delta')} \qquad (111)$$

From these formulæ we learn the conditions necessary for the accurate determination of a. In the first place, if the rate of the clock is not well determined, the interval between the observations must be as brief as possible, so that t and t' will be but little affected by the error in δT. The right ascensions of the two stars must therefore differ as little as possible; or, if one of them is observed at its lower culmination, they must differ by nearly 12^h. In the next place, it is evident that the larger the factor $\tan\delta - \tan\delta'$ in the denominator of (111), the less effect will errors in t' and t have upon the deduced value of a. Therefore, if both stars are observed at the upper culminations, one must be as near to the pole and the other as far from it as possible. Finally, the right ascensions α and α' must be accurately known, and, therefore, only fundamental stars should be used, or those whose places are annually given in the Ephemeris.

If one of the stars is observed at its lower culmination, we have only to use $180° - \delta'$ and $12^h + \alpha'$ for its declination and right ascension, and still use the equations (110) and (111) without change of notation (Art. 128). In this case the factor $\tan\delta - \tan\delta'$ will become $\tan\delta + \tan\delta'$ (taking δ' here to signify the proper declination); and this will be the greater, the nearer *both* stars are to the pole. All the most favorable conditions can therefore be best fulfilled by two circumpolar stars, both as near to the pole as possible and differing in right ascension by nearly 12^h.

If we can rely upon the stability of the instrument and the clock rate for 12^h, we may observe the same star at both its upper and lower culminations, and then, putting $180° - \delta' = \delta$, the formula becomes

$$a = \frac{\alpha' - \alpha - (t' - t)}{2 \cos \varphi \tan \delta} \qquad (112)$$

here α' is the apparent right ascension of the star at the lower ulmination increased by 12^h, and t' is the corrected time for the ower culmination.

If the object of the observer is to re-determine the right ascensions of the fundamental stars themselves, it is plain that he must have an instrument of the greatest stability, and for the determination of the azimuth must rely upon upper and lower culminations of the same star; for the difference $\alpha' - \alpha$ in (112) may be accurately computed by the formulæ for precession and nutation, although the absolute values of α and α' may be but approximately known.

To find n *directly.*—Having observed two stars under the conditions above given, let t and t' be the clock times reduced for rate to the assumed epoch T_0 as before, but further corrected only for collimation; that is, put

$$\left.\begin{aligned} t &= T + \delta T\,(T - T_0) + c' \sec \delta \\ t' &= T' + \delta T\,(T' - T_0) + c' \sec \delta' \end{aligned}\right\} \qquad (113)$$

then, by Bessel's formula, Art. 126,

$$\begin{aligned} \alpha &= t + \Delta T_0 + m + n \tan \delta \\ \alpha' &= t' + \Delta T_0 + m + n \tan \delta' \end{aligned}$$

whence

$$n = \frac{(t' - t) - (\alpha' - \alpha)}{\tan \delta - \tan \delta'} \qquad (114)$$

For a single circumpolar star observed at its upper and lower culminations,

$$n = \frac{(t' - t) - (\alpha' - \alpha)}{2 \tan \delta} \qquad (115)$$

We then find m by (85); namely,

$$m = b \sec \varphi - n \tan \varphi \qquad (116)$$

If we reduce our observations by Bessel's or Hansen's formula, it will be unnecessary to find a. If it is required, however, it may now be found by the equation

$$a = b \tan \varphi - n \sec \varphi \qquad (117)$$

EXAMPLE.—On May 25, 1854, with the meridian circle of the U. S. Naval Academy, the upper and lower transits of *Polaris* and the transit of *α Arietis* were observed, and the clock times reduced to the middle thread were as follows:

	T	
Polaris U. C.	$1^h\ 14^m\ 48^s.24$	(Clamp East.)
α Arietis	2 8 9.13	"
Polaris L. C.	13 14 40.12	"

With the spirit level and mercury collimator, there were found $b = +\ 0^s.004$, $c = -\ 0^s.203$. The hourly rate of the clock on sidereal time was $\delta T = -\ 0^s.224$. The longitude of the instrument was $5^h\ 5^m\ 55^s$ W. of Greenwich, and the latitude $\varphi = 38°\ 58'\ 52''.5$. Find the constants a, m, and n.

From the Nautical Almanac for this date the right ascensions and declinations of the stars, reduced to the time of the observations, are

	α	δ	Nat. tan δ
Polaris U. C.	$1^h\ 5^m\ 29^s.41$	$88°\ 31'\ 39''$	38.902
α Arietis	1 58 56.05	22 46 7	0.420
Polaris L. C.	13 5 29.75	91 28 21	$-$ 38.902

We find for the constant of diurnal aberration for the given latitude, $0^s.021 \cos\varphi = 0^s.016$, and hence $c' = -\ 0^s.203 - 0^s.016 = -\ 0^s.219$. Computing $c' \sec\delta$, $b \cos(\varphi - \delta) \sec\delta$ for each star, and reducing the times for rate to 0^h, the values of t, according to (110), are found as follows:

	T	Red. for rate to 0^h	Corr. for collim.	Corr. for level.	t
Polaris U. C.	$1^h\ 14^m\ 48^s.24$	$-\ 0^s.28$	$-\ 8^s.52$	$+\ 0^s.10$	$1^h\ 14^m\ 39^s.54$
α Arietis,	2 8 9.13	$-$ 0.48	$-$ 0.24	0.00	2 8 8.41
Polaris L. C.	13 14 40.12	$-$ 2.97	+ 8.52	$-$ 0.09	13 14 45.58

To exemplify the use of the formula (111), we will first take *Polaris* U. C., and *α Arietis* (accenting the quantities for the second star). We find

$$\alpha' - \alpha = 53^m\ 26^s.64 \qquad t' - t = 53^m\ 28^s.87$$

$$\tan\delta - \tan\delta' = 38.482$$

and hence, by (111),

$$a = \frac{-\ 2^s.23}{38.482 \cos\varphi} = -\ 0^s.075$$

To exemplify the use of (111) in the case of two stars, one above and the other below the pole, we will take *α Arietis* and *Polaris* L. C., for which we find

$$\alpha' - \alpha = 11^h\,6^m\,33^s.70 \qquad t' - t = 11^h\,6^m\,37^s.17$$
$$\tan\delta - \tan\delta' = 39.322$$

whence

$$a = \frac{-\,3^s.47}{39.322\cos\varphi} = -\,0^s.114$$

To exemplify the use of (112), we will take *Polaris* U. C. and L. C., for which we have

$$\alpha' - \alpha = 12^h\,0^m\,0^s.34 \qquad t' - t = 12^h\,0^m\,6^s.04$$
$$2\tan\delta = 77.80$$

whence

$$a = \frac{-\,5^s.70}{77.80\cos\varphi} = -\,0^s.094$$

We adopt this last determination of a, and then, by (80), we find

$$m = -\,0^s.056 \qquad n = +\,0^s.076$$

But, where m and n are required, it is preferable to find n directly from the observations, and for this purpose we do not correct the times for level. Thus, correcting the times according to (113), we find t as follows:

	T	Red. for rate to 0^h.	Corr. for coll.	t
Polaris U. C.	$1^h\,14^m\,48^s.24$	$-\,0^s.28$	$-\,8^s.52$	$1^h\,14^m\,39^s.44$
α Arietis,	2 8 9.13	− 0.48	− 0.24	2 8 8.41
Polaris L. C.	13 14 40.12	− 2.97	+ 8.52	13 14 45.67

Taking *Polaris* U. C. and *α Arietis*, we find, by (114),

$$n = \frac{+\,2^s.33}{38.482} = +\,0^s.061$$

Taking *α Arietis* and *Polaris* L. C., we find, by the same formula,

$$n = \frac{+\,3^s.56}{39.322} = +\,0^s.091$$

Finally, from *Polaris* U. C. and L. C., we find, by (115),

$$n = \frac{+5^s.89}{77.804} = +0^s.076$$

agreeing exactly with the value above found from the same observations. We now find m by (116), which gives as before $m = -0^s.056$. And then, if a is required, we find, by (117), $a = -0^s.094$.

THE CLOCK CORRECTION.

152. Having determined all the instrumental constants, the clock correction is found from the transit of any known star by the formula

$$\Delta T = \alpha - (T + \tau)$$

in which T is the clock time of the star's transit over the middle thread, or the mean thread, and τ is the reduction of this thread to the meridian, computed by either (81), (86), or (87).

The finally adopted value of ΔT will be the mean of all the values thus found from a number of stars; and this mean will be the value corresponding to the mean of all the times of observation. But the observations thus grouped together for a determination of ΔT should not extend over so great a period of time that the clock rate cannot be regarded as constant during that period.

The clock rate is found by comparing the corrections ΔT, $\Delta T'$, corresponding to two times T, T', or

$$\delta T = \frac{\Delta T' - \Delta T}{T' - T}$$

The value ΔT_0 of the clock correction for an assumed epoch T_0 will be found by taking

$$\Delta T_0 = \Delta T + \delta T(T_0 - T)$$

It is evident, from HANSEN's formula (86), that an error in the determination of n (or of a, which involves n) will have the less effect upon τ and ΔT the less the difference between the observer's latitude and the star's declination. Hence, assuming that b and c can be found with greater precision than n, it is expedient to use for *clock stars* only fundamental stars which pass near to the zenith. If two circumzenith stars are observed, such that the mean of the tangents of their declinations is equal to the tangent of the latitude, the mean value of ΔT will be wholly free from any error in n.

An error in c will be eliminated, either wholly or in part, by taking the mean of the two values of ΔT found in the two positions of the rotation axis, since the sign of c, and, consequently, also that of any error in c, is changed by reversing the axis. An error in the assumed value of the correction p, for inequality of pivots, will also be removed in this manner; but, since the coefficient of b does not change its sign for different stars, nor when the instrument is reversed, there is no method of eliminating an unknown error of b. It is necessary, therefore, that the astronomer give particular attention to the precise determination of this constant.

(For the determination of the clock correction by a transit of the sun, see Art. 155).

DETERMINATION OF THE RIGHT ASCENSIONS OF STARS.

153. The principal application of the transit instrument in the observatory is the determination of the apparent right ascensions of the celestial bodies. The instrumental constants and the clock correction and rate being found from known stars as above explained, the right ascension of any other star is immediately deduced from the time of its transit by (82), in which we may substitute (86) or (87). The form in which the observations are reduced will be best learned by referring to any of the printed observations of the principal observatories.

In making a catalogue of stars, the instrument is clamped at a certain declination, and all the stars within a zone of the breadth of the field of the telescope are observed as they cross the threads. In this case, it will be expedient to find the clock correction from fundamental stars nearly in the parallel of declination upon which the instrument is set. For if we have found ΔT from a star whose right ascension is α, by the formula

$$\Delta T = \alpha - (T + \tau)$$

the right ascension of another star will be

$$\begin{aligned} \alpha' &= T' + \Delta T + \delta T(T' - T) + \tau' \\ &= \alpha \quad + (T' - T)(1 + \delta T) + (\tau' - \tau) \end{aligned}$$

that is, it will be equal to the right ascension of the fundamental star increased by the clock interval corrected for rate, and for the difference $\tau' - \tau$ of the instrumental corrections; and if the declinations are the same, we shall have $\tau' - \tau = 0$, and all the errors of the instrument will be eliminated. Since, in this application, the absolute clock correction is not required, we may substitute in (82) m' for $\Delta T + m$, and m' will be found directly from the fundamental stars by the formula

$$m' = \alpha - (T + n \tan \delta + c' \sec \delta)$$

The right ascensions will then be obtained by adding to the observed times the correction $m' + n \tan \delta + c' \sec \delta$, and it will not be necessary to separate m' into its constituents ΔT and m. Since m' involves the rate of the clock, its hourly variation will be taken into account in precisely the same manner as that of ΔT. This mode of reduction was adopted by BESSEL for his Königsberg Zone observations.

The mean right ascensions for the beginning of the year or for any assumed epoch, are found, from the apparent right ascensions, by the formula (692) of Vol. I.

For the determination of the absolute right ascensions of the fundamental stars, see Chapter XII. Vol. I.

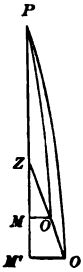
TRANSITS OF THE MOON, THE SUN, AND THE PLANETS.

154. *Transits of the moon.*—The hour angle of the moon's limb, when on a side thread, is affected by parallax; and the time required by the moon to pass from this thread to the meridian differs from that required by a star in consequence of the moon's proper motion in right ascension. If δ is the true declination of the moon, δ' the apparent declination as affected by parallax, ϑ' the apparent east hour angle of the moon's limb at the instant of the observed transit over a thread whose equatorial interval from the middle thread is i, then, since δ' is the declination of the observed point on the thread, we have

$$\vartheta' = m + n \tan \delta' + (i + c') \sec \delta'$$

Thus ϑ' is known, but to reduce the observation we must find the true hour angle ϑ. Let PM, Fig. 43, be the meridian, P the pole, Z the geocentric zenith of the place of observation, O the true place of the moon, O' its apparent place; and denote the true and apparent

Fig. 43.



zenith distances ZO and ZO' by z and z'. We have $MPO = \vartheta$, $MPO' = \vartheta'$, and drawing OM, $O'M'$ perpendicular to the meridian, we find

$$\sin MZO = \frac{\sin MO}{\sin ZO} = \frac{\sin MO'}{\sin ZO'}$$

or

$$\frac{\sin \vartheta \cos \delta}{\sin z} = \frac{\sin \vartheta' \cos \delta'}{\sin z'}$$

whence

$$\vartheta = \vartheta' \cdot \frac{\sin z \cos \delta'}{\sin z' \cos \delta}$$

Now, if

λ = the moon's increase of right ascension in one second of sidereal time,

the sidereal time required by the moon to describe the hour angle ϑ is $\frac{\vartheta}{1 - \lambda}$; and, therefore, T being the clock time of transit of the limb over the thread, the right ascension of the limb at the instant of its transit over the meridian will be

$$\alpha = T + \Delta T + \frac{\vartheta}{1 - \lambda}$$

and if we put

S = the moon's geocentric apparent semidiameter,

the hour angle of the moon's centre when the limb is on the meridian will be $\pm \frac{S}{15 \cos \delta}$, and the time required by the moon to describe this hour angle will be $\pm \frac{S}{15 (1 - \lambda) \cos \delta}$. Hence the formula for computing the right ascension of the centre at the instant of the transit of the centre over the meridian is

$$\alpha = T + \Delta T + \frac{\vartheta}{1 - \lambda} \pm \frac{S}{15 (1 - \lambda) \cos \delta}$$

in which the upper or the lower sign will be used according as the first or the second limb is observed. If then we substitute the values of ϑ and ϑ', and put

$$F = \frac{\sin z}{\sin z'} \cdot \frac{1}{(1 - \lambda) \cos \delta} \tag{118}$$

we have

$$\alpha = T + \Delta T + iF + (m + n \tan \delta' + c' \sec \delta') F \cos \delta' \pm \frac{S}{15(1-\lambda)\cos \delta} \quad (119)$$

To compute the factor F conveniently, put

$$A = \frac{\sin z}{\sin z'} \qquad B = \frac{1}{1 - \lambda}$$

then

$$F = AB \sec \delta$$

The value of A may be developed in a simple form. If we put φ' = the reduced or geocentric latitude of the place of observation, ρ = its geocentric distance, π = the moon's equatorial horizontal parallax, we have $z = \varphi' - \delta$, and

$$\sin (z' - z) = \rho \sin \pi \sin z'$$

whence

$$A = \frac{\sin z}{\sin z'} = \cos (z' - z) - \rho \sin \pi \cos z'$$

or, neglecting the square of the parallax,

$$A = 1 - \rho \sin \pi \cos (\varphi' - \delta)$$

which is the form employed by BESSEL, who gives the value of $\log A$, in Table XIII. of the *Tabulæ Regiomontanæ*, with the argument $\log [\rho \sin \pi \cos (\varphi' - \delta)]$. For a particular observatory, where these reductions are frequent, it is more convenient to prepare a special table, adapted to the latitude, giving $\log A$ with the arguments δ and π. In BESSEL'S table, there are also given the values of $\log B$ with the argument "change of the moon's right ascension in 12^h of mean time," and the argument is expressed in degrees and minutes of arc; but as the change in one minute, expressed in seconds of time, which I shall denote by $\Delta\alpha$, is given in the American Ephemeris, I shall take

$$\lambda = \frac{\Delta\alpha}{60.1643} \qquad B = \frac{60.1643}{60.1643 - \Delta\alpha}$$

where 60.1643 is the number of sidereal seconds in one minute of mean time. The following table gives the values of $\log B$ computed by this formula:

Argument $\Delta\alpha$ = change of the moon's right ascension in one minute of mean time.

$\Delta\alpha$	$\log B$	$\Delta\alpha$	$\log B$	$\Delta\alpha$	$\log B$
$1^{s}.65$	0.01208	$2^{s}.05$	0.01506	$2^{s}.45$	0.01806
1 .70	0.01245	2 .10	0.01543	2 .50	0.01843
1 .75	0.01282	2 .15	0.01580	2 .55	0.01881
1 .80	0.01319	2 .20	0.01618	2 .60	0.01919
1 .85	0.01356	2 .25	0.01655	2 .65	0.01956
1 .90	0.01394	2 .30	0.01693	2 .70	0.01994
1 .95	0.01431	2 .35	0.01730	2 .75	0.02032
2 .00	0.01468	2 .40	0.01768	2 .80	0.02070

This table will be useful also in computing the term

$$\frac{S}{15\,(1-\lambda)\cos\delta} = \tfrac{1}{15}\,SB\sec\delta$$

The reduction of an observed transit of the moon is then as follows. The transit over each thread is reduced to the middle thread (or mean thread) by adding the correction iF to the observed times, and the mean of the several results is taken as the clock time of transit of the limb over the middle (or mean) thread; or this time may be found by multiplying the mean of the equatorial intervals of the observed threads by F and adding the product to the mean of the observed times. This time is then reduced to the meridian by adding the correction $(m + n\tan\delta' + c'\sec\delta')F\cos\delta'$ or $(m\cos\delta' + n\sin\delta' + c')F$, in which we may take $\delta' = \delta - \pi\sin(\varphi' - \delta)$. Then, adding the clock correction, we have the right ascension of the limb at the instant of its transit over the meridian. Finally, adding or subtracting the term $\dfrac{S}{15\,(1-\lambda)\cos\delta}$, we have the right ascension of the moon's centre at the instant of its transit over the meridian.

When the moon has been observed on all the threads, the computation of F by the above method may be dispensed with, as an approximate value, sufficient for computing the reduction to the meridian, may be inferred from the observed times on the first and last thread. For, calling the observed interval between these threads I, and the equatorial interval i, we have $I = iF$, whence

$$F = \frac{I}{i}$$

If we omit the factor $1 - \lambda$ throughout, the right ascension obtained is that which corresponds to the instant of the observation instead of the instant of meridian passage.

EXAMPLE.—The transit of the moon's first limb was observed at the U. S. Naval Academy on May 29, 1855, as follows:

	Thread.	Clock.
	I	$15^h\ 3^m\ 57^s.5$
	II	4 10.3
(Clamp east.)	III	4 23.2
	IV	4 36.2
	V	4 49.0
	VI	5 1.8
	VII	5 14.6

For the Naval Academy we have $\varphi' = 38° 47' 38''$, and $\log \rho = 9.99943$; and the longitude from Greenwich is $5^h\ 5^m\ 57^s$.

The constants of the transit instrument were $m = +\ 0^s.251$, $n = -\ 0^s.162$, $c = +\ 0^s.093$; and hence (Art. 126) $c' = +\ 0^s.093 - 0^s.016 = +\ 0^s.077$. The clock correction to sidereal time was $+\ 1^m\ 25^s.11$. The equatorial intervals of the threads from the middle thread were

i_1	i_2	i_3	i_5	i_6	i_7
$+\ 35^s.65$	$+\ 23^s.72$	$+\ 11^s.78$	$-\ 11^s.77$	$-\ 23^s.77$	$-\ 35^s.67$

From the American Ephemeris we find for the culmination at the Naval Academy on May 29, 1855,

$$\pi = 57'\ 46''.1 \qquad S = 15'\ 46''.5$$
$$\delta = -17°\ 58'\ 53'' \qquad \Delta\alpha = 2^s.2147$$

To illustrate the method of reducing the observations to the middle thread, we will first find the factor F by direct computation. We have $\varphi' - \delta = 56°\ 46'\ 31''$, $\log \rho \sin \pi \cos (\varphi' - \delta) = 7.96355$; and hence

$$\begin{aligned} \log A &= 9.99599 \\ \log B &= 0.01629 \\ \log \sec \delta &= \underline{0.02175} \\ \log F &= 0.03403 \end{aligned}$$

Multiplying the equatorial intervals by F, we find the reductions of the several threads to the middle thread to be

I	II	III	V	VI	VII
$+\ 38^s.56$	$+\ 25^s.65$	$+\ 12^s.74$	$-\ 12^s.73$	$-\ 25^s.71$	$-\ 38^s.58$

The clock times of transit over the middle thread, according to the observations on the several threads, were, therefore,

I	15^h	4^m	$36^s.06$
II			35.95
III			35.94
IV			36.20
V			36.27
VI			36.09
VII			36.02
Mean $T =$	15	4	36.08

To compute the instrumental correction, we have $\pi \sin(\varphi' - \delta) = 48'.3$, whence $\delta' = -18° \ 47'.2$, $m + n \tan \delta' + c' \sec \delta' = +0^s.387$, and therefore

$$(m + n \tan \delta' + c' \sec \delta') F \cos \delta' = +0^s.40$$

Applying this term to the above mean, we have

$$\begin{array}{lrrr}
\text{Clock time of transit of the limb} = & 15^h & 4^m & 36^s.48 \\
\text{Clock correction,} \qquad \Delta T = & +\ & 1 & 25.11 \\
\hline
\text{R. A. of the limb at transit} = & 15 & 6 & 1.59 \\
\dfrac{S}{15(1 - \lambda) \cos \delta} = & & 1 & 8.88 \\
\hline
\text{R. A. of moon's centre at transit, } \alpha = & 15 & 7 & 10.47
\end{array}$$

The factor F might have been approximately deduced from the first and last observations, which give the interval $I = 77^s.1$, and the equatorial interval between the extreme threads is $i = 35^s.65 + 35^s.67 = 71^s.32$, whence

$$\log F = \log \frac{77.1}{71.32} = 0.0338$$

which is sufficiently accurate for reducing the instrumental correction.

The "sidereal time of the semidiameter passing the meridian," or $\dfrac{S}{15(1 - \lambda) \cos \delta}$, may be taken from the table of Moon Culminations given in the Ephemeris.

The clock correction employed in deducing the moon's right ascension should be deduced from stars as nearly as possible in the same parallel of declination. (See Art. 153.) The "moon culminating stars" are stars lying nearly in the moon's path whose positions have been carefully determined for this purpose. (See Vol. I. Art. 229.)

155. *Transits of the sun or a planet.*—The formula (119) is applicable in general to any celestial body; but, in the case of the sun and planets, the parallax is so small that its effect upon the time of transit over a side thread is inappreciable: so that we may take simply

$$F = \frac{1}{(1-\lambda)\cos\delta} = B\sec\delta$$

and, consequently, also put δ for δ'. The formula for computing the right ascension of the centre of the sun or a planet over any given thread is, therefore,

$$\alpha = T + \Delta T + iB\sec\delta + (m + n\tan\delta + c'\sec\delta)B \pm \tfrac{1}{15}SB\sec\delta \qquad (120)$$

in which (λ denoting the change of right ascension in one sidereal second) we have

$$B = \frac{1}{1-\lambda}$$

The logarithm of B may be readily computed. Putting $\Delta\alpha$ for the change of right ascension in one hour of mean time (which change is given in the Ephemeris for the sun), we have, since one mean hour is equal to 3610 sidereal seconds,

$$\lambda = \frac{\Delta\alpha}{3610}$$

$$^{*}\log B = -\log\left(1 - \frac{\Delta\alpha}{3610}\right)$$

$$= \Delta\alpha \cdot \frac{M}{3610}$$

in which $M = 0.43429$, the modulus of the common system of logarithms. Performing the division of M by 3610, we find

$$\log B = 0.00012 \times \Delta\alpha \qquad (121)$$

in which $\Delta\alpha$ must be expressed in seconds of time.

In the British Nautical Almanac, the change of right ascension $\Delta\alpha$ in one hour of longitude is given for each planet. In this case, we have

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* By the formula $\log(1 - x) = -M(x + \frac{1}{2}x^2 + \&c.)$, where the square and higher powers of x are so small as to be inappreciable in the present case.

$$B = 1 + \frac{\Delta\alpha}{3600}$$

the logarithm of which may also be found by (121) with sufficient accuracy, that is, within a unit of the fifth decimal place.

The term $\frac{1}{15} SB \sec\delta$, or "the sidereal time of the semidiameter passing the meridian," is given in the Ephemeris for the sun and each of the planets. When both limbs have been observed on all the threads, this term is not required, since the mean of all the observations is evidently the time of the passage of the centre over the mean of the threads. If this mean is to be reduced to the middle thread, there will remain the small correction $\Delta iB \sec\delta$ to be applied (Art. 133), for which we may take $\Delta i \sec\delta$. We may also put $m + n \tan\delta + c' \sec\delta$ instead of $(m + n \tan\delta + c' \sec\delta) B$, unless m, n, and c' are unusually great.

The reduction of transits of the sun observed with a sidereal clock is greatly facilitated by the use of Table XII. of BESSEL's *Tabulæ Regiomontanæ*, which contains every thing necessary for the purpose, for each day of the *fictitious year* (Vol. I. Art. 406).

156. *Transits of the sun observed with a mean time chronometer.*— A mean time chronometer is often used with the portable transit instrument, and transits of the sun are then observed solely for the purpose of determining the chronometer correction. In this case, the mean motion of the sun corresponds with that of the chronometer, and therefore the factor B may be put equal to unity, unless we wish to obtain extreme precision by taking into account the small difference between the mean motion of the sun and its actual motion at different seasons of the year, a degree of precision quite superfluous in the use of a portable instrument. If we put

E = the equation of time for the instant of transit, positive when additive to apparent time,
$S' = \frac{1}{15} S \sec\delta$ = the mean time of the sun's semidiameter passing the meridian, which may be taken from the Ephemeris,
τ = the reduction to the meridian, found either by (82), (86), or (87),
T = the observed chronometer time of the transit of the sun's limb over a thread whose equatorial interval is i,
ΔT = the chronometer correction to mean time,
t = the chronometer time of the transit of the sun's centre,

then we have

$$t = T + i \sec \delta \pm S' + \tau \qquad (122)$$

and

$$12^h + E = t + \Delta T$$

or

$$\Delta T = 12^h + E - t \qquad (123)$$

Example.—On May 17, 1856, the transit of the sun was observed at the Naval Academy with a portable instrument as below (Clamp West):

Thread.	Mean time chronometer.	
	1st Limb.	2d Limb.
I	$11^h\ 55^m\ 42^s.2$	$11^h\ 57^m\ 56^s.6$
II	55 57 .4	lost
III	56 12 .0	58 26 .7
IV	lost	58 41 .7
V	56 42 .3	lost

There had been found $a = + 0^s.35$, $b = - 0^s.27$, $c = - 0^s.12$. The thread intervals from middle thread were

i_1	i_2	i_4	i_5
$+ 28^s.25$	$+ 14^s.15$	$- 14^s.27$	$- 28^s.31$

The longitude being $5^h\ 5^m\ 57^s$ west of Greenwich, we find from the Ephemeris for the transit over this meridian,

$$\delta = + 19° 29'.1 \qquad S' = 67^s.24 \qquad E = - 3^m\ 49^s.71$$

The reductions of the several threads to the middle thread, or the values of $i \sec \delta$, are, therefore,

I	II	IV	V
$+ 29^s.97$	$+ 15^s.01$	$- 15^s.14$	$- 30^s.03$

Applying these to the observed times, and also the quantity $\pm S'$, we have the chronometer time of the transit of the sun's centre over the middle thread, as deduced from the several threads, as follows:

	Thread.	Chronometer.
1st Limb,	I	$11^h\ 57^m\ 19^s.41$
	II	19.65
	III	19.24
	V	19.51
2d Limb,	I	19.33
	III	19.46
	IV	19.32
	Mean =	11 57 19.42

The latitude being $\varphi = 38° 58'.9$, we find, by (87), $\tau = -0^s.27$, and hence, finally,

$$\begin{array}{rl} t = & 11^h\ 57^m\ 19^s.15 \\ 12^h + E = & 11\ \ 56\ \ 10.29 \\ \hline \Delta T = & -\ \ \ 1\ \ \ 8.86 \end{array}$$

157. *Correction of the transit of the moon or a planet when the defective limb has been observed.*—Let us consider the general case of a spheroidal planet partially illuminated. The transit of the observed limb is reduced to that of the centre by employing instead of S in (119) the perpendicular distance from the centre of the planet to that tangent to the limb which lies in the direction of the transit threads, or, in the case of meridian transits, the perpendicular upon the declination circle which is tangent to the limb. The formulæ for computing this perpendicular, in general, are discussed in Vol. I., *Occultations of Planets*, where we have found that in all practical cases the formulæ (628) of p. 580 may be considered as rigorous. In those formulæ the angle ϑ is the angle which the required perpendicular makes with the axis of the planet, so that, p being the angle which this axis makes with a declination circle, we have here

$$\vartheta = 270° - p \qquad \text{or} \qquad \vartheta = 90° - p$$

according as the first or second limb is observed. The values of p as well as of V and c required are found as in Vol. I. Arts. 351, 352.

But this rigorous process will seldom be required; and when we regard the planet as spherical, the formulæ can be simplified as follows. For a spherical planet we make $c = 1$, and substitute the value $90° - p$ for ϑ, which applies to the 2d limb, whence, by Vol. I. formulæ (628) and (623),

$$\sin \chi = \cos p \sin V$$

or

$$\left.\begin{aligned} \sin\chi &= \frac{R}{R'}\cos D\sin(\alpha' - A) \\ s'' &= s\cos\chi = \frac{s_0}{r}\cos\chi \end{aligned}\right\} \quad (124)$$

where α' and A are the right ascensions of the planet and the sun respectively (and $\alpha' - A$ is therefore in the present case the sun's hour angle at the time of the observation); $D =$ the sun's declination; R, $R' =$ the heliocentric distances of the earth and the planet respectively; $s =$ the apparent semidiameter of the planet at the time of the observation; $s_0 =$ the mean semidiameter (Vol. I. p. 578); $r =$ the geocentric distance of the planet; and $s'' =$ the required perpendicular. For the moon we may put $R = R'$.

The above value of $\sin\chi$ is deduced for the second limb, and, therefore, by Vol. I. Art. 354, it will be positive when the second limb is defective. Since we should have to substitute $270° - p$ for ϑ, or $-\cos p$ for $\sin\vartheta$, in the case of the first limb, which would only change the sign, it follows that *the value of* $\sin\chi$ *computed by the above formula will be positive or negative according as the 2d or the 1st limb is defective.*

The value of s'' is to be substituted for S in (119).

EFFECT OF REFRACTION IN TRANSIT OBSERVATIONS.

158. Since the refraction changes the zenith distance, its effect upon the time of transit over a side thread has the same form as that of the parallax. If then z and z' denote respectively the true and apparent zenith distances, the time required by the star to describe the interval i is iF, where F is found by (118); or, denoting this time by I', and putting $\lambda = 0$,

$$I' = \frac{i}{\cos\delta}\cdot\frac{\sin z}{\sin z'}$$

Now, the refraction is represented by the formula $r = k\tan z'$, k being nearly constant; and for values of z not greater than 85°, we may here assume $k = 58''$, and $z = z' + k\tan z'$, whence we find

$$\frac{\sin z}{\sin z'} = 1 + k\sin 1'' = 1.00028$$

Hence the error in computing the interval by the formula $I = i \sec \delta$ is $I \times .00028$, which amounts to $0^s.01$ when $I = 36^s$; and this is as great an interval as is ever used for an equatorial star. The error of observation for other stars increases with the interval I, or as the value of $\sec \delta$: so that the error produced by neglecting the refraction is always much less than the probable error of observation. Moreover, the error is wholly eliminated when the star is observed on all the threads, or on an equal number on each side of the middle thread.

If, for any special purpose, it becomes necessary to correct an observation on an extreme thread for refraction, we can take, as a very accurate formula,

$$I' = i \sec \delta \, (1 + k \sin 1'')$$

k being found by **Bessel's** Refraction Table (Table II.), and, for a near approximation,

$$I' = i \sec \delta \times 1.00028$$

MERIDIAN MARK.

159. For a fixed instrument, it is desirable to have a permanent meridian mark by which the azimuth of the telescope may be frequently verified. A triangular aperture (for example) in a metallic plate mounted upon a firm pier, with a sky background, makes a good day mark, the thread of the telescope being brought into coincidence with it by bisecting the vertical angle of the triangle. If the mark is sufficiently near, a light may be placed behind it for night observations. A simple mark like this, however, must be so remote as to be distinctly defined in the telescope without a change of the stellar focus, and even for instruments of moderate power this requires a distance of upwards of a mile.

It is found, however, that the apparent direction of these distant marks is often subject to changes from the anomalous lateral refractions which take place in the lower strata of the atmosphere, produced chiefly by variations of temperature. If a sheet of water intervenes, the mark is found to be especially unsteady. It was to remedy this difficulty that **Rittenhouse** first proposed the plan of placing the mark comparatively near to the instrument, but in the focus of a lens which receives the divergent rays from the mark and transmits them to the telescope in parallel lines; a suggestion which has resulted in various important improvements in the methods of investigating instrumental errors, such as the collimating telescopes, the mercury collimator, &c., which have already been fully treated of in the preceding pages. The apparent direction of the mark will be that of the line joining the optical centre of the lens and the mark. At Pulkowa, the lens for this purpose is placed on a pier within the transit room, and has the extraordinary focal length of about 556 feet,* which is, therefore, the distance of the mark from the pier. The mark consists of a circular aperture $\frac{1}{16}$ of an inch in diameter, in a metallic plate, presenting in the telescope a planetary disc of only $2''$ in diameter, which can be bisected by the thread of the telescope with the greatest precision. The merit of such a mark depends on the stability of the two points, the mark and the lens, which determine the direction of its optical line. These points, mounted as they are upon solid stone piers, are not liable to greater relative changes than the piers of the telescope itself, and therefore the changes of direction of their optical line will be less than those of the telescope in the proportion of the focal length of the lens to the length of the rotation axis of the telescope, which in this case was as 556 feet to 3.61 feet, or as 154 : 1. Now, according to STRUVE,† the diurnal changes in the direction of the axis of a well mounted transit instrument are seldom more than one or two seconds of arc; but $\frac{1}{154}$ of a second of arc is a quantity absolutely imperceptible even in the best transit telescopes. Two marks of the same kind were used by STRUVE, one north and the other south of the telescope, and they served not only as meridian marks, but as collimators according to the method of Art. 145.

In the same manner, one of the collimators of the Greenwich transit circle is used as a meridian mark, although it is within the transit room. In this case, the advantage gained is comparatively small.

It is not necessary that the mark be precisely in the meridian of the instrument. It is sufficient if it is so near to it that its deviation in azimuth can be measured with the telescope micrometer. Let A be its azimuth west of north. Direct the telescope to it, and measure its distance m from the middle thread, giving he measure the positive sign when the mark, as seen in the field, is to the apparent west of the thread; then, a being the azimuth constant of the telescope determined by stars, and c the collimation constant, we have

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* *Description de l'Observatoire de Poulkova*, p. 126. † Ibid. p. 128.

$$A = a - m - c \tag{125}$$

So long as the values of A thus found appear to vary only within the limits of the probable errors of observation, their mean is to be taken as expressing the constant position of the mark, and during this period the azimuths of the transit instrument will be found at any time by the formula

$$a = A + m + c$$

If the instrument is reversed and the micrometer distance of the mark west of the middle thread is now m', we have

$$a = A + m' - c$$

which, combined with the former equation, gives

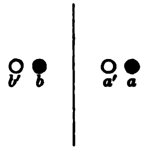
$$\left.\begin{aligned} a &= A + \tfrac{1}{2}(m + m') \\ c &= \tfrac{1}{2}(m' - m) \end{aligned}\right\} \tag{126}$$

which last equation gives c with its proper sign for the first position of the instrument.

PERSONAL EQUATION.

160. It is often found that two observers, both of acknowledged skill, will differ in the time of transit of a star observed by "eye and ear," by a quantity which is nearly the same for all stars. Such a *constant* difference does not necessarily prove a want of skill in subdividing the second according to the method of Art. 121, but may proceed from a discordance between the eye and the ear, which affects the judgment as to the point of the field to which the clock beats are to be referred. Thus, if a and b, Fig. 44, are the true positions of a star at two successive beats of the clock, we may suppose the observer to allow a certain interval of time to elapse after each beat before he associates it with the star's position (possibly in some cases he may anticipate the beat): so that he refers the beats to two different points a' and b', whose distance from each other is, however, the same

Fig. 44.



as that of *a* and *b*. The ratio in which the distance *a′b′* is divided he may still estimate correctly.

The distance between *a* and *a′* may be called the *absolute personal equation* of the observer, and, if it could be determined might be applied as a correction to all his observations. But, s long as his observations are not combined with those of anothe observer, the existence of such an error cannot be discovered nor is it then of any consequence. For the process of dete mining the right ascension of an unknown star consists esse tially in applying to the right ascension of a known star th *difference* of the clock times of the transit of the two stars (co rected for instrumental errors and rate), and this difference wi evidently be the same as if the observer had no personal equatio

In order to combine the observations of two individuals—fo example, to deduce the right ascension of an unknown st whose transit is observed by A, from the time of transit of known star observed by B—it is necessary to know the differenc of their absolute equations,—*i.e.* their *relative personal equatio* Thus, if the times observed by A are later than those observe by B by the quantity E, then B's observations may be reduce to A's (that is, to what they would have been if observed by A by increasing them all by E.

The relative personal equation may be found by the following methods:

First Method.—Let one observer note a star's transit over the first three or four threads, and the other observer its transit over the remaining threads. Reduce the observations of each to the middle thread (or to any assumed thread) by applying the known equatorial intervals multiplied by sec δ. The difference between the mean results for the two observers will be a value of their required personal equation. The mean of the values found from twenty or thirty (or more) such observations will be adopted, provided the probable error of such a determination (as found from the discrepancies of the individual results) is not greater than the equation itself; in which case the difference between them should, of course, be regarded as accidental, and the use of a constant equation would introduce error instead of eliminating it. This remark may be necessary to guard inexperienced observers against an incautious adoption of an equation derived from insufficient data. We may also remark here that constant personal equations are more apt to exist between trained observers than between inexperienced ones, the former having by practice acquired a fixed *habit* of observation.

Second Method.—The preceding method is liable to the objection that as the second observer takes the place of the first in a somewhat hurried manner, his usual habit of observation may be disturbed. To obviate this, let each observer independently determine the clock correction by fundamental stars; then the difference of these corrections, both reduced for clock rate to the same epoch, will be the personal equation. The equation thus found involves the errors of the stars' places and of the clock rate. The first will be inconsiderable if only fundamental stars are used, but may be entirely eliminated by the observers' exchanging stars on a following day and taking the mean of the two results. The effect of error in the rate will be insensible if the stars are so distributed that the means of the right ascensions of the stars of the two groups employed by the two observers are nearly equal.

Third Method.—An equatorial telescope is sometimes used for the purpose, as follows. Two transit threads of the micrometer are adjusted in the direction of a declination circle, and the telescope is directed towards a point in advance of any star not far from the meridian, and clamped. The observer A notes the transit of the star over the first thread, and the observer B the transit over the second thread. The telescope is then moved forward again in advance of the star, and clamped. The observer B now notes the transit over the first thread, and A the transit over the second thread. This gives one determination of their personal equation; for, putting $E =$ the reduction of B's observation to A's, and $I =$ the interval of the threads for the observed star, M and M' the observed intervals, we have

$$M = I + E \qquad\qquad M' = I - E$$

whence

$$E = \frac{M - M'}{2}$$

This process being repeated a number of times, M will be the mean of all the intervals when A begins, and M' the mean of those when B begins.

This method is also open to the objection that the observers succeed each other so rapidly that their usual habit of deliberate observation is likely to be disturbed. Moreover, if their personal equation is required to reduce their observations made with a transit instrument, it should be determined with this instrument; for it is possible that the equation may not be the same with instruments of different powers.

The same clock, also, should be used in determining the personal equation that is used in the observations, for it is very probable that the peculiarity of the clock-beat affects the equation.*

It is one of the advantages of the American (the electro-chronographic) method of recording transits that the personal equation is very much reduced: still it is not wholly destroyed. The same methods may be employed to determine its amount as when the observations are made by eye and ear.

It may also be remarked that not only should the same telescope and the same clock be employed in determining the personal equation, as in the observations to which it is to be applied, but also the observer's general *physical* condition should be as nearly as possible the same. Even the posture of the body has been found to have some effect upon the observer's estimate of the time of transit; and it can hardly be doubted that the personal equation will fluctuate more or less with the observer's health, or the condition of his nervous system.

That the personal equation depends upon no organic defect of either the eye or the ear, but upon an acquired habit of observation, seems to follow from the fact that it is usually greatest in the case of the most practised observers. In 1814 there was no personal equation between those eminently skilful astronomers BESSEL and STRUVE; but in 1821 they differed by $0^s.8$, and in 1823 by a whole second; a progressive increase indicating the gradual formation of certain fixed habits of observation. So far from invalidating the results of either observer, this fact would indicate that their absolute personal equations were in all probability very constant for moderate intervals of time, and therefore had no appreciable effect upon their results so long as these results did not depend upon a combination of their observations with those of other observers.

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* BESSEL found that with a chronometer beating half seconds he observed transits $0^s.49$ later than with a clock beating whole seconds.

PERSONAL SCALE.

161. Prof. PEIRCE has called attention to the fact that experienced observers often acquire a fixed erroneous habit of estimating particular fractions of the second. Thus, when a star is really at 0ˢ.3 from a thread, one observer may have a habit of calling it 0ˢ.4, while another may incline rather to 0ˢ.2; or, again, when the fraction is less than 0.1, one invariably takes 0.1, while the other as invariably neglects it and puts 0.0. Thus each observer is conceived to have his own *personal scale* for the division of the second.

In a very large number of individual transits over threads by the same observer, there is, according to the doctrine of probabilities, the same chance for the occurrence of each of the decimals .0, .1, .2, &c., if the *observations are perfectly made, or if the errors of the observers are purely accidental;* otherwise, one or more of these decimals will occur more frequently than the rest. Hence, by simply counting the number of times each decimal occurs in a very large number of observations by the same observer, the personal scale of this observer may be determined.

It is easily shown that the effect of an erroneous personal scale is to increase or diminish the mean result of a large number of observations by a constant quantity. For example, suppose that in 1000 observations of a certain observer the fraction 0.3 appears but 20 times, while 0.4 appears 180 times, and that each of the other fractions appears 100 times. Then, since each fraction should appear 100 times, the mean of any large number of observations by this observer will probably be too great by the quantity

$$\frac{(0.4 \times 180 + 0.3 \times 20) - (0.4 \times 100 + 0.3 \times 100)}{1000} = 0.008$$

The effect, therefore, being constant, will be combined with the personal equation determined from a large number of observations, and may be regarded as always forming a part of it. Hence it follows that the application of the personal equation, which involves the errors of the personal scale, does not necessarily eliminate the observer's constant error from *each* observation, but that it probably does eliminate it from the mean of a large number of observations.

PROBABLE ERROR OF A TRANSIT OBSERVATION.

162. That part of the error in the observed time of transit of a star which is independent of the personal equation and other constant errors, and is irregular or accidental, may be distinguished as the *probable error;* and it will be the only error of observation which will affect the final result, if the observations of two observers are not combined. It may be determined for each observer by comparing the several values of the thread intervals given by his observations. Let

I = the observed interval of two threads whose equatorial interval is i;

then, since we should have $i = I \cos \delta$, each observation furnishes a value of i; and from a great number of values the probable error r of each single determination is deduced by the formula*

$$r = 0.6745 \sqrt{\frac{\Sigma(v^2)}{m-1}}$$

in which the values of v are the residuals found by subtracting the known value of i from each value found from observation, and m is the number of observations.

Now put

ε = the probable error of the observed time of transit of an *equatorial* star over a thread;

then, since the time of transit over each thread is affected by this error, we have

$$2\varepsilon^2 = r^2$$

whence

$$\varepsilon = 0.6745 \sqrt{\frac{\Sigma(v^2)}{2(m-1)}}$$

EXAMPLE.—From the transit observations made by Mr. ELLIS at the Greenwich Observatory in 1843, the observed intervals between the successive threads (*i.e.* from 1st to 2d, from 2d to 3d, &c.) were found as in the following table: the true equatorial intervals being those given in the fourth column. The difference between the computed and the true equatorial interval (v) is given in the fifth column, and the last column gives v^2.

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* Appendix, Art. 17

1843.	Observed I	Computed $i = I \sec \delta$	True i	v	v^2
March 8.	$13^s.8$	$12^s.79$	$12^s.89$	$-0^s.10$	0.0100
ν Tauri	13 .8	.79	.76	+ .03	9
$\delta = +22^\circ\ 27'$	14 .0	.93	.87	+ .06	36
	14 .0	.93	.91	+ .02	4
	13 .7	.66	.88	— .22	484
	13 .6	.57	.86	— .29	841
	13 .8	.85	.89	— .04	16
ι Tauri	13 .8	.85	.76	+ .09	81
$\delta = +21^\circ\ 21'$	13 .9	.94	.87	+ .07	49
	13 .9	.94	.91	+ .03	9
	13 :8	.85	.88	— .03	9
	13 .7	.76	.86	— .10	100
	13 .7	.65	.89	— .24	576
μ Geminor.	14 .0	.93	.76	+ .17	289
$\delta = +22^\circ\ 35'$	14 .0	.93	.87	+ .06	36
	14 .0	.93	.91	+ .02	4
	13 .9	.84	.88	— .04	16
	13 .8	.74	.86	— .12	144
	$m = 18,$			$\Sigma(v^2) = 0.2803$	

Hence we find, by the above formula,

$$\varepsilon = 0^s.06$$

Taking a much greater number of the observations made by Mr. ELLIS of stars from the 3d to the 5th magnitude, I found $\varepsilon = 0^s.056$, which is probably smaller than will be found for most observers. In the case of another well trained observer, I found $\varepsilon = 0^s.08$.

In the same manner, from a large number of Mr. ELLIS's observations of the moon I found his probable error in observing the transit of the first limb over a single thread to be $0^s.074$, and for the second limb $0^s.071$. In the case of another observer, I found for the first limb $0^s.078$, and for the second limb $0^s.094$.

If we assume, then, that for moderately skilful observer $\varepsilon = 0^s.08$ for a star, the probable error of the mean of the observations over seven threads will be $0^s.08 \div \sqrt{7}$, or only $0^s.03$, the star being in the equator. For the declination δ the probable error will be $0^s.03 \sec \delta$.

The probable error thus found is the accidental error, composed of the error of the observer in estimating the fractions of a second (including the errors of his personal scale), and of the error arising from unsteadiness of the star; but it must not be taken as the measure of the degree of precision in the deduced right ascension or time.*

163. The error of the right ascension derived from a single complete transit is composed of the following errors:

1st. The undetermined instrumental errors, depending upon the errors in the determination of the constants a, b, and c;
2d. The errors of the assumed clock correction and rate;
3d. The error arising from irregularity of the clock;
4th. The error in the observer's personal equation, arising from an imperfect determination of the equation, or from fluctuations in its value, depending on the observer's physical and mental condition;
5th. The accidental error of observation, composed of the observer's error in estimating the fractions of a second, and of errors arising from unsteadiness of the star;
6th. The error arising from an atmospheric displacement of the star, which may possibly be constant during the transit over the field of the telescope, and may be called the *culmination error.*

We may form an estimate of the total effect of all these sources of error by examining the several values of the right ascension of a fundamental star deduced from different culminations, and reduced for precession and nutation to a common epoch. Thus, there were found from the different observations of the transit of *α Arietis*, in the year 1852 at the Greenwich Observatory; the following values of its mean right ascension on Jan. 1, 1852. Putting $\alpha = 1^h\ 58^m\ 50^s + x$, the values of x were—

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[*] In this connection see the remarks of BESSEL in the Berlin Jahrbuch for 1823, p. 166.

x	x	x	x
$0^s.40$	$0^s.34$	$0^s.59$	$0^s.37$
.44	.31	.42	.34
.39	.42	.42	.34
.39	.45	.46	.59
.42	.53	.33	24
.40	.35	.32	31

The mean is $x = 0^s.40$; and from the differences between this mean and the several values of x we deduce $r = 0^s.057$ as the probable error of a single determination of the right ascension of this star. In the same manner, I find from the observations of γ *Ceti* during the same year $r = 0^s.063$, and for α *Ursæ Majoris* $r = 0^s.131$. If these be multiplied by the respective values of $\cos\delta$, we have $0^s.053$, $0^s.063$, $0^s.063$, the mean of which, or $0^s.06$, expresses nearly the probable error of a single determination of an equatorial star with the transit circle of the Greenwich Observatory in 1852. A larger number of stars should be examined to determine this error with certainty; but the above will suffice to illustrate the mode of proceeding. It must not be forgotten, however, that this instrument is never reversed, and all its results *may* be affected by small constant errors peculiar to the several stars.

If we denote the probable error of observation, or the 5th of the above enumerated errors, by ε, and the combined effect of all the rest by ε_1, we have

$$r^2 = \varepsilon^2 + \varepsilon_1^2$$

whence, taking $r = 0^s.06$, and $\varepsilon = 0^s.03$, as above found, we deduce $\varepsilon_1 = 0^s.052$: so that if ε were reduced to zero—that is, if the observations were made *perfectly*—the right ascension determined by a single transit would be improved by only $0^s.01$. Hence it follows that *an increase of the number of threads for the purpose of reducing the error of observation would be attended by no important advantage.*

Bessel thought five threads sufficient.

164. We see from these principles that the weight of an observed transit is not to be assumed to vary as the number of threads, as it would do were there no culmination error or unknown instrumental errors. For practical purposes it will be sufficient to regard the probable error of a transit as composed

only of the error of observation and the culmination error. The latter will then be the quantity denoted above by ε_1; and, if we now put

ε = the probable error of a transit over a single thread,
n = the number of threads observed,
r = the probable error of the observed right ascension,

we shall have

$$r^2 = \varepsilon_1^2 + \frac{\varepsilon^2}{n}$$

If then we also put

E = the probable error of an observation whose weight is unity,
p = the weight of the given observation,

we shall have, according to the theory of least squares,

$$p = \frac{E^2}{\varepsilon_1^2 + \frac{\varepsilon^2}{n}} \tag{127}$$

The unit of weight is arbitrary, and hence E also is arbitrary. If N is the total number of threads in the reticule, and a complete observation on them all is to have the weight unity, we shall have

$$E^2 = \varepsilon_1^2 + \frac{\varepsilon^2}{N}$$

and the formula will become

$$p = \frac{\varepsilon_1^2 + \frac{\varepsilon^2}{N}}{\varepsilon_1^2 + \frac{\varepsilon^2}{n}} \tag{128}$$

If we substitute the values $\varepsilon_1 = 0.052$, $\varepsilon = 0.09$, which are sufficiently accurate for an approximate estimation of the weights of observations, we shall find, very nearly,*

$$p = \frac{1 + \frac{3}{N}}{1 + \frac{3}{n}} \tag{129}$$

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* See also Vol. I. Art. 236, where a slightly different formula is obtained.

This will be a very convenient formula in practice in cases where there is no reason to depart from the above assumed values of ε_1 and ε. The observer who has determined these quantities for himself will, of course, employ (128) directly.

It may be useful to illustrate, by the aid of this formula, the proposition announced at the end of the preceding article. If $N = 7$ and $E = 0^s.062$, the weights and probable errors of observations on one or more threads will be as below:

n	p	$\frac{E}{\sqrt{p}}$
1	0.36	$0^s.104$
2	0.57	0 .082
3	0.71	0 .073
4	0.82	0 .069
5	0.90	0 .065
6	0.95	0 .063
7	1.00	0 .062
25	1.25	0 .055
∞	1.43	0 .052

We see that the advantage of seven threads over five is almost insignificant, and BESSEL's opinion is confirmed.

165. The probable error of a single transit of a star recorded by the electro-chronograph does not appear to be much less than that of one observed by eye and ear by experienced observers;* but it must be remembered that it takes but a short time to acquire the requisite skill in the use of the chronograph, while the small probable errors by eye and ear above adduced are evidences of long training. The personal equation, however, is much less in the use of the chronograph, and probably more constant. It is not unlikely that a considerable portion of the total error of a determination of right ascension, as above found, is the result of variations in the observer's personal equation; and, if so, the substitution of the chronograph for eye and ear will carry these determinations to a still more remarkable degree of accuracy.

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* See Dr. B. A. GOULD's Report in the U. S. Coast Survey Report for 1857, p. 307.

APPLICATION OF THE METHOD OF LEAST SQUARES TO THE DETERMINATION OF THE TIME WITH A PORTABLE TRANSIT INSTRUMENT IN THE MERIDIAN.

166. In the use of the portable transit instrument in the field, it is not always possible to mount it so firmly that its azimuth and level can be absolutely relied upon as constant for a whole day. Frequently it is necessary to take all the observations at a given place within a few hours. We must then observe such stars as are available at the time, and so conduct the observations and their reduction as to obtain the *most probable* result.

First, as to the observations.—The instrument having been brought very near to the meridian (see Art. 125), a number of stars must be observed in both positions of the rotation axis, and, in general, about the same number of stars in each position. Among these must be included at least one circumpolar star, and, if possible, two or three, one or more being below the pole. The level should be observed at the beginning and end of the series, and before and after each reversal of the axis.

Secondly, as to the computation.—We assume that the thread intervals have been well determined, as also the value of a division of the level. If they have not been found *before* the observations, they must, of course, be determined subsequently, only observing that no change of the instrument has occurred which might change the value of the thread intervals. The mean of all the level determinations should be adopted as the constant value of b for all the observations, unless the differences of the several values are greater than the probable errors of observations made with the particular spirit-level used, in which case it will be better to interpolate a value of b for each star from the actually observed values. The chronometer time T of transit over the middle thread or the mean thread being found for each star by employing the thread intervals when necessary, we shall suppose that observation has furnished only T and b for each star. The rate δT of the chronometer is also supposed to be approximately known. The constants a and c, and the clock correction ΔT, are then to be found by a proper combination of the observations. Let us put in formula (87), for each star,

A = the azimuth factor = $\sin(\varphi - \delta)\sec\delta$,
B = the level factor = $\cos(\varphi - \delta)\sec\delta$,
C = the collimation factor = $\sec\delta$;

also, let each observation be reduced to some assumed time T_0, and put

$$\Delta T_0 = \text{the chronometer correction at the time } T_0,$$

whence

$$\Delta T = \Delta T_0 + \delta T(T - T_0)$$

Let

$$\vartheta = \text{an assumed approximate value of } \Delta T_0$$
$$\Delta\vartheta = \text{the required correction of } \vartheta$$

so that

$$\vartheta + \Delta\vartheta = \Delta T_0$$

then the formula (82) becomes

$$\alpha = T + \vartheta + \Delta\vartheta + \delta T(T - T_0) + Aa + Bb + Cc$$

in which every thing is known except the small quantities $\Delta\vartheta$, a, and c. If we now put*

$$t = T + \delta T(T - T_0) + Bb$$
$$w = \vartheta - (\alpha - t)$$

then, since $\alpha - t$ and ϑ are each nearly equal to the clock correction, w is a small residual, and the equation is

$$Aa + Cc + \Delta\vartheta + w = 0 \tag{130}$$

Each star gives an equation of condition of this form, and from all these equations the most probable values of a, c, and $\Delta\vartheta$ will be found by the method of least squares. The sign of the term Cc will be changed when the axis of the instrument is reversed.

If the observations are extended over a number of hours, it will not always be safe to assume that the azimuth a has been constant during the whole time. We may then divide the observations into two groups, in one of which the azimuth will be denoted by a and in the other by a'. The normal equations, formed by combining all the equations in the usual manner, will then involve the four unknown quantities a, a', c, and $\Delta\vartheta$.

To determine the mean error of the resulting value of $\Delta\vartheta$, it must be remembered that when a and c have been eliminated by

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* For greater precision (not always required in the use of a portable instrument), we may allow for the diurnal aberration. Since α requires the correction $+ 0^s.021 \cos\phi \sec\delta$, we have merely to take

$$t = T + \delta T(T - T_0) + Bb - 0^s.021 \cos\phi \sec\delta$$

successive substitution, taking care to introduce no new factor into the equations, the coefficient of $\Delta\vartheta$ in the resulting final equation will be the weight p of the value of $\Delta\vartheta$ thus determined.* Then, substituting the values of a, c, and $\Delta\vartheta$ in the equations of condition, and denoting the residual in each by v, we have the mean error of a single observation by the formula

$$\varepsilon = \sqrt{\frac{[vv]}{m - \mu}}$$

in which $[vv]$ = the sum of the squares of the residuals, m = the number of observations, and μ = the number of unknown quantities.

The mean error of $\Delta\vartheta$ and ΔT_0 will be

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{p}}$$

and if we wish the *probable* errors, we multiply the mean errors by 0.6745.

If any residuals are so large as to throw a doubt upon the observations, such doubtful observations may be examined by PEIRCE's Criterion.†

If an observation consists of transits over only a portion of the threads, it may be well to give it a diminished weight, multiplying its equation of condition by the square root of the weight found by (129).

If the collimation constant c has been previously determined, we have only to include the term Cc in the quantity t; thus, putting

$$t = T + \delta T(T - T_0) + Bb + Cc$$
$$w = \vartheta - (\alpha - t)$$

the equation for each star will be

$$Aa + \Delta\vartheta + w = 0 \qquad (131)$$

and the determination of a and $\Delta\vartheta$ from these equations is then exceedingly simple.

EXAMPLE.—The following observations were taken on the United States North-Western Boundary Survey with a portable transit instrument in the meridian. The stars were mostly elected from the British Association Catalogue, and are coneniently designated by their numbers in this catalogue. But eir apparent places have been derived from the more reliable thority the Greenwich Twelve Year Catalogue. The apparent ce of *α Ursæ Majoris* is derived from the American Ephemeris. her stars from the British Association Catalogue, observed the same evening, have been excluded because they are not en in the later catalogues.

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* See Appendix. † See Appendix, Arts. 57–60.

Camp Simiahmoo.—1857, July 27. Latitude 49° 0′ N.

Star.	L	Threads.							Mean.	Level.
		I	II	III	IV	V	VI	VII		
B. A. C. 6390	W.	$15^s.3$	$43^s.6$	$12^s.3$	$40^s.8$	$9^s.2$	$37^s.9$	$6^s.2$	22^h 4^m $40^s.76$	$+ 0^d.75$
" 6434	"	27 .2		15 .2	38 .8	2 .3	26 .3	50 .1	22 10 38 .68	"
" 6441	"	23 .8	47 .8	11 .3	35 .3	58 .6	22 .6	46 .3	22 11 35 .10	"
" 6489	"	21 .3	46 .8	12 .5	37 .8	2 .8			22 18 37 .58	"
" 6836 }	"	30 .3	33 .3	38 .2	42 .0				23 13 41 .63	"
" " }	E.	52 .3	48 .2	43 .9					23 13 40 .49	— 0 .70
" 3232 S. P.	"	32 .1	36 .9	43 .2	49 .4	54 .5	59 .9	5 .9	0 46 48 .84	— 0 .51
" 3346 S. P.	"	39 .7	22 .7	7 .0	50 .1	33 .6	16 .9	0 .3	1 5 50 .04	— 0 .48
" 7686	"	53 .1	40 .0	26 .7	13 .9	0 .7	48 .6	35 .3	1 22 14 .04	— 0 .44
" 7778	"	48 .3	8 .9	29 .2	49 .4	10 .3	30 .8	51 .2	1 34 49 .73	— 0 .42
" 3647 S. P.	"	26 .8	20 .7	17 .5	11 .8	7 .3	1 .9	57 .0	1 57 11 .86	— 0 .38
α Urs. Maj. S. P.	"	32 .7	19 .8	7 .9	55 .0	42 .6	30 .0	17 .4	2 19 55 .06	— 0 .33

he threads are numbered from the end of the axis at which the luminating lamp is placed, and the seconds of the chronometer re recorded, not in the order of observation, but in the columns ppropriated to the several threads. The column "Mean" gives he time of passage over the mean of the threads, employing in he case of the defective transits the following equatorial interals from the mean:

$$\overset{i_1}{+65^s.82} \quad \overset{i_2}{+44^s.05} \quad \overset{i_3}{+21^s.84} \quad \overset{i_4}{-0^s.08} \quad \overset{i_5}{-22^s.00} \quad \overset{i_6}{-43^s.79} \quad \overset{i_7}{-65^s.85}$$

here the signs are given for Lamp West. The column marked gives the position of the lamp end of the axis. The value of e division of the level was $0^s.105$. Only one observation of e level was made during the observations "lamp west." Two servations of the level were made during the observations amp east," one near the beginning, the other near the end, of the series, from which those given in the table are obtained by interpolation.

Stars observed at their lower culminations are marked S. P. (sub polo).

The chronometer was sidereal, and its rate was losing $0^s.4$ daily.

A first computation of the observations having shown that th observations lamp west and lamp east give very different result the presumption is that in reversing the axis the observer di turbed the instrument, a supposition rendered still more probabl by the change of level. It will, therefore, be proper to comput the observations upon the supposition of a different azimuth f the two positions of the axis.

The apparent places of the stars on the given date were a follows:

Star.	α	δ
B. A. C. 6390	$18^h\ 39^m\ 38^s.71$	+ 39° 31′
" 6434	18 45 35.70	− 22 55
" 6441	18 46 31.91	− 22 51
" 6489	18 53 34.36	− 30 5
" 6886	19 48 41.61	+ 69 58
" 3232	9 21 46.76	+ 70 29
" 3346	9 40 48.22	+ 59 44
" 7686	21 57 14.44	+ 72 28
" 7778	22 9 49.07	+ 56 18
" 3647	10 32 9.78	+ 66 30
α Urs. Maj.	10 54 53.21	+ 62 31

The observed times of transit are to be reduced for the chronometer's rate to some common epoch, which we shall here assume to be $T_0 = 0^h$ by the chronometer. The assumed correction of the chronometer at this time will be

$$\vartheta = -\ 3^h\ 25^m\ 0^s.$$

The formation of the equations of condition for the first and last stars is as follows:

	L. W.	L. E.
	B. A. C. 6390.	α Ursæ Maj. S. P.
δ	$+39°\ 31'$	$117°\ 29'$
$\varphi - \delta$	$+\ 9\ \ 29$	$-68\ \ 29$
log sec δ	0.1127	n0.3358
log cos $(\varphi - \delta)$	9.9940	9.5644
log sin $(\varphi - \delta)$	9.2169	n9.9686
log cos $(\varphi - \delta)$ sec δ = log B	0.1067	n9.9002
log sin $(\varphi - \delta)$ sec δ = log A	9.3296	0.3044
A	+ 0.214	+ 2.016
sec $\delta = C$	+ 1.296	− 2.166
b	$+\ 0^s.08$	$+\ 0^s.03$
Observed mean	$22^h\ 4^m\ 40^s.76$	$2^h\ 19^m\ 55^s.06$
Rate to 0^h	− 0.03	+ 0.04
Bb	+ 0.10	+ 0.02
iurnal ab. = $-0^s.021 \cos\varphi \sec\delta$	− 0.02	+ 0.03
t	22 4 40.81	2 19 55.15
α	18 39 38.71	22 54 53.21
$\alpha - t$	−3 25 2.10	−3 25 1.94
Assumed ϑ	−3 25 0.	−3 25 0.
w	+ 2.10	+ 1.94

Denoting the azimuth of the instrument for *L. W.* by a, and at for *L. E.* by a', and changing the sign of c for *L. E.*, the quations of condition for these two stars are, therefore,

$$+\ 0.214\, a + 1.296\, c + \Delta\vartheta + 2^s.10 = 0$$
$$+\ 2.016\, a' + 2.166\, c + \Delta\vartheta + 1.94 = 0$$

The equations for the other stars being found in the same manner, we have then:

1. $+\ 0.214\, a + 1.296\, c + \Delta\vartheta + 2^s.10 = 0$
2. $+\ 1.032\, a + 1.086\, c + \Delta\vartheta + 2.96 = 0$
3. $+\ 1.031\, a + 1.085\, c + \Delta\vartheta + 3.17 = 0$
4. $+\ 1.135\, a + 1.156\, c + \Delta\vartheta + 3.19 = 0$
5. $-\ 0.732\, a + 2.056\, c + 0.707\, \Delta\vartheta + 0.15 = 0$
6. $-\ 0.732\, a' - 2.056\, c + 0.707\, \Delta\vartheta - 0.97 = 0$

7. $+ 2.606\,a' + 2.993\,c + \Delta\vartheta + 2.22 = 0$
8. $+ 1.879\,a' + 1.984\,c + \Delta\vartheta + 1.91 = 0$
9. $- 1.322\,a' - 3.319\,c + \Delta\vartheta - 0.58 = 0$
10. $- 0.229\,a' - 1.802\,c + \Delta\vartheta + 0.58 = 0$
11. $+ 2.264\,a' + 2.508\,c + \Delta\vartheta + 2.18 = 0$
12. $+ 2.016\,a' + 2.166\,c + \Delta\vartheta + 1.94 = 0$

where the 5th and 6th equations have been multiplied by $\sqrt{\tfrac{1}{2}}$, thus giving each but one-half the weight of an ordinary observation, because the star was observed on but half the threads.*

The normal equations are

$$\begin{aligned}
3.998\,a + \quad 0 \quad + 2.325\,c + 2.894\,\Delta\vartheta + 10.283 &= 0 \\
0 \quad + 21.848\,a' + 27.881\,c + 6.697\,\Delta\vartheta + 19.569 &= 0 \\
2.325\,a + 27.881\,a' + 51.969\,c + 9.153\,\Delta\vartheta + 36.352 &= 0 \\
2.894\,a + 6.697\,a' + 9.153\,c + 11.000\,\Delta\vartheta + 19.090 &= 0
\end{aligned}$$

from which we find

$$\begin{aligned}
a &= - 1^s.681 \\
a' &= - 0\ .083 \\
c &= - 0\ .423 \\
\Delta\vartheta &= - 0\ .891 \text{ with the weight } p = 6.775
\end{aligned}$$

This example is instructive in several respects. The instrument was reversed upon the star B. A. C. 6836 for the purpose of deducing the value of *c*. But, upon the supposition that the azimuth remained unchanged during the reversal, we find $c = - 0^s.267$. The danger of disturbing the instrument in reversing the axis is, of course, greater with small instruments, and always requires great caution. Again, the observer neglected to observe the level immediately before and after the reversal, the values of *b* given in the table being inferred from observations taken at the time of the transits of Nos. 1, 7, and 11. If the level had been observed more frequently, as it should be, the disturbance of the azimuth might have been suggested to the observer himself, who, however, appears not to have suspected it.

But we shall obtain still further instruction from this example by substituting the values of a, a', c, $\Delta\vartheta$ in the original equations of condition. The residuals v will exhibit to us the anomalous observations. We find :

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* To proceed more accurately, we should have computed, by (129), the weights of the *four* defective observations, the 2d, 4th, 5th, and 6th. We should have found the weights 0.95, 0.89, 0.82, 0.71 respectively.

No.	v	vv
1.	$+0^s.302$	0.0912
2.	-0.125	.0156
3.	$+0.086$	.0074
4.	-0.098	.0096
5.	-0.120	.0144
6.	-0.669	.4475
7.	-0.153	.0234
8.	$+0.024$	.0006
9.	$+0.043$	.0018
10.	$+0.470$	.2209
11.	$+0.040$	.0016
12.	-0.034	.0012
		$[vv] = 0.8352$

ice, the number of observations being denoted by $m = 12$, the number of unknown quantities in our equations by 4, we have the mean error of an observation of the weight y,

$$\epsilon = \sqrt{\frac{[vv]}{m-\mu}} = 0^s.323$$

large residuals of Nos. 6 and 10 point them out as probably malous; but, before rejecting them, we will apply PEIRCE's terion. Since Table X. is adapted only to the cases of one two unknown quantities, we shall have to employ Table X.A. nmencing with the hypothesis of but one doubtful observa-, we assume for a first trial $x = 1.5$.

		1st Approx.	2d Approx.
$m = 12, \mu = 4, n = 1$	x	1.5	1.78
Table X.A.	$\log T$	8.5051	8.5051
" "	$\log R$	9.3973	9.3464
	$\log \frac{T}{R}$	9.1078	9.1587
$\frac{2n}{m-n} = \frac{2}{11}$	$\log \lambda^2 = \log\left(\frac{T}{R}\right)^{\frac{2}{11}}$	9.8378	9.8470
	$1 - \lambda^2$	0.3117,	0.2970
$\frac{m-\mu-n}{n} = 7,$	$x^2 - 1 = 7(1-\lambda^2)$	2.1819	2.0790
	x^2	3.1819	3.0790
	x	1.78	1.76
			$x\epsilon = 0^s.568$

The residual 0.669 surpasses the limit $0^s.568$, and hence the 6th observation is to be rejected. We must then pass to the hypothesis of two doubtful observations, for which we commence by assuming $\varkappa = 1.5$, and then with $n = 2$ we find $\varkappa = 1.49$, $\varkappa\varepsilon = 0^s.481$. Hence the 10th observation is *not* to be rejected. Thus the only observation to be rejected as anomalous is the 6th; and our hypothesis of a disturbed state of the instrument produced by reversal is confirmed.

If we now form normal equations from the remaining eleven equations of condition, we shall find the values of the unknown quantities to be

$$a = -1^s.636$$
$$a' = -0.092$$
$$c = -0.367$$
$$\Delta\vartheta = -0.999 \text{ with weight } p = 5.963$$

and these values substituted in the equations of condition give the residuals and mean errors as follows:

No.	v	$[vv]$
1.	$+0^s.276$	0.0762
2.	-0.126	.0159
3.	$+0.086$	.0074
4.	-0.089	.0079
5.	-0.114	.0130
7.	-0.120	.0144
8.	$+0.010$	.0001
9.	-0.239	.0571
10.	$+0.264$	.0697
11.	$+0.051$	.0026
12.	-0.040	.0016
$m - \mu = 7$		$[vv] = 0.2659$

$$\varepsilon = \sqrt{\frac{[vv]}{m-\mu}} = 0^s.195$$

The 10th observation is now well represented, and the Criterion does not reject any of them.

The mean error of $\Delta\vartheta$ is

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{p}} = 0^s.08$$

and the probable error $0^s.05$.

Hence we have, finally, the chronometer correction at 0^h,

$$\Delta T_0 = \vartheta + \Delta\vartheta = -3^h\,25^m\,1^s.00 \pm 0^s.05$$

THE TRANSIT INSTRUMENT IN ANY VERTICAL PLANE.

167. The formulæ (78) and (79) apply to any position of the instrument. When the instrumental constants m and n are known, or when a and b are given, from which m and n can be found by (78), the formula (79) determines the apparent east hour angle τ of the observed object at the time of its transit over any given thread whose distance from the collimation axis is c. The constants are found by combining observations of stars near to and remote from the pole, as will be illustrated hereafter. When the transits over several threads have been observed, each may be separately reduced by the general formulæ; but it is necessary also to have the means of reducing them all to a common instant. I shall, therefore, here consider the most general case of an observation of the moon's limb on any given thread, and investigate the formula for reducing it to the middle thread, or to the collimation axis of the instrument. This general formula will be applicable to any other object which has a proper motion and a sensible diameter. Let

Θ = the sidereal time of the observed transit of the moon's limb over the given thread,
i = the equatorial interval of the thread from the middle thread,
α, δ = the true R.A. and decl. of the moon's centre at the time Θ,
α', δ' = the apparent R.A. and declination,
s = the moon's geocentric semidiameter,
s' = the moon's apparent semidiameter.

At the instant the moon's limb touches the thread whose distance from the middle thread is i, the centre of the moon is at the distance $i \pm s'$ from the middle thread, and, consequently, at the distance $c + i \pm s'$ from the collimation axis of the telescope. The apparent east hour angle of the moon's centre at this instant is

$$\tau = \alpha' - \Theta$$

Putting then $c + i \pm s'$ for c and $\alpha' - \Theta$ for τ in (79), we have

$$\begin{aligned}\sin(c + i \pm s') &= -\sin n \sin \delta' - \cos n \cos \delta' \sin(\Theta - \alpha' + m) \\ &= -\sin n \sin \delta' - \cos n \cos m \cos \delta' \sin(\Theta - \alpha') \\ &\quad - \cos n \sin m \cos \delta' \cos(\Theta - \alpha')\end{aligned}$$

where the apparent declination and right ascension are employed, since it is the moon's apparent place which is observed. To introduce the geocentric quantities, let

π = the moon's equatorial horizontal parallax,
ρ, φ' = the earth's radius and reduced latitude of the place of observation,
Δ, Δ' = the moon's distance from the centre of the earth and the observer respectively;

then, putting

$$f = \frac{\Delta'}{\Delta}$$

we find from Vol. I., equations (132),

$$\begin{aligned}f \cos \delta' \sin(\Theta - \alpha') &= \cos \delta \sin(\Theta - \alpha) \\ f \cos \delta' \cos(\Theta - \alpha') &= \cos \delta \cos(\Theta - \alpha) - \rho \sin \pi \cos \varphi' \\ f \sin \delta' &= \sin \delta \qquad\qquad\quad\; - \rho \sin \pi \sin \varphi'\end{aligned}$$

Substituting these values, we obtain

$$\begin{aligned}f(c + i \pm s') \sin 1'' = &- \sin n \sin \delta - \cos n \cos \delta \sin(\Theta - \alpha + m) \\ &+ \rho \sin \pi \sin \varphi' \sin n + \rho \sin \pi \cos \varphi' \cos n \sin m\end{aligned} \tag{132}$$

The right ascension and declination are, however, variable, and we should introduce into the formula their values for some assumed epoch. Let this epoch be the sidereal time, Θ_0, which is the common instant to which the observations on the several threads are to be reduced. Let

α_0, δ_0 = the true right ascension and declination at the time Θ_0,
$\Delta\alpha$ = the increase of the right ascension in one minute of mean time,
$\Delta\delta$ = the increase of the declination (towards the north) in one minute of mean time,

and put

$$I = \Theta_0 - \Theta = \text{the required reduction,}$$

$$\lambda = \frac{\Delta\alpha}{60.164} = \text{the increase of } \alpha \text{ in } 1^s \text{ of sidereal time}$$

$$\lambda' = \frac{\Delta\delta}{60.164} = \quad \text{"} \quad \delta \quad \text{"} \quad \text{"}$$

then, if I is expressed in seconds of arc, we have

$$\alpha = \alpha_0 - \lambda I, \qquad \delta = \delta_0 - \tfrac{1}{15}\lambda' I$$

$$\Theta - \alpha = \Theta_0 - \alpha_0 - (\Theta_0 - \Theta) + (\alpha_0 - \alpha) = \Theta_0 - \alpha_0 - (1 - \lambda) I$$

$$\sin(\Theta - \alpha + m) = \sin(\Theta_0 - \alpha_0 + m)$$
$$- (1 - \lambda)\cos[\Theta_0 - \alpha_0 + m - \tfrac{1}{2}(1 - \lambda) I]\, 2\sin\tfrac{1}{2} I$$

[in which $(1 - \lambda)\sin\frac{1}{2} I$ is put for $\sin\frac{1}{2}(1 - \lambda) I$]

$$\sin\delta = \sin\delta_0 - \frac{\lambda'}{15}\cos\delta_0 \, . \, 2\sin\tfrac{1}{2} I$$

$$\cos\delta = \cos\delta_0 + \frac{\lambda'}{15}\sin\delta_0 \, . \, 2\sin\tfrac{1}{2} I$$

Substituting these values, our formula becomes (omitting a term multiplied by the exceedingly small quantity $\frac{1}{15}\lambda' \sin^2\frac{1}{2} I$)

$$\begin{aligned} f(c + i \pm s')\sin 1'' = & - \sin n \sin\delta_0 - \cos n \cos\delta_0 \sin(\Theta_0 - \alpha_0 + m) \\ & + \rho \sin\pi \sin\varphi' \sin n + \rho \sin\pi \cos\varphi' \cos n \sin m \\ & + (1 - \lambda)\cos n \cos\delta_0 \cos[\Theta_0 - \alpha_0 + m - \tfrac{1}{2}(1 - \lambda) I]\, 2\sin\tfrac{1}{2} I \\ & + \tfrac{1}{15}\lambda' [\sin n \cos\delta_0 - \cos n \sin\delta_0 \sin(\Theta_0 - \alpha_0 + m)]\, 2\sin\tfrac{1}{2} I \end{aligned} \tag{133}$$

In this formula, we may consider I as the only quantity which varies with the time; for, although f, s', and π vary slightly, their variations will not usually be sensible, or, if sensible for a single thread, their effect will disappear when the epoch is nearly the mean of all the observed times.

If now Θ_0 is the time of transit of the moon's centre over the great circle of the instrument, this formula gives

$$\left.\begin{aligned} 0 = & - \sin n \sin\delta_0 - \cos n \cos\delta_0 \sin(\Theta_0 - \alpha_0 + m) \\ & + \rho \sin\pi \sin\varphi' \sin n + \rho \sin\pi \cos\varphi' \cos n \sin m \end{aligned}\right\} \tag{134}$$

Subtracting this from (133), and, for brevity, putting

$$t' = \Theta_0 - \alpha_0 + m$$
$$R = \sin n \cos\delta_0 - \cos n \sin\delta_0 \sin t$$

we find

$$2 \sin \tfrac{1}{2} I = \frac{f(c + i \pm s') \sin 1''}{(1 - \lambda) \cos n \cos \delta_0 \cos [t - \frac{1}{2}(1 - \lambda) I] + \frac{1}{15} \lambda' R}$$

This is equivalent to the formula given by Sawitsch (*Pract. Astron.*, Vol. I. p. 303); but he has not observed that the expression for R may be put under a much more simple form. In so small a term as $\frac{1}{15}\lambda' R$, we need not consider the effect of the parallax upon the factor R; but when we neglect the parallax we have, by (184),

$$0 = -\sin n \sin \delta_0 - \cos n \cos \delta_0 \sin t$$

Multiplying this by $\sin \delta_0$, and subtracting the product from $R \cos \delta_0$, we find

$$R \cos \delta_0 = \sin n, \qquad \text{or} \qquad R = \sin n \sec \delta_0$$

It is also to be observed that by the formula (246) of Vol. I. we have

$$fs' = s = \text{the true semidiameter.}$$

Hence our formula becomes

$$2 \sin \tfrac{1}{2} I = \frac{f(c + i) \sin 1'' \pm s \sin 1''}{(1 - \lambda) \cos n \cos \delta_0 \cos [t - \frac{1}{2}(1 - \lambda) I] + \frac{1}{15} \lambda' \sin n \sec \delta_0} \qquad (185)$$

or, when I is small, as it usually is,

$$I = \frac{f(c + i) \pm s}{(1 - \lambda) \cos n \cos \delta_0 \cos [t - \frac{1}{2}(1 - \lambda) I] + \frac{1}{15} \lambda' \sin n \sec \delta_0} \qquad (185^*)$$

This formula, then, gives the reduction of the observed time of transit of the moon's limb over any given thread to the time of transit of the moon's centre over the great circle of the instrument.

If we omit s in the numerator of the second member, I becomes the reduction to the time of transit of the limb over the great circle of the instrument.

If we omit $fc \pm s$, I becomes the reduction to the time of transit of the limb over the middle thread.

The factor f is determined rigorously by (137), Vol. I.; but it generally suffices to take

$$f = \frac{\sin \zeta}{\sin \zeta'}$$

which is very nearly exact, according to (101) of Vol. I. The finder of the instrument will give the apparent zenith distance ζ', and the difference between this and the true zenith distance ζ will be found with sufficient accuracy by the formula

$$\sin(\zeta' - \zeta) = \rho \sin \pi \sin(\zeta' - \gamma)$$

in which, a being the azimuth constant of the instrument,

$$\gamma = (\varphi - \varphi') \cos a$$

or, very nearly,

$$\gamma = (\varphi - \varphi') \cos n \cos m$$

For the sun or a planet we can always put $\lambda' = 0$ and $\zeta = \zeta'$, and the formula becomes

$$I = \frac{c + i \pm s}{(1 - \lambda) \cos n \cos \delta_0 \cos(t - \tfrac{1}{2} I)} \tag{136}$$

For a fixed star, we further put $\lambda = 0$, $s = 0$, $t = \Theta_0 - \alpha + m$, and the formula becomes for stars near the pole,

$$2 \sin \tfrac{1}{2} I = \frac{(c + i) \sin 1''}{\cos n \cos \delta \cos(t - \tfrac{1}{2} I)} \tag{137}$$

and for other stars,

$$I = \frac{c + i}{\cos n \cos \delta \cos(t - \tfrac{1}{2} I)} \tag{137*}$$

In all cases, we must carefully observe the sign of I in the denominator of the second member. I will be negative when the observed time is later than the time to which the reduction is made, and then $-\tfrac{1}{2} I$ will be essentially positive. An approximate value of I must first be found by neglecting I in the second member, and then a more precise value by the complete formula.

If the azimuth a and the level b are given, m and n must first be found by (78), in which, however, we may usually neglect b when our object is merely to reduce the several threads to a common instant.

168. For a fixed star, another formula has been given by **Hansen**. We have

$$\begin{aligned}\sin(c + i) &= -\sin n \sin \delta - \cos n \cos \delta \sin(t - I) \\ &= -\sin n \sin \delta - \cos n \cos \delta \sin t \cos I + \cos n \cos \delta \cos t \sin I\end{aligned}$$

If the reduction is made to the collimation axis, we have

$$0 = -\sin n \sin \delta - \cos n \cos \delta \sin t$$

which, subtracted from the above, gives

$$\sin (c + i) = 2 \cos n \cos \delta \sin t \sin^2 \tfrac{1}{2} I + \cos n \cos \delta \cos t \sin I$$

whence

$$\sin I = \frac{\sin (c + i)}{\cos n \cos \delta \cos t} - 2 \tan t \sin^2 \tfrac{1}{2} I \qquad (138)$$

which is a rigorous formula. We see also that t may be found by the formula

$$\sin t = -\tan n \tan \delta \qquad (139)$$

169. *To deduce the moon's right ascension from an observed transit in any given position of the instrument.*—We first find the clock time of transit of the moon's centre over the great circle of the instrument, from each thread, by applying to the observed time the reduction given by the formula (135). Let T_0 be the mean of the resulting times, and ΔT_0 the corresponding correction of the clock; then we have $\Theta_0 = T_0 + \Delta T_0$, and from (134) we deduce

$$\sin (\Theta_0 - \alpha_0 + m) = -\tan n \tan \delta_0 + \rho \sin \pi \left(\frac{\sin \varphi' \tan n + \cos \varphi' \sin m}{\cos \delta_0} \right) \qquad (140)$$

in which α_0 and δ_0 are the true right ascension and declination at the sidereal time Θ_0.

If it is preferred, we may first find the apparent right ascension by the formula

$$\sin (\Theta_0 - \alpha_0' + m) = -\tan n \tan \delta_0'$$

and deduce the true right ascension by applying the parallax computed by Art. 102, Vol. I; but it will then be necessary to compute the apparent declination δ_0'.

It will be easy to deduce from (140) the formula for the case where the instrument is in the meridian, which has already been given in Art. 154.

The constants m and n, above supposed to be known, may be found from the transits of two stars as in the next article.

FINDING THE TIME WITH A PORTABLE TRANSIT INSTRUMENT OUT OF THE MERIDIAN.

170. The number of Nautical Almanac stars near the pole is so small, that the observer in the field, when pressed for time, cannot always wait for their transits over the meridian, and must then either employ catalogue stars whose places are not so well determined, or have recourse to extra-meridian observations. If the transit instrument is mounted so as to be readily revolved in azimuth and clamped in any assumed position (as is the case with the "universal instruments"), it may be directed at once to a fundamental star near the pole, and then, its rotation axis being levelled, its collimation axis will describe a vertical circle not far from the meridian. The transit of any star over this circle being observed, the general equations of Art. 123 will enable us to find the hour angle of this star, and hence the time, when we have determined the constants m and n for the assumed position of the instrument.

The stars best adapted for the purpose in the northern hemisphere are *Polaris* (*α Ursæ Minoris*) and *δ Ursæ Minoris*, one of these being always near the meridian when the other is most remote from it; and it will be advisable always to employ that which is nearest to the meridian. In the southern hemisphere, the best star is *σ Octantis*, which is less than 1° from the pole; but, as it is of the 6th magnitude, it may be necessary, with small instruments, to use either *β Hydri* or *β Chamæleontis.*

To take the observation, make the axis approximately level, and turn the telescope upon the circum-polar star. The star moving very slowly, set the instrument, so that a few minutes must elapse before the star will cross the middle thread. During this interval, apply the spirit level and determine the constant b. Observe the transit of the star over the middle thread by the chronometer. The instrument now remaining clamped in azimuth, revolve the telescope upon its axis, and observe the transit of an equatorial star over all the threads. Then determine the constant b again, and employ the mean of its two values.

In order to eliminate an error of collimation, the rotation axis is to be reversed, and another similar observation is to be taken, the instrument being set at a new azimuth slightly in advance of the polar star as before. Each observation of a pair of stars must, of course, be separately reduced. We may, however, combine each transit of the polar star with the transits of several equatorial stars.

The collimation constant should have been made as small as possible before the observations; but, in any case, we shall assume that its value is known.

To reduce the observations, we must first find the constants which determine the position of the instrument. For this purpose, we use only the observations on the middle thread. Let then T' and T be the observed chronometer times of transit of the polar and equatorial star respectively over the middle thread, reduced for rate to an assumed time T_0; and let ΔT_0 be the chronometer correction at this time; α', α, the right ascensions, δ', δ, the declinations; τ', τ, the east hour angles, or reductions to the meridian; $90° - m$, and n, the hour angle and declination of the point in which the rotation axis produced towards the west meets the celestial sphere; c the collimation constant: then we have, by (79),

$$\left.\begin{aligned}\sin(\tau - m) &= \tan n \tan \delta + \sin c \sec n \sec \delta \\ \sin(\tau' - m) &= \tan n \tan \delta' + \sin c \sec n \sec \delta'\end{aligned}\right\} \quad (141)$$

in which we have

$$\begin{aligned}\tau &= \alpha - (T + \Delta T_0) \\ \tau' &= \alpha' - (T' + \Delta T_0)\end{aligned}$$

If we could put $c = 0$, these equations would give us m and n by a very simple transformation; but, retaining c, we can still reduce them to the form they would have if c were zero.* For this purpose, let m' and n' be approximate values of m and n, determined by the conditions

$$\begin{aligned}\sin(\tau - m') &= \tan n' \tan \delta \\ \sin(\tau' - m') &= \tan n' \tan \delta'\end{aligned}$$

from which we shall find n' and then the correction to reduce it to n. Put

$$\gamma = \tfrac{1}{2}(\tau' - \tau) \qquad\qquad \lambda = \tfrac{1}{2}(\tau' + \tau) - m'$$

then γ is known from the observation, since we have

$$\gamma = \tfrac{1}{2}[\alpha' - T' - (\alpha - T)] \quad (142)$$

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* This transformation is given by HANSEN, *Astr. Nach.*, Vol. XLVIII. p. 115.

We have then

$$\lambda - \gamma = \tau - m' \qquad\qquad \lambda + \gamma = \tau' - m'$$

and hence

$$\sin(\lambda - \gamma) = \tan n' \tan \delta \qquad\qquad \sin(\lambda + \gamma) = \tan n' \tan \delta'$$

the sum and difference of which give

$$\begin{aligned} 2\sin\lambda\cos\gamma\cos\delta\cos\delta' &= \tan n' \sin(\delta' + \delta) \\ 2\cos\lambda\sin\gamma\cos\delta\cos\delta' &= \tan n' \sin(\delta' - \delta) \end{aligned}$$

If, therefore, we make

$$\left.\begin{aligned} L\sin\lambda &= \frac{\sin(\delta' + \delta)}{\cos\gamma} \\ L\cos\lambda &= \frac{\sin(\delta' - \delta)}{\sin\gamma} \end{aligned}\right\} \qquad (143)$$

these equations will give us λ and L, and then we shall have

$$\tan n' = \frac{2\cos\delta\cos\delta'}{L} \qquad (144)$$

It is to be observed that α' is always to be regarded as greater than T', and in finding γ by (142) the difference $\alpha' - T'$ is to be found by increasing α' by 24^h when necessary, but $\alpha - T$ will be positive or negative. This makes γ less than 180°, and, since $\lambda + \gamma (= \tau' - m')$ must be less than 360°, it follows that λ must also be less than 180°. Hence, L will have the same sign as $\cos\gamma$, and n' will be negative when $\gamma > 90°$.

Now, we have $\tau - m = \tau - m' + (m' - m)$, and, since $m' - m$ is very small,

$$\sin(\tau - m) = \sin(\tau - m') + \sin(m' - m)\cos(\tau - m')$$

which, substituted in the first equation of (141), gives

$$\begin{aligned} \sin c = \sin(\tau - m')\cos n\cos\delta &- \sin n\sin\delta \\ &+ \sin(m' - m)\cos(\tau - m')\cos n\cos\delta \end{aligned}$$

To simplify this, let us put

$$\sin w = \frac{\sin\delta}{\cos n'}$$

from which and the equation

$$\sin(\tau - m') = \tan n' \tan\delta$$

there follows also

$$\cos w = \cos(\tau - m')\cos\delta$$

for, if we add together the squares of the first and third of these equations, the sum is reduced by means of the second to the identical equation $1 = 1$. By substituting the values of $\sin(\tau - m')$, $\cos(\tau - m')$, and $\sin\delta$, which these equations give, in the expression for c, it becomes

$$\sin c = \sin(n' - n)\sin w + \sin(m' - m)\cos n\cos w$$

In the same manner, if for the polar star we take

$$\sin w' = \frac{\sin\delta'}{\cos n'} \qquad \cos w' = \cos(\tau' - m')\cos\delta'$$

we shall have

$$\sin c = \sin(n' - n)\sin w' + \sin(m' - m)\cos n\cos w'$$

Combining these two values of $\sin c$, we have

$$\sin c\,(\cos w - \cos w') = \sin(n' - n)\sin(w' - w)$$

whence

$$\sin(n' - n) = \sin c\cdot\frac{\sin\frac{1}{2}(w' + w)}{\cos\frac{1}{2}(w' - w)}$$

or, putting $n' - n = \nu$,

$$\left.\begin{aligned}\nu &= c\cdot\frac{\sin\frac{1}{2}(w' + w)}{\cos\frac{1}{2}(w' - w)}\\ n &= n' - \nu\end{aligned}\right\}\quad(145)$$

The angles w' and w here required are found by the equations

$$\tan w' = \frac{\tan\delta'}{\cos(\lambda + \gamma)\cos n'} \qquad \tan w = \frac{\tan\delta}{\cos(\lambda - \gamma)\cos n'} \quad (146)$$

observing that for a negative value of $\tan w'$, w' is to be taken in the 2d quadrant, but that for a negative value of $\tan w$, w is to be taken numerically less than 90°, and with the negative sign.

To find m, we have, by eliminating a from (78),

$$\sin m\cos n\cos\varphi + \sin n\sin\varphi = \sin b$$

whence

$$\sin m = -\tan n\tan\varphi + \frac{\sin b}{\cos n\cos\varphi}$$

If then we take

$$\left.\begin{aligned}\sin\mu &= -\tan n \tan\varphi \\ \beta &= \frac{b}{\cos n \cos\mu \cos\varphi} \\ m &= \mu + \beta\end{aligned}\right\} \qquad (147)$$

we have

The constants being thus found, we proceed to find the correction of the chronometer by the equatorial star. We must first reduce the transits over the several threads to the collimation axis, which may here be done by the formula (138), omitting the last term, which is insensible when the instrument is so near the meridian as we here suppose it to be. If, therefore, we first find t by the formula

$$\sin t = -\tan n \tan\delta \qquad (148)$$

and then put

$$F = \cos n \cos\delta \cos t$$

we must apply to the observed time on each thread the correction

$$I = \frac{i}{F} \qquad (149)$$

(where i is the equatorial interval of a thread from the middle thread), and to the mean of the results we must apply also the correction $\frac{c}{F}$ to reduce to the collimation axis. Let the resulting time, reduced for rate to the assumed epoch T_0, be denoted by (T). Then, if Θ_0 is the true sidereal time at the same instant, we have

$$\Theta_0 = (T) + \Delta T_0$$

and, by Art. 167,

$$t = \Theta_0 - \alpha + m$$

whence we derive*

$$\Delta T_0 = \alpha - (T) + t - m \qquad (150)$$

If we wish to take into account the diurnal aberration, we must add to the right ascension of each star the correction $0^s.021 \cos\varphi \sec\delta \cos\tau$.

171. In the above, we have supposed c to be given. To investigate the effect of an error in the assumed value of c, let $c + \Delta c$

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* It is easily seen that the general formula (150) reduces to HANSEN's formula (86) when the instrument is in the meridian.

be its true value; then the correction of n corresponding to Δc is, by (145),

$$\Delta n = -\Delta c \frac{\sin\frac{1}{2}(w' + w)}{\cos\frac{1}{2}(w' - w)}$$

and, by differentiating the expressions (147), (148), and (149), we find the corresponding corrections of m, t, and I to be

$$\Delta m = -\Delta n \cdot \frac{\tan\varphi}{\cos^2 n \cos m} = \Delta c \cdot \frac{\sin\frac{1}{2}(w' + w)\tan\varphi}{\cos\frac{1}{2}(w' - w)\cos^2 n \cos m}$$

$$\Delta t = -\Delta n \cdot \frac{\tan\delta}{\cos^2 n \cos t} = \Delta c . \frac{\sin\frac{1}{2}(w' + w)\tan\delta}{\cos\frac{1}{2}(w' - w)\cos^2 n \cos t}$$

$$\Delta I = \frac{\Delta c}{\cos\delta \cos n \cos t}$$

The correction of the quantity $(T) - t + m$ will be composed of the corrections of I (by which (T) is obtained), of m, and of t. Denoting the whole correction by $\Delta\tau$, we have

$$\Delta\tau = \Delta I - \Delta t + \Delta m$$

Substituting the values of the corrections, we find

$$\Delta\tau = \frac{\Delta c}{\cos n}\left[\frac{1}{\cos w} - \frac{\sin\frac{1}{2}(w' + w)\tan w}{\cos\frac{1}{2}(w' - w)} + \frac{\sin\frac{1}{2}(w' + w)\tan\varphi}{\cos\frac{1}{2}(w' - w)\cos n \cos m}\right]$$

By observing that $\frac{1}{2}(w' - w) = \frac{1}{2}(w' + w) - w$, the first two terms within the parentheses become

$$\frac{\cos\frac{1}{2}(w' - w) - \sin\frac{1}{2}(w' + w)\sin w}{\cos\frac{1}{2}(w' - w)\cos w} = \frac{\cos\frac{1}{2}(w' + w)}{\cos\frac{1}{2}(w' - w)}$$

whence

$$\Delta\tau = \frac{\Delta c}{\cos n \cos\frac{1}{2}(w' - w)}\left[\cos\frac{1}{2}(w' + w) + \sin\frac{1}{2}(w' + w)\frac{\tan\varphi}{\cos n \cos m}\right]$$

Finally, if we put

$$\tan\varphi' = \frac{\tan\varphi}{\cos n \cos m} \tag{151}$$

the expression becomes*

$$\Delta\tau = \Delta c \cdot \frac{\cos\left[\frac{1}{2}(w' + w) - \varphi'\right]}{\cos n \cos\varphi' \cos\frac{1}{2}(w' - w)} \tag{152}$$

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* As given by HANSEN, *Astr. Nach.*, Vol. XLVIII. p. 120.

If we denote the coefficient of Δc in this equation by C, and the true chronometer correction by ΔT, the first computed correction being (ΔT), we have

$$\Delta T = (\Delta T) - C \Delta c \tag{153}$$

For another observation in the reversed position of the axis the coefficient of Δc computed by (152) being denoted by C', and the computed chronometer correction by $(\Delta T')$, we have, since the sign of Δc is changed,

$$\Delta T = (\Delta T') + C' \Delta c \tag{154}$$

and, combining the two results, we can determine both ΔT and Δc. If we have taken a number of stars in each position, we can treat all the equations of this kind by the method of least squares.

172. The designation "equatorial star," in the preceding explanations, has been used to designate the star from which the chronometer correction has been deduced; but it is by no means necessary that this star should be very near the equator. A star which passes near the zenith will be preferable, since an error in the determination of n will then have little or no effect upon the computed time.

EXAMPLE.*—In 1843, August 17, at Cronstadt, latitude $\varphi = 59° 59'.5$, the following observations were taken. The value of one division of the level was $0^s.113$. The correction for inequality of pivots was $p = +0^s.14$ for *circle west.* The equatorial intervals of the threads, numbered from the circle end of the axis, were

i_1	i_2	i_4	i_5
$+34^s.50$	$+18^s.74$	$-16^s.14$	$33^s.33$

The assumed collimation constant was $c = -0^s.33$ for *circle west.* The chronometer correction was approximately $\Delta T = +40^s$; its losing rate, $1^s.72$, or $\delta T = +1^s.72$ daily.

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* SAWITSCH, *Pract. Astron.*, Vol. I. p. 348.

1st position of the instrument: *Circle West.*

		E.	W.	
Level.	Direct	-12.0	$+27.0$	$B = +0^s.52$
	Reversed	-17.8	$+21.2$	$p = +0.14$

Mean $B = +4^d.6$ $\qquad b = +0.66$

Transits observed with chronometer "Haut No. 19."

Thread.	I	II	III	IV	V
α Urs. Min.	—	—	$17^h\ 23^m\ 10^s.0$	—	—
β Draconis	$38^s.0$	$3^s.9$	17 28 35.0	$1^s.4$	$29^s.3$

		E.	W.	
Level.	Direct	-18.0	$+21.0$	$B = +0^s.49$
	Reversed	-12.4	$+26.8$	$p = +0.14$

Mean $B = +4^d.35$ $\qquad b = +0.63$

2d position: *Circle East.*

		E.	W.	
Level.	Direct	-18.4	$+21.0$	$B = +0^s.24$
	Reversed	-17.4	$+23.1$	$p = -0.14$

Mean $B = +2^d.08$ $\qquad b = +0.10$

Thread.	V	IV	III	II	I
α Urs. Min.	—	—	$17^h\ 52^m\ 45^s.5$	—	—
γ Draconis	$8^s.1$	$35^s.8$	17 55 1.4	$31^s.6$	$57^s.1$

		E.	W.	
Level.	Direct	-16.2	$+23.6$	$B = +0^s.30$
	Reversed	-18.3	$+21.5$	$p = -0.14$

Mean $B = +2^d.65$ $\qquad b = +0.16$

For the given date we find, from the Nautical Almanac,

	α	δ
α *Urs. Min.*	$1^h\ 3^m\ 45^s.70$	$88^\circ\ 28'\ 24''.2$
β *Draconis,*	$17\ \ 26\ \ 55\ .73$	$52\ \ 25\ \ 25\ .5$
γ *Draconis,*	$17\ \ 53\ \ \ 0\ .35$	$51\ \ 30\ \ 51\ .0$

Computation of the observations, *circle west.*—We shall reduce the observed times for the chronometer rate to the common epoch $T_0 = 18^h$. To allow for the diurnal aberration, we take for the approximate times of the observation of α *Ursæ Minoris* and β *Draconis*, $17^h\ 24^m$ and $17^h\ 29^m$, which, subtracted from the respective right ascensions, give for their eastern hour angles, or the values of τ, $7^h\ 40^m$ and $-\ 0^h\ 2^m$, and hence the values of $0^s.021 \cos\varphi \sec\delta \cos\tau$ for the two stars are $-\ 0^s.17$ and $+\ 0^s.02$, which are to be added to the right ascensions. The corrected quantities are then:

$$\begin{array}{lrlrl}
\alpha\ \textit{Urs. Min.} & \alpha' = & 1^h\ \ 3^m\ 45^s.53 & T' = 17^h\ 23^m\ \ 9^s.96 & \delta' = \ \ 88^\circ\ 28'\ 24''.2 \\
\beta\ \textit{Draconis,} & \alpha\ = & \underline{17\ \ 26\ \ 55\ .75} & T\ = \underline{17\ \ 28\ \ 34\ .96} & \delta\ = \underline{\ \ 52\ \ 25\ \ 25\ .5} \\
 & \alpha' - T' = & 7\ \ 40\ \ 35\ .57 & & \delta' + \delta = 140\ \ 53\ \ 49\ .7 \\
 & \alpha\ - T\ = & \underline{-\ \ \ \ 1\ \ 39\ .21} & & \delta' - \delta = \ \ 36\ \ \ 2\ \ 58\ .7 \\
 & 2\gamma = & 7\ \ 42\ \ 14\ .78 = 115^\circ\ 33'\ 41''.7 & & \\
 & & \gamma = \ \ 57^\circ\ 46'\ 50''.9 & &
\end{array}$$

Hence, by the formulæ (143) and (144),

$$\begin{array}{rlrlrl}
\log\sin(\delta'+\delta) & 9.799833 & \log\sin(\delta'-\delta) & 9.769736 & \log\cos\delta' & 8.425554 \\
\log\cos\gamma & \underline{9.726857} & \log\sin\gamma & \underline{9.927378} & \log\cos\delta & 9.785199 \\
\log L\sin\lambda & 0.072976 & \log L\cos\lambda & 9.842358 & \log 2 & \underline{0.301030} \\
\log\tan\lambda & 0.230618 & \log\cos\lambda & \underline{9.704899} & & 8.511783 \\
\lambda = & 59^\circ\ 32'\ 39''.2 & \log L & 0.137459 & & \\
 & & \log 2\cos\delta'\cos\delta & \underline{8.511783} & & \\
n' = & +\ 1^\circ\ 21\ \ 22''.8 & \log\tan n' & 8.374324 & &
\end{array}$$

By the formulæ (145) and (146),

$$\begin{array}{rlrl}
\lambda + \gamma = & 117^\circ\ 20' & \lambda - \gamma = & 1^\circ\ 46' \\
\log\sec(\lambda+\gamma) & n0.3380 & \log\sec(\lambda-\gamma) & 0.0002 \\
\log\sec n' & 0.0001 & \log\sec n' & 0.0001 \\
\log\tan\delta' & \underline{1.5743} & \log\tan\delta & \underline{0.1138} \\
\log\tan w' & n1.9124 & \log\tan w & 0.1141 \\
w' = & 90^\circ\ 42' & w = & 52^\circ\ 27' \\
\tfrac{1}{2}(w'+w) = & 71\ \ \ 35 & \tfrac{1}{2}(w'-w) & 19\ \ \ 8 \\
 & & \log\sin\tfrac{1}{2}(w'+w) & 9.9772 \\
 & & \log\sec\tfrac{1}{2}(w'-w) & 0.0247 \\
c = -\ 0^s.33 = -\ 4''.95 & & \log c & \underline{n0.6946} \\
\nu = -\ 4\ .97 & & \log\nu & n0.6965 \\
n' - \nu = n = +\ 1^\circ\ 21'\ 27''.8 & & &
\end{array}$$

By the formulæ (147):

$$\begin{array}{lr} \log(-\tan n) & n8.374769 \\ \log\tan\varphi & \underline{0.238415} \\ \log\sin\mu & n8.613184 \\ \mu = & -\ 2^\circ\ 21'\ \ 7''.0 \\ \beta = & \underline{+\ 19\ .4} \\ m = & -\ 2\ \ 20\ \ 47\ .6 \end{array} \qquad \begin{array}{lr} \multicolumn{2}{r}{b = +\ 0^s.645 = +\ 9''.68} \\ \log b & 0.9859 \\ \log\sec n & 0.0001 \\ \log\sec\mu & 0\ 0004 \\ \log\sec\varphi & \underline{0.3009} \\ \log\beta & 1.2873 \end{array}$$

The constants of the instrument being thus found, we proceed to find the chronometer correction by *β Draconis*. We first find t and the thread intervals by (148) and (149):

$$t = -\ 1^\circ\ 45'\ 54''.6 \qquad \begin{array}{lr} \log\tan n & 8.374769 \\ \log\tan\delta & \underline{0.113823} \\ \log\sin t & n8.488592 \end{array} \qquad \begin{array}{lr} \log\cos n & 9.99988 \\ \log\cos\delta & 9.78520 \\ \log\cos t & \underline{9.99979} \\ \log F & 9.78487 \end{array}$$

$$\begin{array}{lrrrr} & \text{I} & \text{II} & \text{IV} & \text{V} \\ \log i & 1.53782 & 1.27277 & n1.20790 & n1.52284 \\ \log I & 1.75295 & 1.48790 & n1.42303 & n1.73797 \\ I & +\ 56.62 & +\ 30^s.75 & -\ 26^s.49 & -\ 54^s.70 \end{array} \qquad \begin{array}{lr} \multicolumn{2}{r}{c = -0^s.33} \\ \log c & n9.518 \\ \log\dfrac{c}{F} & n9.733 \\ \multicolumn{2}{r}{\dfrac{c}{F} = -0^s.54} \end{array}$$

Applying these reductions, we have, for the time of passage over the middle thread, and the chronometer correction by (150),

$$\begin{array}{rrrr} & \multicolumn{3}{c}{\beta\ \textit{Draconis.}} \\ & 17^h & 28^m & 34^s.62 \\ & & & 34\ .65 \\ & & & 35\ .00 \\ & & & 34\ .91 \\ & & & \underline{34\ .60} \\ & 17 & 28 & 34\ .76 \\ \dfrac{c}{F} = & & - & 0\ .54 \\ \text{Red. for rate to } 18^h = & & - & \underline{0\ .04} \\ (T) = & 17 & 28 & 34\ .18 \\ \alpha = & 17 & 26 & \underline{55\ .75} \\ \alpha - (T) = & - & 1 & 38\ .43 \\ t - m = +\ 0^\circ\ 34'\ 53''.0 = & + & 2 & \underline{19\ .53} \\ \text{Chron. correction at } 18^h = \Delta T_0 = & + & & 41\ .10 \end{array}$$

Computation of the observations, *circle east.*—This being in all respects similar to the above, we shall only put down the principal results. The approximate hour angles (τ) of α *Ursæ Minoris* and γ *Draconis* are $7^h\ 10^m$ and $-\ 0^h\ 3^m$, whence the correction of the right ascensions for diurnal aberration are $-\ 0^s.12$ and $+\ 0^s.02$. Reducing the times for rate to 18^h, we find

α *Urs. Min.* $\alpha' = 1^h\ 3^m\ 45^s.58$ $\quad T' = 17^h\ 52^m\ 45^s.49$ $\quad \delta' = 88°\ 28'\ 24''.2$

γ *Draconis* $\alpha = 17\ \ 53\ \ 0.37$ $\quad T = 17\ \ 55\ \ 1.39$ $\quad \delta = 51\ \ 30\ \ 51\ .0$

whence

$\gamma = 54°\ 7'\ 38''.3$

$n' = +\ 1\ \ 26\ \ 2\ .5$

$\lambda + \gamma = 110°\ 4'$

$w' = 90\ \ 31$

$\nu = +\ 5''.0$

$n = +\ 1°\ 25'\ 57''.5$

$\mu = -\ 2\ \ 28\ \ 54\ .7$

$m = -\ 2\ \ 28\ \ 50\ .8$

$t = -\ 1\ \ 48\ \ 9\ .6$

$\lambda = 55°\ 55'\ 54.''2$

$c = +\ 0^s.33 = +\ 4''.95$

$\lambda - \gamma = 1°\ 48'$

$w = 51\ \ 32$

$b = +\ 0^s.13 = +\ 1''.95.$

$\beta = +\ 3''.9$

$\log F = 9.79366$

For the reductions of the threads for γ *Draconis*, we find

	V	IV	II	I
I	$+\ 53^s.60$	$+\ 25^s.96$	$-\ 30^s.14$	$-\ 55^s.48$

$\frac{c}{F} = +\ 0^s.53$

and hence

γ *Draconis.*

Transit over middle thread $= 17^h\ 55^m\ 1^s.59$

$\frac{c}{F} = +\ 0.53$

Red. for rate to $18^h = -\ 0.01$

$(T) = 17\ \ 55\ \ 2.11$

$\alpha = 17\ \ 53\ \ 0.37$

$\alpha - (T) = -\ 2\ \ 1.74$

$t - m = +\ 2\ \ 42.75$

$\Delta T_0 = +\ 41.01$

The mean value derived from the observations in both positions of the instrument is, therefore,

$$\Delta T_0 = +\ 41^s.06 \text{ at } 18^h.$$

In general, however, unless the declinations of the two stars are nearly equal, the true value of ΔT_0 will not be the mean of the values found in the two positions; but we shall have to proceed as follows.

To estimate the effect of an error in the assumed value of c in this computation, we might here put $\varphi' = \varphi$ in (152), since n and m are here small; but, for the sake of illustration, we shall use the complete formulæ. We find

	Circle West.		*Circle East.*
$\varphi' =$	$60^\circ\ 1'.2$		$60^\circ\ 1'.4$
$\frac{1}{2}(w' + w) - \varphi' =$	$11\ \ 34$		$11\ \ 0$
$\log \cos [\frac{1}{2}(w' + w) - \varphi']$	9.9911		9.9919
$\log \sec \frac{1}{2}(w' - w)$	0.0247		0.0257
$\sec n$	0.0001		0.0001
$\sec \varphi'$	0.3013		0.3013
$\log C$	0.3172	$\log C'$	0.3190
$C =$	$+ 2.075$	$C' =$	$+ 2.084$

Hence

$$\begin{array}{ll} \text{(Circle west)} & \Delta T_0 = +\ 41^s.10 - 2.075\,\Delta c \\ \text{(Circle east)} & \Delta T_0 = +\ 41\ .01 + 2.084\,\Delta c \end{array}$$

whence

$$\Delta c = + \frac{0^s.09}{4.159} = +\ 0^s.0216$$

$$\begin{array}{ll} \text{(Circle west)} & \Delta T_0 = +\ 41^s.10 - 0^s.04 = +\ 41^s.06 \\ \text{(Circle east)} & \Delta T_0 = +\ 41\ .01 + 0\ .05 = +\ 41\ .06 \end{array}$$

This result agrees with the mean value found before, because here the declinations of the stars were nearly equal, and the position of the instrument with respect to the meridian was nearly the same in both observations.

As the value of c is often but imperfectly known, it will be best always to take a pair of stars in each position of the axis, and then to compute the two clock corrections upon the supposition of $c = 0$. The true correction will then be found by computing $C\Delta c$ as above, and the value of Δc will be the true value of c. Thus, in the preceding example, if we had first taken $c = 0$, we should have found from β *Draconis* $(\Delta T) = +\ 40^s.42$, and from γ *Draconis* $(\Delta T') = +\ 41^s.70$, and, computing the coefficients C and C' as above, we should have had

$$\begin{array}{ll} \text{(Circle west)} & \Delta T_0 = +\ 40^s.42 - 2.075\,c \\ \text{(Circle east)} & \Delta T_0 = +\ 41\ .70 + 2.084\,c \end{array}$$

whence

$$c = \frac{-1^s.28}{4.159} = -\ 0^s.308$$

$$\begin{array}{ll} \text{(Circle west)} & \Delta T_0 = +\ 40^s.42 + 0^s.64 = +\ 41^s.06 \\ \text{(Circle east)} & \Delta T_0 = +\ 41\ .70 - 0\ .64 = +\ 41\ .06 \end{array}$$

APPLICATION OF THE METHOD OF LEAST SQUARES TO THE DETERMINATION OF THE TIME WITH A PORTABLE TRANSIT INSTRUMENT IN THE VERTICAL CIRCLE OF A CIRCUMPOLAR STAR.

173. We here suppose the observations to be made essentially as directed in Art. 170, with this difference, however, that we shall not restrict the observation of the star near the pole to its transit over the middle thread. The instrument being brought near the vertical of a circumpolar star: 1st, the transit of this star over *any one* of the threads is observed; 2d, the transits of a number of equatorial stars are observed; 3d, the axis of the instrument is reversed, and the transit of the polar star again observed over one thread; and 4th, the transits of a number of equatorial stars are observed. The level is read for each star. If, however, the circumpolar star has passed all the threads by the time the axis has been reversed, the azimuth of the instrument must be changed, so as to bring the star near a thread; then, clamping the instrument in azimuth, the transit over this thread will be observed, and also the transits of a set of equatorial stars as before. In this case the observations, being made in two different vertical circles, must be separately computed according to the following method. It is hardly necessary to observe that the observations of the equatorial stars may either precede or follow that of the circumpolar star, as may happen to be most convenient. In this method, we form an equation of condition from the observation of each star, and all those for which the azimuth of the instrument is the same are combined by the method of least squares.

Let c denote the collimation constant for the mean of the threads, and i the equatorial distance of a thread from the mean; then, τ denoting the hour angle of the star when observed on the thread, $i + c$ must be substituted for c in our fundamental equation (79); and, since this quantity is always sufficiently small, we shall put it in the place of its sine. Thus, we have for each thread

$$c + i = -\sin n \sin \delta + \cos n \cos \delta \sin (\tau - m)$$

When several threads are observed, the mean of the observed times corresponds to that point of the field which we call the mean of the threads only when the instrument is in the meridian. When the instrument is not in the meridian, two methods of procedure offer themselves. The first is that which has been used in the preceding articles, and consists in reducing each thread either to the middle or the mean thread by means of the computed intervals. But to compute these intervals we must, as has been seen, know the position of the instrument. The second method, which we owe to BESSEL, is not only more simple in practice, but is wholly independent of the position of the instrument; and, as it will be useful both in the present problem and in that of finding the latitude by transits over the prime vertical, I shall treat of it here.

If we denote the number of observed threads by q, we have q equations of the above form, i and τ being different in each. The mean of these equations is

$$c + \frac{1}{q}\Sigma i = -\sin n \sin \delta + \cos n \cos \delta \, \frac{1}{q}\Sigma \sin (\tau - m)$$

where Σ is the usual summation sign. Now let

$T =$ the mean of the observed times on the several threads,
$T - I =$ the observed time on any thread;

then I is the interval found by subtracting each observed time from the mean of all, and, consequently, the algebraic sum of all these intervals is zero. Also let

$\vartheta =$ the clock correction,
$t = \alpha - (T + \vartheta)$

then for each thread we have

$$\tau = \alpha - (T - I + \vartheta) = t + I$$

$$\sin (\tau - m) = \sin (t - m + I) = \sin (t - m) \cos I + \cos (t - m) \sin I$$

$$\frac{1}{q}\Sigma \sin (\tau - m) = \sin (t - m) \frac{1}{q}\Sigma \cos I + \cos (t - m) \frac{1}{q}\Sigma \sin I$$

Let k and $\varkappa$ be determined by the conditions

$$\frac{1}{k}\cos \varkappa = \frac{1}{q}\Sigma \cos I$$

$$\frac{1}{k}\sin \varkappa = -\frac{1}{q}\Sigma \sin I$$

then we have

$$\frac{1}{q}\Sigma \sin (\tau - m) = \frac{1}{k}\sin (t - \varkappa - m)$$

Hence, putting

$$\left.\begin{aligned} \tau_1 &= t - x = \alpha - (T + x + \vartheta) \\ i_0 &= \frac{1}{q}\,\Sigma i \end{aligned}\right\} \quad (155)$$

our equation becomes

$$c + i_0 = -\sin n \sin \delta + \frac{\cos n \cos \delta \sin (\tau_1 - m)}{k}$$

Thus, x and k being found, we find τ_1 by using the corrected time $T + x$ instead of T, as in (155), and then this single equation represents the mean of the q equations. We may bring this equation still nearer in form to that for each thread, by substituting

$$\gamma \cos \delta_1 = \frac{1}{k} \cos \delta$$

$$\gamma \sin \delta_1 = \sin \delta$$

which give

$$\frac{c + i_0}{\gamma} = -\sin n \sin \delta_1 + \cos n \cos \delta_1 \sin (\tau_1 - m) \qquad (156)$$

where γ is so nearly equal to unity (as will presently appear) that, as the divisor of the small term $c + i_0$, it may usually be omitted. Thus, the mean equation is precisely of the form for one thread, when we use both a corrected mean time and a corrected declination. The quantities x and δ_1, or else x and log k, are readily found by the aid of tables such as Tables VIII. and VIII.A at the end of this volume, the construction of which is as follows. The equations which determine k and x may be written thus:

$$\frac{1}{k} \cos x = 1 - \frac{1}{q}\,\Sigma\, 2 \sin^2 \tfrac{1}{2} I$$

$$\frac{1}{k} \sin x = \frac{1}{q}\,\Sigma (I - \sin I)$$

for, since $\Sigma I = 0$, this last equation is the same as the one before given. But the quantity $I - \sin I$ is of the order I^3, and therefore extremely small, so that we may put $\cos x = 1$, and hence

$$\frac{1}{k} = 1 - \frac{1}{q}\,\Sigma\, 2 \sin^2 \tfrac{1}{2} I$$

$$x = \frac{1}{q}\,\Sigma (I - \sin I)$$

and since

$$\tan \delta_1 = k \tan \delta$$

we have*

$$\delta_1 = \delta + \frac{k-1}{k+1} \cdot \frac{\sin 2\delta}{\sin 1''} + \left(\frac{k-1}{k+1}\right)^2 \frac{\sin 4\delta}{2 \sin 1''} + \&c.$$

or, substituting the value of k,

$$\delta_1 = \delta + \frac{\frac{1}{q} \Sigma \sin^2 \frac{1}{2} I}{1 - \frac{1}{q} \Sigma \sin^2 \frac{1}{2} I} \cdot \frac{\sin 2\delta}{\sin 1''} + \&c.$$

Bessel gives† a table from which with the argument I we find $I - \sin I$ in seconds, and $\frac{\sin^2 \frac{1}{2} I}{\sin 1''}$. The means of the tabular quantities taken for the several values of I are respectively $\varkappa$ and the numerator of the coefficient of 2δ. A small subsidiary table corrects for the neglect of the denominator. In the tables at the end of this volume I have adopted a different arrangement. By the logarithmic formula

$$\log (1 - x) = - M (x + \tfrac{1}{2} x^2 + \&c.)$$

in which $M = 0.4342945$, we find

$$\log k = - \log \frac{1}{k} = M \left[\frac{1}{q} \Sigma\, 2 \sin^2 \tfrac{1}{2} I + \tfrac{1}{2} \left(\frac{1}{q} \Sigma\, 2 \sin^2 \tfrac{1}{2} I \right)^2 + \&c. \right]$$

where the second term of the series will mostly be inappreciable. The approximate value of $\log k$, neglecting this term, will be

$$\log k = \frac{1}{q} \Sigma\, 2 M \sin^2 \tfrac{1}{2} I$$

and, employing this value in the second term, the complete value will be

$$\log k = \frac{1}{q} \Sigma\, 2 M \sin^2 \tfrac{1}{2} I + \frac{(\log k)^2}{2 M}$$

Table VIII. gives, in the column $\log k$, the value of $2M \sin^2 \frac{1}{2} I$ corresponding to each interval I. The mean value of $\log k$, which is required in reducing several threads, will be found by taking the mean of the several values from the table. When

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[*] Pl. Trig., Art. 254.

[†] *Astron. Nach.*, Vol. VI. p. 245.

extreme precision is desired, this mean is to be increased by the small correction given in Table VIII.A, which contains the value of the term $\frac{(\log k)^2}{2M}$, with the argument "mean log k." The column marked x gives the value of $I -$ sin I in seconds for each value of I; and the mean of the several values is likewise to be taken as the correction of the mean of the observed times T. The sign of I is different for threads on opposite sides of the mean, and the sign of x must be the same as that of I. Hence the mean x will be evanescent when the observed threads are symmetrically disposed about the mean.

These tables, then, effect the reduction of the threads to a single instant in a remarkably simple manner, without requiring a previous knowledge of the position of the instrument. We have only to add x to the mean of the observed times, and to find the corrected declination by the formula

$$\tan \delta_1 = k \tan \delta \tag{157}$$

Then, taking the mean of the equatorial intervals i of the observed threads, we proceed to use equation (156), as representing the mean of all the threads. The divisor γ is found, from the equations which determine γ and δ_1, to be

$$\gamma = \frac{1 - \left(1 - \frac{1}{k}\right)\cos^2 \delta}{\cos(\delta_1 - \delta)}$$

where we may put $\cos(\delta_1 - \delta) = 1$. Since i_0 is zero when all the threads are observed, we may put $\gamma = 1$ in such cases without hesitation, since it is then the divisor only of the very small quantity c. But in the method of observation here adopted we may in all cases put $\gamma = 1$; for we suppose the slow-moving star to be observed on but one thread, in which case we have rigorously $\gamma = 1$; and for the equatorial star (even if we extend this denomination to stars of the declination 50° or 60°) the intervals I will always be less than 2^m, and then the mean log k will always be less than 0.00001, and log γ will be less than 0.00002. We take then, as complete, the equation

$$c + i_0 = -\sin n \sin \delta_1 + \cos n \cos \delta_1 \sin(\tau_1 - m)$$

Substituting $\sin \tau_1 \cos m - \cos \tau_1 \sin m$ for $\sin(\tau_1 - m)$ and then substitut he values of sin n, cos n cos m, cos n sin m, from (78) the e becomes

$$\begin{aligned} c + i_0 = &- b(\sin\varphi\sin\delta_1 + \cos\varphi\cos\delta_1\cos\tau_1) + \cos a\cos\delta_1\sin\tau_1 \\ &+ \sin a(\cos\varphi\sin\delta_1 - \sin\varphi\cos\delta_1\cos\tau_1) \end{aligned}$$

This equation will be satisfied when a is the true value of the azimuth of the instrument and τ_1 has been found by employing the true clock correction ϑ. But, if a and ϑ denote assumed approximate values of these quantities, Δa and $\Delta\vartheta$ their required corrections, and if τ_1 is found by the formula

$$\tau_1 = \alpha - (T_1 + \vartheta) \tag{158}$$

then we must substitute in ab equation $a + \Delta a$ for a, and $\tau_1 - \Delta\vartheta$ for τ_1. We thus n ecting the products of the small quantities b, Δa, and $\Delta\vartheta$,

$$\begin{aligned} c + i_0 = &-b(\sin\varphi\sin\delta_1 + c \quad s \quad \cos\tau_1) \\ &+ \cos a\cos\delta_1\sin \quad a(\quad s\,\varphi\sin\delta_1 - \sin\varphi\cos\delta_1\cos\tau_1) \\ &- \Delta a\sin a\cos\delta_1 si \quad a\cos \quad (\cos\varphi\sin\delta_1 - \sin\varphi\cos\delta_1\cos\tau_1) \\ &- \Delta\vartheta\cos\delta_1(\cos a\cos \quad n\, a\sin\varphi\sin\tau_1) \end{aligned}$$

To adapt this for con nd A be the zenith distance and azimuth of the here whose declination is δ_1 and hour angle τ_1: then we have (Vol. I. Art. 14)

$$\left.\begin{aligned} \cos z &= \sin\varphi\sin\delta_1 + \cos\varphi\cos\delta_1\cos\tau_1 \\ \sin z\cos A &= -\cos\varphi\sin\delta_1 + \sin\varphi\cos\delta_1\cos\tau_1 \\ \sin z\sin A &= \cos\delta_1\sin\tau_1 \end{aligned}\right\} \tag{159}$$

and our equation becomes

$$\begin{aligned} c + i_0 = &- b\cos z - \sin(a - A)\sin z - \Delta a\cos(a - A)\sin z \\ &- \Delta\vartheta\cos\delta_1(\cos a\cos\tau_1 + \sin a\sin\varphi\sin\tau_1) \end{aligned}$$

Here $a - A$ must be of the same order as $c + i_0$, and therefore may also be put for its sine, and its cosine may be put $= 1$. In the coefficient of $\Delta\vartheta$ we may put $\cos\delta$ for $\cos\delta_1$. Transposing the equation, and collecting the known terms, by putting

$$h = i_0 + b\cos z + (a - A)\sin z \tag{160}$$

we obtain the equation of condition

$$c + \Delta a\sin z + \Delta\vartheta\cos\delta(\cos a\cos\tau_1 + \sin a\sin\varphi\sin\tau_1) + h = 0 \tag{161}$$

in which the sign of c must be changed when the axis of the instrument is reversed. It must also be observed that, (as in meridian observations where $z = \varphi - \delta$), $\sin z$ must be negative when the star is north of the zenith: this sign, however, will be given by the equations (159) if attention is paid to the signs of the other quantities. To compute z and A by logarithms, let g and G be determined by the conditions

$$\begin{aligned} g \sin G &= \sin \delta_1 \\ g \cos G &= \cos \delta_1 \cos \tau_1 \end{aligned}$$

then

$$\begin{aligned} \cos z &= g \cos (\varphi - G) \\ \sin z \cos A &= g \sin (\varphi - G) \\ \sin z \sin A &= \cos \delta_1 \sin \tau_1 \end{aligned}$$

or (observing that $\tan \delta_1 = k \tan \delta$)

$$\left.\begin{aligned} \tan G &= \frac{k \tan \delta}{\cos \tau_1} \\ \tan A &= \frac{\tan \tau_1 \cos G}{\sin (\varphi - G)} \\ \tan z &= \frac{\tan (\varphi - G)}{\cos A} \end{aligned}\right\} \qquad (162)$$

in which G and A are to be taken less than 90°, positive or negative according to the sign of their tangents, and the sign of $\tan z$ will be determined by that of $\tan (\varphi - G)$.

If we put

$$\tan F = \tan \tau_1 \sin \varphi \qquad (163)$$

the coefficient of $\Delta\vartheta$ may be computed under the form

$$P = \frac{\cos \delta \cos \tau_1 \cos (a - F)}{\cos F} \qquad (164)$$

The whole process of forming the equation of condition for each star is, therefore, as follows:

1st. Find x and $\log k$ from Table VIII., and add x to the mean of the observed times on the several threads. Call the resulting time T_1, and find

$$\tau_1 = \alpha - (T_1 + \vartheta)$$

in which ϑ is the *assumed* clock correction reduced to the time T_1.

2d. Compute A, z, P by the equations (162), (163), and (164), and h by the equation

$$h = i_0 + b \cos z + (a - A) \sin z$$

in which i_0 is the mean of the equatorial intervals of the observed threads from the mean thread, b is the inclination of the rotation axis, and a is the assumed azimuth of the instrument.

Then the equation of condition is

$$\pm c + \Delta a \sin z + P . \Delta\vartheta + h = 0$$

in which the sign of c is to be determined by the position of the rotation axis of the instrument.

From all the equations thus formed, the most probable values of c, Δa, and $\Delta\vartheta$ will be found by the method of least squares.

If the azimuth of the instrument has been changed during the observations, these must be divided into two sets, and two different assumed azimuths a, a', with the corrections Δa and $\Delta a'$, will be used in the formation of the equations.

It is hardly necessary to remark that all the quantities i_0, b, $a - A$, c, Δa, $\Delta\vartheta$ are expressed in the same unit, either of time or arc: the latter will perhaps be most convenient.

Example.—The following observations were taken by Bessel with a very small portable instrument, to determine the time.

Munich, 1827, June 27.

Circle East.	I	II	III	IV	V	Level.
χ Scorpii	$8^m 12^s.2$	$7^m 52^s.5$	$11^h \ldots^m \ldots^s.$	....	....	$-1^d.080$
ε Ophiuchi	14 22 .4	14 2 .6	11 13 43 .2	$13^m 22^s.7$	$13^m 1^s.6$	$-0 .608$
α Ursæ Minoris	....	....	11	20 3 .2	....	$-0 .079$
Circle West.						
α Ursæ Minoris	....	....	$13^h 19^m 52^s.8$	....	....	$+1^d.583$
* *a (Anon.)*	$21^m 35^s.5$	$21^m 56^s.2$	13 22 16 .2	$22^m 37^s.0$	$22^m 58^s.8$	$+1 .67$
24 *Scuti Sob.*	26 11 .4	26 31 .6	13 26 52 .3	27 12 .8	27 34 .4	$+1 .83$

The azimuth of the instrument was changed between the tw sets of observations, *circle east* and *circle west*.

The place of observation was in the garden of Dr. Steinheil house, where the latitude was $\varphi = 48° 8' 40''$.

The chronometer was a pocket mean time chronometer o

KESSEL. Its correction to sidereal time at 12^h (chronometer time) was assumed to be $\vartheta = 5^h\ 1^m\ 3^s.00$, and its rate on sidereal time was $+\ 9^s.19$ per hour (losing).

The equatorial intervals of the threads from the mean thread were as follows for *circle west:*

I	II	III	IV	V
$+ 598''.08$	$+ 303''.09$	$+ 6''.19$	$- 294''.91$	$- 612''.46$

The value of one division of the level was $4''.49$. The pivots were of unequal thickness, the correction for which had previously been found to be $- 1''.89$ for circle west.

The apparent places of the stars on the given date were as follows:

	α	δ
χ *Scorpii*	$16^h\ 2^m\ 36^s.71$	$-\ 9°\ 36'\ 34''.2$
ε *Ophiuchi*	16 9 13 .90	− 4 16 8 .9
α *Ursæ Minoris*	0 59 5 .28	+ 88 23 2 .5
* *a (Anon.)*	18 18 8 .49	+ 14 52 36 .7
24 *Scuti Sob.*	18 19 24 .11	− 14 39 56 .0

The reduction of the observations of χ *Scorpii* and ε *Ophiuchi* on the several threads to a mean will serve to illustrate the mode of using our Table VIII., although in this case the quantity x is quite insensible and log k nearly so. We have, then,

Circle East.	T	I	x	log k	i
χ *Scorpii* I.	$11^h\ 8^m\ 12^s.2$	$-\ 9^s.85$	0.00	0.0000001	$- 598''.08$
II.	7 52 .5	+ 9 .85	0.00	1	− 303 .09
Means	11 8 2 .35	0 .00	0.00	0.0000001	− 450 .59
ε *Ophiuchi*	11 14 22 .4	$-\ 39^s.90$	0.00	0.0000018	$- 598.''08$
	14 2 .6	− 20 .10		5	− 303 .09
	13 43 .2	− 0 .70		0	− 6 .19
	13 22 .7	+ 19 .80		5	+ 294 .91
	13 1 .6	+ 40 .90	0.00	19	+ 612 .46
Means	11 13 42 .50	0 .00	0.00	0.0000009	0 .00

To form the equations of condition for the three stars observed, circle east, we now find by the formulæ (158, &c.)*

	χ Scorpii.	*ε Ophiuchi.*	*α Ursæ Min.*
$T + x = T_1$	$11^h\ 8^m\ 2^s.35$	$11^h\ 13^m\ 42^s.50$	$11^h\ 20^m\ 3^s.20$
Assumed ϑ	+ 5 1 3 .00	+ 5 1 3 .00	+ 5 1 3.00
Rate to 12^h	— 7 .96	— 7 .09	— 6.12
$T_1 + \vartheta$	16 8 57 .39	16 14 38 .41	16 21 0.08
α	16 2 36 .71	16 9 13 .90	0 59 5.28
τ_1	— 6 20 .68	— 5 24 .51	+ 8 38 5.20
(in arc)	— 1° 35′ 10″.2	— 1° 21′ 7″.65	129° 31′ 18″.0
log sec τ_1	0.000166	0.000121	*n*0.196290
log tan δ	*n*9.228677	*n*8.873022	1.549573
log k	0.000000	0.000001	0.000000
log tan G	*n*9.228843	*n*8.873144	*n*1.745863
G	— 9° 36′ 47″.2	— 4° 16′ 13″.2	— 88° 58′ 17″.3
$\phi - G$	57 45 27 .2	52 24 53 .2	137 6 57 .3
log tan τ_1	*n*8.442337	*n*8.372975	*n*0.083561
log cos G	9.993858	9.998793	8.254067
log cosec $(\phi - G)$	0.072734	0.101030	0.167161
log tan A	*n*8.508929	*n*8.472798	*n*8.504789
log cos A	9.999774	9.999808	9.999778
log tan $(\phi - G)$	0.200130	0.113683	*n*9.967894
log tan z	0.20036	0.11387	*n*9.96812
log sin z	9.92733	9.89904	*n*9.83296
log cos z	9.72697	9.78517	9.86484
A	— 1° 50′ 55″.85	— 1° 42′ 4″.85	— 1° 49′ 52″.74
Assumed a	— 1 42 0 .		
$a - A$	+ 8′ 55″.85	+ 4″.85	+ 7′ 52″.74
b	— 2 .96	— 0 .84	+ 1.54
$(a - A) \sin z$	+ 453″.29	+ 3″.84	— 321″.80
$b \cos z$	— 1 .58	— 0 .51	+ 1 .13
i_0	— 450 .59	0 .00	+ 294 .91
h	+ 1″.12	+ 3″.33	— 25″.76
l. tan τ_1 sin ϕ = l. tan F	*n*8.314394	*n*8.245032	*n*9.955618
F	— 1° 10′ 54″	— 1° 0′ 26″	— 42° 4′ 39″
$a - F$	— 31 6	— 41 34	40 22 39
log cos δ	9.99386	9.99879	8.45025
log cos τ_1	9.99983	9.99988	*n*9.80371
log cos $(a - F)$	9.99998	9.99997	9.88184
log sec F	0.00009	0.00007	0.12946
log P	9.99376	9.99873	*n*8.26526

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* We have neglected the diurnal aberration, as an insensible quantity in observations with so small an instrument.

Hence the equations of condition, circle east, are:

$$\begin{array}{lllll} \chi \textit{ Scorpii} & -c + 0.8459\,\Delta a & + 0.9857\,\Delta\vartheta & + 1''.12 & = 0 \\ \epsilon \textit{ Ophiuchi} & -c + 0.7926\,\Delta a & + 0.9971\,\Delta\vartheta & + 3\,.33 & = 0 \\ \alpha \textit{ Urs. Min.} & -c - 0.6807\,\Delta a & - 0.0184\,\Delta\vartheta & - 25\,.76 & = 0 \end{array}$$

In the same manner, we find for the stars observed, circle west,

	a Ursæ Min.	**a*	*24 Scuti Sob.*
$T_1 + \vartheta$	$18^h\ 21^m\ 8^s.03$	$18^h\ 23^m\ 32^s.34$	$18^h\ 28^m\ 8^s.81$
τ_1	$99^\circ\ 29'\ 18''.75$	$-\ 1^\circ\ 20'\ 57''.75$	$-\ 2^\circ\ 11'\ 10''.5$
$\log k$	0.000000	0.000001	0.000001
$\log \tan A$	n8.617903	n8.618105	n8.618199
$\log \sin z$	n9.82674	9.73943	9.94926
$\log \cos z$	9.87007	9.92217	9.65941
A	$-\ 2^\circ\ 22'\ 32''.22$	$-\ 2^\circ\ 22'\ 36''.20$	$-\ 2^\circ\ 22'\ 38''.05$
Assumed a'	$-\ 2\ \ 22\ \ 40\ .$		
$a' - A$	$-\ \ 7\ .78$	$-\ \ 3\ .80$	$-\ \ 1\ .95$
b	$+\ \ 5\ .22$	$+\ \ 5\ .61$	$+\ \ 6\ .36$
$(a' - A) \sin z$	$+\ \ 5\ .22$	$-\ \ 2\ .09$	$-\ \ 1\ .74$
$b \cos z$	$+\ \ 3\ .87$	$+\ \ 4\ .69$	$+\ \ 2\ .90$
i_a	$+\ \ 6\ .19$	$0\ .00$	$0\ .00$
h	$+\ \ 15\ .28$	$+\ \ 2\ .60$	$+\ \ 1\ .16$
$\log P$	n7.74071	9.98501	9.98544

and hence the equations for these stars are

$$\begin{array}{lllll} \alpha \textit{ Urs. Min.} & +c - 0.6710\,\Delta a' & - 0.0055\,\Delta\vartheta' & + 15''.28 & = 0 \\ {}^*a & +c + 0.5488\,\Delta a' & + 0.9661\,\Delta\vartheta & + 2\,.60 & = 0 \\ 24 \textit{ Scuti Sob.} & +c + 0.8897\,\Delta a' & + 0.9670\,\Delta\vartheta & + 1\,.16 & = 0 \end{array}$$

The six equations involve four unknown quantities, which might be determined from the four normal equations formed in the usual manner. But, where the number of equations is so little greater than that of the unknown quantities, it is not worth while to employ this method. We can here obtain the same result by eliminating Δa from the first set and $\Delta a'$ from the second, and then combining the resulting equations for the determination of c and $\Delta\vartheta$. Thus, substituting the values of Δa

and $\Delta\alpha'$ found from the equations for *α Ursæ Min.* in the e tions of the other two stars in the two groups respectively, have the four equations

χ Scorpii $\quad -2.2427\,c + 0.9629\,\Delta\vartheta - 30''.89 = 0$
ε Ophiuchi $\quad -2.1642\,c + 0.9757\,\Delta\vartheta - 26\ .66 = 0$
**a* $\quad +1.8179\,c + 0.9616\,\Delta\vartheta + 15\ .10 = 0$
24 *Scuti Sob.* $\quad +2.3259\,c + 0.9597\,\Delta\vartheta + 21\ .42 = 0$

from which we derive the normal equations

$$\begin{aligned} 18.4281\,c - 0.2908\,\Delta\vartheta + 204''.25 &= 0 \\ -\ 0.2908\,c + 3.7249\,\Delta\vartheta - \ \ 20\ .68 &= 0 \end{aligned}$$

which give

$$\begin{aligned} \Delta\vartheta &= +\ \ 4''.69 = +\ 0^s.31 \\ c &= -\,11''.01 = -\,0^s.73 \end{aligned}$$

Hence we have, finally,

$$\vartheta = +\,5^h\,1^m\,3^s.31$$

By the four time stars, severally, we have $3^s.43$, $3^s.18$, $3^s.34$, 3

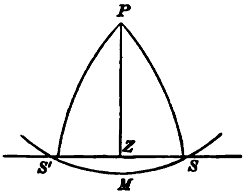
The methods which have here been given, for finding time with a transit instrument out of the meridian, are inten for the use of observers in the field who have but little tim adjust their instruments and wish to collect all the data possi reserving their reduction for a future time. The greater la of these reductions, compared with those of meridian obse tions, is often more than compensated by the saving of time the field.

DETERMINATION OF THE GEOGRAPHICAL LATITUDE BY A TRAN INSTRUMENT IN THE PRIME VERTICAL.

174. The transit instrument is said to be in the prime vert when the great circle described by its collimation axis is in prime vertical. The rotation axis is then perpendicular to plane of the prime vertical, and lies in the intersection of planes of the meridian and horizon. We owe to Bessel the plication of the instrument in this position to the determina of the latitude of the place of observation.

The fundamental principle of the method may be br stated as follows.* Let PZ, Fig. 45, be the meridian; SZS' the prime vertical of the observer; SMS' the diurnal circle of a star which crosses the meridian between the zenith and the equator. Such a star crosses the prime vertical above the horizon at two points S and S' on opposite sides of the zenith and at equal distances from the meridian. If then we observe the transits at these two points with an instrument perfectly adjusted in the prime vertical, and note the times by a clock whose rate is well known, we determine the hour angle $ZPS' = t$, which is equal to one-half the elapsed sidereal time between the two observations; and, therefore, in the right triangle PZS' we know this angle and the hypothenuse $PS' = 90° - \delta$, from which we find the side $PZ = 90° - \varphi$; whence the formula

Fig. 45.



$$\tan \varphi = \tan \delta \sec t$$

in which φ is the latitude. It is evident that only those stars can be observed on the prime vertical whose declinations are between 0 and φ. The nearer the observations to the zenith, that is, the less the difference between the declination and the latitude, the less the effect of errors in the observed times upon the value of sec t, and, consequently, upon the computed latitude.

The advantage of this method of finding the latitude lies chiefly in the facility with which all the instrumental errors may be eliminated by using the instrument alternately in opposite positions of the rotation axis, reversing it, either between the observations on two different stars or between observations of the same star, or using it in one position on one night and in the reverse position on the same stars on another night. Different methods of reduction apply to these several methods of observation, which will be hereafter investigated. We must first show how to place the instrument in or near the prime vertical.

175. *Approximate adjustment in the prime vertical*—The middle thread must be carefully adjusted in the collimation axis, or as nearly so as possible. Then compute the sidereal time of passing the prime vertical for some star whose declination is small,

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* See also Vol. I. Arts. 192 and 193.

that is, a star which passes the prime vertical at a low altitude. If t = the hour angle in the prime vertical, δ = the declination, and φ = the assumed latitude, we have

$$\cos t = \tan \delta \cot \varphi$$

and, if α = the star's right ascension, Θ = the sidereal time of passing the prime vertical,

$$\Theta = \alpha \mp t \begin{Bmatrix} - \text{ for east transit} \\ + \text{ `` west ``} \end{Bmatrix}$$

At this time, therefore, by the clock (allowing for the correction of the clock), bring the middle thread upon the star, observing to keep the rotation axis as nearly horizontal as possible. The zenith distance at which the star will be observed may also be previously computed, to facilitate the finding. For this purpose we have

$$\cos z = \frac{\sin \delta}{\sin \varphi}$$

which gives the true zenith distance, from which we should subtract the refraction in the case of very low stars.

After the instrument has thus been brought near the prime vertical by one star, the rotation axis should be carefully levelled, and the adjustment verified by another star. In the first adjustment the frame of the instrument would be moved; but in the second only the V which is provided with a small motion in azimuth. When the instrument is provided with a graduated horizontal circle, the most satisfactory method is to adjust it first in the meridian and then revolve it in azimuth 90°.

In preparing for an observation on the extreme threads, we must know the interval required by the star to pass from one of these to the middle thread. It will be shown hereafter that if i = the equatorial interval of the sidereal thread from the middle, the corresponding star interval I, near the prime vertical, will be nearly

$$I = \frac{i}{\sin \varphi \cos \delta \sin t} = \frac{i}{\sin \varphi \sin z}$$

and it is easily shown that when the hour angle t becomes $t \pm I$, the zenith distance becomes $z \pm 15\, I \cos \varphi$, where the factor 15 is used to reduce I from time to arc. The first observation on a side thread at the east transit will, therefore, be expected about I

seconds before the time of transit already computed, and at a greater zenith distance by about $15 I \cos\varphi$; while the first observation at the west transit will also be expected I seconds before the time of transit computed, but nearer the zenith by about $15 I \cos\varphi$. These simple calculations are accurate enough for the purpose of preparing for the observation. When the intervals of the threads are not known at first, they will be obtained accurately enough from the early observations for subsequent use in finding stars.

For stars whose declination is very nearly equal to the latitude, the zenith distance and hour angle on the prime vertical may be more accurately computed by the formulæ

$$\sin z = \frac{\sqrt{\sin(\varphi - \delta)\sin(\varphi + \delta)}}{\sin\varphi} \qquad \sin t = \frac{\sin z}{\cos\delta}$$

176. *Correction for inclination of the axis.*—When the rotation axis is in the meridian, but is inclined to the horizon, the great circle described by the collimation axis is still perpendicular to the meridian, but intersects it in a point whose angular distance from the zenith of the observer is precisely equal to the inclination of the rotation axis. This point may be called *the zenith of the instrument;* and the great circle described by the collimation axis, *the prime vertical of the instrument.* If we put

φ' = latitude of the zenith of the instrument,
φ = " " observer,
b = inclination of the rotation axis, positive when north end is elevated,

we have

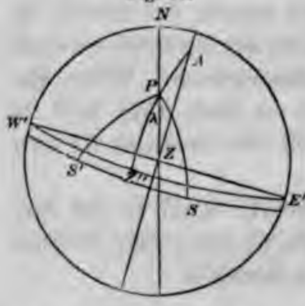
$$\varphi = \varphi' + b$$

and the only consideration of the level correction required in this case is to apply it directly to the latitude found from the instrument by the same methods that are used when the axis is truly horizontal.

But if the rotation axis is not in the meridian, nor the middle thread in the collimation axis, the simple solution given in Art. 174 requires some modification. I proceed now to consider the instrument in the most general manner, with deviations in azimuth, level, and collimation, and to show how to eliminate the effects of these deviations.

177. *To find the latitude from the observed times of transit of a given star over a given thread east and west of the meridian, the rotation axis being in the same position at both observations.*—Let the rotation axis lie in the vertical circle ZA, Fig. 46, and suppose the north end elevated, so that the great circle of the instrument is $E'Z''W'$, and a thread at the distance c south of the collimation axis describes the small circle SS'. Let A be the point in which the rotation axis produced meets the celestial sphere, and through A and the pole P draw the great circle APZ''. This great circle is perpendicular to $E'Z''W'$, and the observations of the star on the thread at S and S' are equally distant from it. We may call PZ'' the meridian, $E'Z''W'$ the prime vertical, and Z'' the zenith *of the instrument.*

Fig. 46.



Now, the equations (78) and (79) of Art. 123, being entirely general, apply to the instrument in this position, but it is convenient to make some modifications of the notation. The point A being now near the north point of the horizon, its azimuth is nearly zero and its hour angle nearly 180°. If we put

$$\begin{aligned}&\text{the azimuth of } A = 90° + (a) = -a, \text{ or } (a) = -(90° + a)\\&\text{the hour angle of } A = 90° - m = 180° + \lambda, \text{ or } m = -(90° + \lambda)\end{aligned}$$

where we distinguish the a of the equations (78) by enclosing it in brackets; then a is the small azimuth of the rotation axis reckoned from the north towards the east, and λ is the hour angle of the meridian of the instrument (or, as we might call it, the west longitude of the instrument); and the substitution of these quantities in equations (78) gives

$$\left.\begin{aligned}\cos n \cos \lambda &= -\sin b \cos \varphi + \cos b \cos a \sin \varphi\\ \cos n \sin \lambda &= \cos b \sin a\\ \sin n &= \sin b \sin \varphi + \cos b \cos a \cos \varphi\end{aligned}\right\} \quad (165)$$

and as τ in (79) is the hour angle east of the meridian, while it is here more convenient to reckon it, in the usual manner, towards the west, we shall change its sign, so that the factor $\sin(\tau - m)$ will become

$$\sin(-\tau + 90° + \lambda) = \cos(\tau - \lambda)$$

d the equation (79) will become

$$\sin c = -\sin n \sin \delta + \cos n \cos \delta \cos (\tau - \lambda) \qquad (166)$$

›r the convenience of future reference, I shall here recapitulate e notation used in these our fundamental equations: namely,

φ = the latitude of the place of observation, *positive* when *north;*
δ = the declination of the star, *positive* when *north;*
τ = the hour angle of the star;
a = the azimuth of the rotation axis, *positive* when *east of north;*
b = the inclination of the rotation axis, *positive when the north end is elevated;*
c = the collimation constant of a thread, *positive* when the thread is *north** of the collimation axis;
λ = the longitude of the meridian of the instrument, *positive* when *west;*
n = the declination of the north end of the axis.

further, when the star is observed at both the east and west ınsits, we put

τ, τ' = the hour angles of the east and west observations, respectively;
T, T' = the clock times of observation;
$\Delta T, \Delta T'$ = the corresponding clock corrections;
α = the right ascension of the star;
2ϑ = the elapsed sidereal time between the east and west observations on the same thread;

e have

$$\tau = T + \Delta T - \alpha \qquad \tau' = T' + \Delta T' - \alpha$$
$$\vartheta = \tfrac{1}{2}(T' + \Delta T' - T - \Delta T)$$
$$\lambda = \tfrac{1}{2}(T' + \Delta T' + T + \Delta T) - \alpha$$

hence $\vartheta = \tau' - \lambda = \lambda - \tau$

'e see that ϑ will be well determined when the clock *rate*, or $\Gamma' - \Delta T$, is known; but to find λ we must also know the clock rrection and the star's right ascension.

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* When the *thread* is north of the prime vertical, the small circle of the sphere ich corresponds to it is *south* of the prime vertical, and *vice versa.*

Now, let h and β be assumed so as to satisfy the conditions

$$h \sin \beta = \sin b$$
$$h \cos \beta = \cos b \cos a$$

then the equations (165) become

$$\left.\begin{aligned} \cos n \cos \lambda &= h \sin (\varphi - \beta) \\ \cos n \sin \lambda &= \cos b \sin a \\ \sin n &= h \cos (\varphi - \beta) \end{aligned}\right\} \quad (167)$$

Substituting in (166) the values of $\cos n$, $\sin n$, given by these equations, and also $\cos (\tau - \lambda) = \cos (\lambda - \tau') = \cos \vartheta$, we have

$$\sin c = - h \cos(\varphi - \beta) \sin \delta + h \sin (\varphi - \beta) \cos \delta \frac{\cos \vartheta}{\cos \lambda}$$

to reduce which we assume h' and φ' to satisfy the conditions

$$\left.\begin{aligned} h' \sin \varphi' &= \sin \delta \\ h' \cos \varphi' &= \cos \delta \frac{\cos \vartheta}{\cos \lambda} \end{aligned}\right\} \quad (168)$$

which transform the preceding equation into

$$\sin c = h h' \sin (\varphi - \varphi' - \beta)$$

whence

$$\sin (\varphi - \varphi' - \beta) = \frac{\sin c}{h h'}$$

But, as c is never more than 15′, and $h' = \frac{\sin \delta}{\sin \varphi'}$ will never be less than ½, while h differs from unity only by a quantity depending upon $\sin^2 a$, the angle $\varphi - \varphi' - \beta$ will never exceed 30′: so that we may write, without sensible error,

$$\varphi - \varphi' - \beta = \frac{c \sin \varphi'}{\sin \delta}$$

To find β, we have

$$\tan \beta = \tan b \sec a$$

or, since b is only a few seconds and a but a few minutes,

$$\beta = b$$

and φ' is determined by (168), which give

$$\tan \varphi' = \tan \delta \sec \vartheta \cos \lambda \qquad (169)$$

and then we have

$$\varphi = \varphi' + b + \frac{c \sin \varphi'}{\sin \delta} \tag{170}$$

It is evident that the factor $\cos \lambda$ in (169) corrects for azimuth deviation, the term b in (170) for inclination of the rotation axis, and the term $\frac{c \sin \varphi'}{\sin \delta}$ for the distance of the thread from the collimation axis.

In these equations, ϑ and λ are obtained from the observed times on the same thread, the rotation axis being in the same position at the two observations. The constant c has then the same sign at both observations, + for north threads,— for south threads; and its value must be known for each thread. We deduce then, by (169) and (170), from each thread separately, a value of the latitude, and take the mean of all the results as the latitude given by the instrument *in this position of the axis.* But if the pivots are unequal the striding level does not give the true value of b directly. (See Art. 137.) Moreover, the constant c is composed of the equatorial interval of the given thread from the middle thread combined with the collimation constant of the middle thread, and will, therefore, involve both the error in the determination of the interval and in the adjustment for collimation.

Now, to eliminate all these instrumental errors, repeat the observations on the same star on a subsequent night in the reverse position of the axis. Let p be the (unknown) correction for inequality of pivots, q the (unknown) correction of c for error in the interval of thread and collimation adjustment; let φ', φ'' be the latitudes given by (169) for the same star on different nights and in reverse positions of the axis; b, b' the inclinations of the rotation axis given by the spirit level. The true inclinations are $b + p$ and $b' - p$, and the true value of the collimation constant for the given thread is $c + q$: so that in the first position of the axis we have

$$\varphi = \varphi' + b + p + (c + q)\frac{\sin \varphi'}{\sin \delta}$$

and in the second position,

$$\varphi = \varphi'' + b' - p - (c + q)\frac{\sin \varphi''}{\sin \delta}$$

and the mean of these is

$$\varphi = \tfrac{1}{2}(\varphi' + b + \varphi'' + b') + \frac{c+q}{2}\left[\frac{\sin\varphi' - \sin\varphi''}{\sin\delta}\right]$$

so that the inequality of pivots is wholly eliminated, and the error of thread and collimation is reduced to the term

$$\frac{q}{2}\left[\frac{\sin\varphi - \sin\varphi''}{\sin\delta}\right] = \frac{q\sin(\varphi' - \varphi'')\cos\varphi}{2\sin\delta} \text{ (nearly)}$$

which for $q = 1''$, $\varphi' - \varphi'' = 1°$, is $0''.008 \cos\varphi \operatorname{cosec}\delta$, and that part of this small quantity which depends on the collimation of the middle thread will have different signs for north and south threads, and will also wholly disappear from the mean. There will remain, therefore, in the result only that part of this term which depends on the errors of the thread intervals. As the thread intervals can easily be determined in the meridian within $1''$, this remaining error in the latitude will be insensible in practice, and we may assume the mean of two nights' observations to be wholly free from the instrumental errors.

There remain yet the errors of observation and of the clock. These affect both the angles ϑ and λ. As λ is always small, their effect will not generally be appreciable in $\cos\lambda$, and their effect in $\sec\vartheta$ will be less the nearer the star is to the zenith; for the clock errors that appear in ϑ are only the variations of *rate*, and the less the interval the less the effect of these upon ϑ, and, at the same time, the less the angle ϑ the less effect will any change in ϑ produce in $\sec\vartheta$.

The expression for the error in φ resulting from an error in ϑ is found by differentiating (169); whence

$$d\varphi \sec^2\varphi' = d\vartheta \tan\delta \sec\vartheta \tan\vartheta \cos\lambda = d\vartheta \tan\varphi' \tan\vartheta$$

or nearly

$$d\varphi = \frac{d\vartheta}{2} \sin 2\varphi \tan\vartheta$$

and $\sin 2\varphi$ is greatest for $\varphi = 45°$, in which case we have $d\varphi = \frac{d\vartheta}{2}\tan\vartheta$. For $\vartheta = 1^h$, $d\varphi = d\vartheta \times 0.13$; or an error in ϑ of $1^s = 15''$ produces an error in φ of less than $2''$. If we assume, then, that ϑ can always be obtained within 1^s, we ought to expect the mean of the latitudes obtained in two nights from the same thread and with the same star to agree with that found in the ame way from any other thread, within 2″, when the observa- ions are taken within one hour of the meridian. This, in fact, s the experience of observers in the use of this method.

Finally, the latitude is affected by an error in the tabulated eclination of the star. When $\varphi < 45°$, the error in the latitude always greater than the error of the declination; but when $> 45°$, the error in the latitude will be less than the error in e declination, if we use stars whose declinations fall between e limits 90° and $90° - \varphi$, as will be seen at once by examining e equation

$$d\varphi = d\delta \cdot \frac{\sin 2\varphi}{\sin 2\delta}$$

ich is found by differentiating (169) with reference to φ and δ. is evident, therefore, that this method is better suited to high itudes than to low ones, although satisfactory results may be tained by it even in latitudes not greater than 30°.

178. Instead of deducing a value of the latitude from each read, it is usually more convenient to reduce the observations the several threads to the middle thread, and then to find the lue of the latitude from the mean. This value will, of course, e the same as the mean of the several values found from the reads individually. I proceed, therefore, to investigate the ormula for reducing the observations on the side threads to the iddle thread.

Let

i = the equatorial interval of any given thread north of the middle thread,

I = the corresponding star interval,

hen, τ being the hour angle of the star when on the middle hread, $\tau - I$ is its hour angle when on the given thread: so that now denoting the collimation constant of the middle thread, d, consequently, $c + i$ being now put for c in (166), we have

$$\sin(i + c) = -\sin n \sin\delta + \cos n \cos\delta \cos(\tau - \lambda - I)$$

hile for the middle thread we have

$$\sin c = -\sin n \sin\delta + \cos n \cos\delta \cos(\tau - \lambda)$$

e difference of these equations gives

$$2\cos(\tfrac{1}{2}i + c)\sin\tfrac{1}{2}i = 2\cos n \cos\delta \sin(\tau - \lambda - \tfrac{1}{2}I)\sin\tfrac{1}{2}I$$

In the first member, since i and c are both small, we may put $2 \cos \frac{1}{2} i \sin \frac{1}{2} i$, or $\sin i$, and hence

$$2 \sin \tfrac{1}{2} I = \frac{\sin i}{\cos n \cos \delta \sin (\tau - \lambda - \frac{1}{2} I)}$$

If the azimuth a of the instrument is even as great as 20′ (and it will always be much less), it is easily shown that log h in (167) will not be less than 9.999993, that is, it will not change the fifth decimal place by a unit in the computation of log cos n; and, as this degree of accuracy is evidently even more than sufficient in computing I, we shall here take $\cos n = \sin (\varphi - b)$, and hence

$$2 \sin \tfrac{1}{2} I = \frac{\sin i}{\sin (\varphi - b) \cos \delta \sin (\tau - \lambda - \frac{1}{2} I)} \tag{171}$$

This very exact formula will be required, however, only where the star is very near the zenith. In most cases we can employ $\sin \varphi$ for $\sin (\varphi - b)$ and put $\frac{1}{2} I$ instead of its sine.

When the star has been observed on the middle thread, both east and west of the meridian, we may find $\tau - \lambda = \vartheta$ with sufficient accuracy for computing the reductions of the threads, by taking the half difference of the observed times on this thread; and hence the formula will be

$$2 \sin \tfrac{1}{2} I = \frac{\sin i}{\sin (\varphi - b) \cos \delta \sin (\vartheta - \frac{1}{2} I)} \tag{172}$$

or, in most cases,

$$I = \frac{i}{\sin \varphi \cos \delta \sin (\vartheta - \frac{1}{2} I)} \tag{172*}$$

In applying these formulæ, the signs of i, I, and ϑ must be carefully observed. Thus, i will be positive for north and negative for south threads; ϑ positive for a star west, and negative for a star east of the meridian. The value of I required in the second member may be found with sufficient accuracy from the observations themselves; and, in order to obtain it with the proper sign, it is to be observed that the observed time on the given thread is always to be subtracted from that on the middle thread.

Having reduced the several observations to the middle thread by adding the values of I thus found, the means of the results for the east and west transits, respectively, will now be denoted by T and T', after which ϑ and λ will be accurately found, and the latitude computed precisely as in the preceding article. The quantity c in equation (170) will now denote the collimation constant of the middle thread.

The level constant should be determined both before and after each transit east and west, and the mean of the four values employed for b, particular care being required in the determination of this quantity, since any error in it affects the resulting latitude by its whole amount.

Example.—The following observations were taken by Hansen in Heligoland with a transit instrument in the prime vertical.* The hours are given only for the middle thread, and the observations on threads VII., VI., and V. are placed immediately below those on I., II., and III., respectively.

1824, July 31.—*Circle North.*

γ *Draconis.*	I. and VII.	II. and VI.	III. and V.	IV.	Level.
East transit	$14^m\ 28^s.8$ $9\ \ 26.$	$13^m\ 36^s.8$ $10\ \ 13.$	$12^m\ 46^s.$ $11\ \ 3.8$	$16^h\ 11^m\ 54^s.$	$-0^d.40$
West "	$27\ \ 35.$ $32\ \ 37.5$	$28\ \ 26.8$ $31\ \ 50.$	$29\ \ 17.5$ $31\ \ 0.$	$19\ \ 30\ \ 9.8$	$-1\ .37$
Clock correction (sidereal) at $14^h\ 22^m = +1^m\ 47^s.40$. Daily rate, $+4^s.12$					

1824, August 3.—*Circle South.*

γ *Draconis.*	I. and VII.	II. and VI.	III. and V.	IV.	Level.
East transit	$8^m\ 57^s.$ $13\ \ 59.$	$9^m\ 47^s.$ $13\ \ 9.5$	$10^m\ 36^s.$ $12\ \ 17.5$	$16^h\ 11^m\ 27^s.5$	$-1^d.50$
West "	$32\ \ 15.$ $27\ \ 14.$	$31\ \ 26.$ $28\ \ 3.$	$30\ \ 36.5$ $28\ \ 55.$	$19\ \ 29\ \ 44.$	$-0\ .03$
Clock correction at $14^h\ 8^m = +1^m\ 59^s.98$. Daily rate, $+4^s.27$					

The threads are numbered from the circle end of the axis, so that for "circle north" stars at the east transit are observed first on thread VII. Their equatorial intervals, as found by observations in the meridian, were—

	I	II	III	V	VI	VII
(Circle north) i,	$+32^s.382$	$+21^s.557$	$+10^s.963$	$-10^s.652$	$-21^s.426$	$-31^s.672$

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* *Astron. Nach.*, Vol. VI. p. 117.

The value of one division of the level was $2''.5$ (of arc).
The collimation constant was $c = +\ 2''.18$ (in arc), *circle north.*
The assumed latitude was $\varphi = 54^\circ\ 10'.8$.
For the given dates, the apparent places of the star were—

γ Draconis	α	δ
1824, July 31,	$17^h\ 52^m\ 34^s.42$	$+\ 51^\circ\ 30'\ 57''.64$
" Aug. 3,	" " $34\ .37$	" " $58\ .04$

We shall first reduce the observations of July 31. To compute the thread intervals, we find an approximate value of ϑ from the observed times on the middle thread, the difference of which is $3^h\ 18^m\ 15^s.8$, and, since in this time the clock rate is $+\ 0^s.6$, we take $2\vartheta = 3^h\ 18^m\ 16^s.4$, and hence

$$(\text{Approx.})\ \vartheta = 1^h\ 39^m\ 8^s.2$$

Taking the differences between the observed times on each side thread and that on the middle thread for both the east and west transits, the mean of the two values for each thread may be used as a sufficiently exact value of I to be used in the second member of (172), namely:

	I	II	III	V	VI	VII
(Approx.) I,	$+\ 2^m\ 34^s.8$	$+\ 1^m\ 42^s.9$	$+\ 0^m\ 52^s.2$	$-\ 0^m\ 50^s.2$	$-\ 1^m\ 40^s.6$	$-\ 2^m\ 27^s.8$
$\vartheta - \frac{1}{2}I$,	$1^h\ 37\ 50\ .8$	$1^h\ 38\ 16\ .7$	$1^h\ 38\ 42\ .1$	$1^h\ 39\ 33\ .3$	$1^h\ 39\ 58\ .5$	$1^h\ 40\ 22\ .1$

whence the reductions to the middle thread are, for the west transit,

$$I,\quad +\ 2^m\ 34^s.97 + 1^m\ 42^s.74 + 0^m\ 52^s.04 - 0^m\ 50^s.16 - 1^m\ 40^s.49 - 2^m\ 28^s.01$$

and the same values, with their signs changed, are used for the east transit. These being applied to the observed times, we have—

	East.		West.
I	$16^h\ 11^m\ 53^s.83$		$19^h\ 30^m\ 9^s.97$
II	$54\ .06$		$9\ .54$
III	$53\ .96$		$9\ .54$
IV	$54\ .00$		$9\ .80$
V	$53\ .96$		$9\ .84$
VI	$53\ .49$		$9\ .51$
VII	$54\ .01$		$9\ .49$
$T =$	$16\ \ 11\ \ 53\ .90$	$T' =$	$19\ \ 30\ \ 9\ .67$
$\Delta T =$	$+\ \ \ 1\ \ 47\ .71$	$\Delta T' =$	$+\ \ \ 1\ \ 48\ .28$
$T + \Delta T =$	$16\ \ 13\ \ 41\ .61$	$T' + \Delta T' =$	$19\ \ 31\ \ 57\ .95$
	$19\ \ 31\ \ 57\ .95$		$16\ \ 13\ \ 41\ .61$
$\frac{1}{2}$ sum $=$	$17\ \ 52\ \ 49\ .78$	$\frac{1}{2}$ diff. $=$	$1\ \ 39\ \ 8\ .17$
$\alpha =$	$17\ \ 52\ \ 34\ .42$	$= \vartheta =$	$24^\circ\ 47'\ 2''.55$
$\lambda =$	$15\ .36$		
$=$	$0^\circ\ \ 3'\ \ 50''$		

Hence, by (169) and (170),

$$\begin{array}{rl} \log\tan\delta & 0.0996440 \\ \log\sec\vartheta & 0.0419648 \\ \log\cos\lambda & \underline{9.9999997} \\ \log\tan\varphi' & 0.1416085 \\ \varphi' = & 54^\circ\ 10'\ 47''.41 \\ \dfrac{c\sin\varphi'}{\sin\delta} = & +\ \ 2\ .26 \\ b = & \underline{-\ \ 2\ .21} \\ \varphi = & 54\ \ 10\ \ 47\ .46 \end{array} \qquad \begin{array}{rl} c = & +\ 2''.18 \\ \log c & 0.3385 \\ \log\sin\varphi' & 9.9089 \\ \log\operatorname{cosec}\delta & \underline{0.1064} \\ \ldots\ldots\ldots & 0.3538 \end{array}$$

For the observations of August 3, we find, from the observed times on the middle thread,

$$(\text{Approx.})\ \vartheta = 1^h\ 39^m\ 8^s.5$$

and from the observed times on the side threads compared with the middle thread,

	I	II	III	V	VI	VII
(Approx.) I,	$-2^m\ 30^s.8$	$-1^m\ 41^s.2$	$-0^m\ 52^s.0$	$+0^m\ 49^s.5$	$+1^m\ 41^s.5$	$+2^m\ 30^s.8$
$\vartheta - \frac{1}{2}I$,	$1^h\ 40\ \ 23.9$	$1^h\ 39\ \ 59.1$	$1^h\ 39\ \ 34.5$	$1^h\ 38\ \ 43.7$	$1^h\ 38\ \ 17.7$	$1^h\ 37\ \ 53.1$

with which we find the true values of I to be as follows:

$$I,\ -2^m\ 31^s.28\ -1^m\ 41^s.10 - 0^m\ 51^s.61 + 0^m\ 50^s.55 + 1^m\ 42^s.10 + 2^m\ 31^s.52$$

Applying these to the observed times, and taking the means, we have—

$$\begin{array}{rl} & \textbf{East.} \\ T = & 16^h\ 11^m\ 27^s.61 \\ \Delta T = & \underline{+\ \ 2\ \ \ 0.35} \\ T + \Delta T = & 16\ \ 13\ \ 27.96 \\ \lambda = & 0^\circ\ \ 0'\ \ 37''. \end{array} \qquad \begin{array}{rl} & \textbf{West.} \\ T' = & 19^h\ 29^m\ 44^s.81 \\ \Delta T' = & \underline{+\ \ 2\ \ \ 0.94} \\ T' + \Delta T' = & 19\ \ 31\ \ 45.75 \\ \vartheta = & 24^\circ\ 47'\ \ 13''.5 \end{array}$$

With these we find, taking now $c = -2''.18$,

$$\begin{array}{rl} \varphi' = & 54^\circ\ 10'\ 50''.25 \\ \dfrac{c\sin\varphi'}{\sin\delta} = & -\ \ 2\ .26 \\ b = & \underline{-\ \ 1\ .91} \\ \varphi = & 54\ \ 10\ \ 46\ .08 \end{array}$$

The mean of the results in the two positions of the instrument is, therefore, $\varphi = 54^\circ\ 10'\ 46''.77$. From numerous observations of the same kind, HANSEN found $\varphi = 54^\circ\ 10'\ 46''.53$.

179. *To find the latitude when the instrument is reversed between the east and west transits of the same star on the same night.*—Reduce the observations to the middle thread, and let T and T' be the mean of the resulting clock times at the east and west transits, respectively. If the middle thread was north of the collimation axis at the east transit, it will be south of that axis at the west transit, and the interval $T' - T$ will be sensibly the same as the interval between the two transits over the collimation axis itself. We may, therefore, compute the latitude precisely as in the preceding method, and regard c as zero. Thus, our formulæ will be

$$\left.\begin{aligned} \vartheta &= \tfrac{1}{2}[(T' + \Delta T') - (T + \Delta T)] \\ \lambda &= \tfrac{1}{2}[\,T' + \Delta T' + T + \Delta T] - \alpha \\ \tan \varphi' &= \tan \delta \sec \vartheta \cos \lambda \\ \varphi &= \varphi' + b \end{aligned}\right\} \quad (173)$$

in which b is the mean of the level determinations in the two positions of the axis, and is, therefore, free from the error of inequality of pivots. This method, then, enables us to obtain from the observations of a single night a value of the latitude free from all the instrumental errors.* We may remark here that the result by this method, as well as the mean of the results of two observations in reverse positions of the axis by the preceding method, is free from errors arising from flexure of the rotation axis.

EXAMPLE.—The following observations were taken at Cronstadt with a transit instrument in the prime vertical, the axis of which was reversed between the east and west transits.

1843, August 9: Cronstadt. Assumed $\phi = 59° 59'.5$.

	Circle South.	I	II	III	IV	V	Level.
E.	*γ Cassiopeæ*	$13^m\ 23^s.$	$17^m\ 46^s.$	$0^h\ 24^m\ 6^s.$	$31^m\ 32^s.$	....	$+5''.36$
	δ Cassiopeæ	20 32 .	23 6 .	0 26 21 .	29 19 .	$32^m\ 44^s.$	$+5\ .56$
	Circle North.						
W.	*γ Cassiopeæ*	....	$1^m\ 2^s.$	$1^h\ 9^m\ 55^s.$	$16^m\ 26^s.$	$20^m\ 21^s.$	$-2''.10$ $-2\ .08$
	δ Cassiopeæ	$57^m\ 36^s.$	0 45 .	2 4 11 .	7 0 .	9 50 .	$-1\ .50$ $-1\ .10$

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* There is a theoretical inaccuracy in finding λ, since this quantity will be affected by the collimation error; but the error will have no sensible effect upon the cosine of so small a quantity, unless c is unusually large. It will, indeed, be always inappreciable when the observer has bestowed ordinary care upon the adjustment of the middle thread.

The level was observed before the east transit of γ *Cassiop.* and after that of δ *Cassiop.*: so that the mean $b = +\,5''.46$ will be used for both stars at the east transit. But at the west transit the level was observed before and after each star: so that for γ *Cassiop.*, at this transit we shall use $b = -\,2''.09$, and for δ *Cassiop.*, $b = -\,1''.30$.

The threads are numbered from the circle end of the axis, and thread I. was first observed at both the east and west transits. The equatorial intervals from the middle thread were—

	I	II	IV	V
(Circle North) i,	$+\,34^s.40$	$+\,18^s.74$	$-\,16^s.14$	$-\,33^s.33$

The collimation constant, as found from observations in the meridian, was $c = +\,4''.50$ (in arc) for "circle south."

The chronometer correction (sidereal) was $+\,30^s.20$ at $0^h\,24^m$; its daily rate, $+\,0^s.90$.

The apparent places of the stars for this date were—

	α	δ
γ *Cassiopeæ*,	$0^h\,47^m\,21^s.49$	$+\,59°\,52'\,2''.3$
δ *Cassiopeæ*,	$1\;\;15\;\;40\,.38$	$+\,59°\,25\;\;6\;.2$

To reduce the observations of γ *Cassiopeæ*, we first find the approximate value of ϑ from the difference of the observed times on the middle thread to be

$$\vartheta = 0^h\,22^m\,54^s.5$$

from which we find, by (172), the reductions of the side threads to the middle thread to be as follows:

	I.	II	IV	V
γ *Cassiop.* E.	$+\,10^m\,43^s.2$	$+\,6^m\,19^s.7$	$-\,7^m\,23^s.9$	——
" W.	——	$+\,8\;\;55\,.6$	$-\,5\;\;32\,.2$	$-\,10^m\,26^s.4$

Applying these, and proceeding by (173), we find,—

	East.		West.
I	$0^h\ 24^m\ 6^s.2$		$-^h\ -^m\ -^s$
II	5.7		$1\ \ 9\ 57.6$
III	6.0		55.0
IV	8.1		53.8
V	—		54.6
$T =$	$0\ 24\ \ 6.5$	$T' =$	$1\ \ 9\ 55.3$
$\Delta T =$	$+30.2$	$\Delta T' =$	$+30.2$
$T + \Delta T =$	$0\ 24\ 36.7$	$T' + \Delta T' =$	$1\ 10\ 25.5$
	$1\ 10\ 25.5$		$0\ 24\ 36.7$
$\frac{1}{2}$ sum $=$	$0\ 47\ 31.1$	$\frac{1}{2}$ diff. $=$	$0\ 22\ 54.4$
$\alpha =$	$0\ 47\ 21.5$	$= \vartheta =$	$5^\circ\ 43'\ 36''.$
$\lambda =$	9.6		
$=$	$0^\circ\ 2'\ 24''.$		

$\log \tan \delta$	0.2362409
$\log \sec \vartheta$	0.0021729
$\log \cos \lambda$	9.9999999
$\log \tan \varphi'$	0.2384137

$$\varphi' = 59^\circ\ 59'\ 29''.78$$

$$b = \tfrac{1}{2}(5''.46 - 2''.09) = \quad +\ 1\ .69$$

$$\varphi = 59\ \ 59\ \ 31\ .47$$

The observations of γ *Cassiopeæ*, reduced in the same manner, give $\varphi = 59^\circ\ 59'\ 30''.98$, and the mean is $\varphi = 59^\circ\ 59'\ 31''.23$.

The preceding methods of reduction leave nothing to be desired when the intervals of the threads are known. When, however, these are unknown, we may resort to one or the other of the following methods, according to the nature of the observation.

180. *To find the latitude from the observed transits of a star over the prime vertical, east and west of the meridian, when the instrument is reversed only between the observations of different nights, the intervals of the threads being unknown.*

Put

$c =$ the distance of any thread from the collimation axis,

$\vartheta_n = \frac{1}{2}$ the elapsed sidereal time between the east and west transits over the same thread when the circle or finder is north,

$\vartheta_s =$ ditto for the same star when the axis is reversed,

$b_n, b_s =$ the level constants in the two positions;

then, by (169) and (170), we shall have

$$\begin{aligned}
\tan \varphi_n &= \tan \delta \sec \vartheta_n \cos \lambda \\
\tan \varphi_s &= \tan \delta \sec \vartheta_s \cos \lambda \\
\varphi &= \varphi_n + b_n + \frac{c \sin \varphi_n}{\sin \delta} \\
\varphi &= \varphi_s + b_s - \frac{c \sin \varphi_s}{\sin \delta}
\end{aligned}$$

The last two equations involve but two unknown quantities, φ and c, both of which may, therefore, be determined. Put

$$\begin{aligned}
\varphi_0 &= \tfrac{1}{2} (\varphi_n + b_n + \varphi_s + b_s) \\
\gamma &= \tfrac{1}{2} (\varphi_n + b_n - \varphi_s - b_s)
\end{aligned}$$

then our equations become

$$\begin{aligned}
\varphi - \varphi_0 &= \gamma + \frac{c \sin \varphi_n}{\sin \delta} \\
\varphi - \varphi_0 &= -\gamma - \frac{c \sin \varphi_s}{\sin \delta}
\end{aligned}$$

Multiplying the first by $\sin \varphi_s$, the second by $\sin \varphi_n$, and adding them together, we find

$$\varphi - \varphi_0 = -\gamma \left[\frac{\sin \varphi_n - \sin \varphi_s}{\sin \varphi_n + \sin \varphi_s} \right] = -\gamma \tan \tfrac{1}{2} (\varphi_n - \varphi_s) \cot \tfrac{1}{2} (\varphi_n + \varphi_s)$$

Since γ is very nearly equal to $\frac{1}{2} (\varphi_n - \varphi_s)$, the second member of this equation involves the square of γ, and is, consequently, an exceedingly small quantity, in computing which we may, evidently, put $\gamma = \frac{1}{2} (\varphi_n - \varphi_s)$ and substitute φ for $\frac{1}{2} (\varphi_n + \varphi_s)$, whereby we obtain

$$\varphi = \varphi_0 - \tfrac{1}{4} \gamma^2 \sin 1'' \cot \varphi$$

This method may, therefore, be expressed by the following equations:

$$\left.\begin{aligned}
\tan \varphi_n &= \tan \delta \sec \vartheta_n \cos \lambda \\
\tan \varphi_s &= \tan \delta \sec \vartheta_s \cos \lambda \\
\varphi_0 &= \tfrac{1}{2} (\varphi_n + b_n + \varphi_s + b_s) \\
\Delta\varphi &= \tfrac{1}{4} (\varphi_n - \varphi_s)^2 \sin 1'' \cot \varphi \\
\varphi &= \varphi_0 - \Delta\varphi
\end{aligned}\right\} \quad (174)$$

in which the assumed value of φ may be used in computing $\Delta\varphi$.

181. In this form of the method, only *pairs* of observations of the same star made on different nights in reverse positions of the axis can be reduced. But it often happens that the observation on a thread is lost, and the corresponding observation on

the same thread in the reverse position of the axis becomes useless. In order to avail ourselves of every observation, we may, after a sufficient number of observations have been made on the same star, determine for this star the mean difference between φ and $\varphi_n + b_n$ and between φ and $\varphi_s + b_s$, and these differences may be used to reduce the observations on the several nights independently of each other. Thus, if we put

$$\begin{aligned} \Delta_n\varphi &= \varphi - (\varphi_n + b_n) = -\tfrac{1}{2}(\varphi_n - \varphi_s + b_n - b_s) - \Delta\varphi \\ \Delta_s\varphi &= \varphi - (\varphi_s + b_s) = +\tfrac{1}{2}(\varphi_n - \varphi_s + b_n - b_s) - \Delta\varphi \end{aligned}$$

each complete pair of observations on two nights furnishes a value of $\Delta_n\varphi$ and $\Delta_s\varphi$, and, the mean of all being taken, any individual observation may be reduced by the formulæ

$$\tan\varphi_n = \tan\delta \sec\vartheta_n \cos\lambda \qquad \varphi = \varphi_n + b_n + \Delta_n\varphi$$

$$\text{or,} \quad \tan\varphi_s = \tan\delta \sec\vartheta_s \cos\lambda \qquad \varphi = \varphi_s + b_s + \Delta_s\varphi$$

This method of reduction is given by Professor PEIRCE.*

182. The quantity λ, which is the difference between the right ascension of the star and the mean of the sidereal times of observation on the same thread east and west of the meridian, should have the same or nearly the same value throughout the series of observations, since any change of sufficient magnitude to affect the value of $\cos\lambda$ sensibly will give different values of φ_n or φ_s, and, consequently also of $\Delta_n\varphi$ or $\Delta_s\varphi$, which are here supposed to be constant. To secure this condition, the stability of the instrument in azimuth must be secured, or it must be verified and corrected from time to time by means of a terrestrial mark to which the middle thread is referred.

183. The factor $\cos\lambda$ may be omitted (not only in this, but in all other methods) throughout the reduction of a series of observations where it can be regarded as constant, and a small correction for the azimuth of the instrument can be applied to the final mean latitude. If we denote this mean by (φ), found by neglecting the factor $\cos\lambda$, the true latitude will be found by the formula

$$\tan\varphi = \tan(\varphi)\cos\lambda$$

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* In a memoir on the latitude of Cambridge, Mass., *Memoirs of Am. Academy of Sciences*, Vol. II. p. 183

or

$$\varphi = (\varphi) - \tfrac{1}{4}\lambda^2 \sin 1'' \sin 2\varphi \qquad (175)$$

If the azimuth deviation a is required, it may be found by the second equation of (167), which gives, very nearly,

$$\sin a = \sin \lambda \sin \varphi \qquad (176)$$

If the azimuth of the instrument is known independently of the observations for latitude, we have, by substituting a for $\lambda \sin \varphi$,

$$\varphi = (\varphi) - \tfrac{1}{2} a^2 \sin 1'' \cot \varphi \qquad (176^*)$$

184. The thread intervals may also be found; for the difference of the equations for φ, Art. 180, gives

$$c = -\frac{(\varphi_n + b_n - \varphi_s - b_s) \sin \delta}{2 \sin \frac{1}{2}(\varphi_n + \varphi_s) \cos \frac{1}{2}(\varphi_n - \varphi_s)}$$

for which we may take

$$c = \frac{(\Delta_n\varphi + \Delta\varphi) \sin \delta}{\sin \varphi \cos \Delta_n\varphi}$$

or, in most cases,

$$c = \frac{(\Delta_n\varphi + \Delta\varphi) \sin \delta}{\sin \varphi} \qquad (177)$$

This will give the distance of each thread (the middle thread included) from the collimation axis, whence we can deduce the distance of each from the middle thread.

Example.—Let us apply this method to the reduction of the observations taken at Heligoland by **Hansen**, given on p. 249.

Beginning with the observations of July 31, "circle north," we find ϑ_n for each thread by taking half the difference of the observed times on this thread, east and west, and correcting for the clock rate in the interval, which is here $+ 0^s.28$. The value of λ may be found accurately enough from the middle thread alone. Thus we have

$$\begin{array}{rl}
\text{Mean of times on middle thread} = & 17^h\ 51^m\ \ 1^s.9 \\
\text{Clock corr.} = & \underline{+\ \ 1\ \ 48\ .0} \\
\text{Sid. time} = & 17\ \ 52\ \ 49\ .9 \\
\text{Star's } \alpha = & \underline{17\ \ 52\ \ 34\ .4} \\
\lambda = & \qquad\quad 15\ .5 = 0^\circ\ 3'\ 52''.
\end{array}$$

Hence we have $\log \tan \delta \cos \lambda = 0.0996437$, which will be used for all the threads, the value of $\log \cos \vartheta_n$ for each thread being subtracted from it to find $\log \tan \varphi_n$, as follows:

Thread.	ϑ_n	log cos ϑ_n	log tan ϕ_n	ϕ_n
I	$1^h\ 36^m\ 33^s.38$	9.9602592	0.1393845	54° 2′ 25″.76
II	1 37 25 .28	9.9595210	0.1401227	5 12 .36
III	1 38 16 .03	9.9587918	0.1408519	7 56 .84
IV	1 39 8 .18	9.9580351	0.1416086	10 47 .43
V	1 39 58 .38	9.9572996	0.1423441	13 33 .16
VI	1 40 48 .78	9.9565540	0.1430897	16 21 .07
VII	1 41 36 .03	9.9558485	0.1437952	18 59 .87

From the observations of August 3, "circle south," we find

Mean of times on middle thread = $17^h\ 50^m\ 35^s.7$
Clock corr. = + 2 0 .6
Sid. time = 17 52 36 .3
α = 17 52 34 .4
log tan δ cos λ = 0.0996457 λ = 1 .9 = 0′ 29″

Thread.	ϑ_s	log cos ϑ_s	log tan ϕ_s	ϕ_s
I	$1^h\ 41^m\ 39^s.29$	9.9557996	0.1438461	54° 19′ 11″.32
II	40 49 .79	9.9565389	0.1431068	16 24 .93
III	40 0 .54	9.9572678	0.1423779	13 40 .80
IV	39 8 .54	9.9580299	0.1416158	10 49 .07
V	38 19 .04	9.9587483	0.1408974	8 7 .11
VI	37 27 .04	9.9594958	0.1401499	5 18 .50
VII	36 37 .79	9.9601968	0.1394489	2 40 .30

With the assumed latitude $\varphi = 54° 10'.8$, we find log $\frac{1}{4}$ sin 1″ cot φ = 3.9419, and the computation of $\Delta\varphi$ for each thread is as follows:

Thread.	$\phi_n - \varphi_s$	$\log(\varphi_n - \varphi_s)^2$	log $\Delta\varphi$	$\Delta\varphi$
I	− 16′ 45″.56	6.0046	9.9465	0″.88
II	− 11 12 .57	5.6556	9.5975	0 .40
III	− 5 43 .96	5.0730	9.0149	0 .10
IV	− 0 1 .64	0.4296	4.3715	0 .00
V	+ 5 26 .05	5.0264	8.9683	0 .09
VI	+ 11 2 .57	5.6426	9.5845	0 .38
VII	+ 16 19 .57	5.9820	9.9239	0 .84

We have $b_n = -2''.21$, $b_s = -1''.91$, $\frac{1}{2}(b_n + b_s) = -2''.06$; and hence the several values of the latitude given by the different threads are found as follows:

Thread.	$\frac{1}{2}(\phi_n + \phi_s)$	ϕ_0	ϕ
I	54° 10′ 48″.54	46″.48	45.60
II	48 .65	46 .59	46.19
III	48 .82	46 .76	46.66
IV	48 .25	46 .19	46.19
V	50 .14	48 .08	47.99
VI	49 .79	47 .73	47.35
VII	50. 09	48. 03	47.19
		Mean =	46.74

Hence $\varphi = 54° 10' 46''.74$; which agrees within 0″.03 with the result found on p. 251. The slight difference is perhaps due to small errors in the thread intervals employed in the former method.

The values of $\Delta_n\varphi$ and $\Delta_s\varphi$ for each thread may be found as follows:

Thread.	$\frac{1}{2}(\phi_n - \phi_s)$	$\frac{1}{2}(\phi_n - \phi_s + b_n - b_s)$	$\Delta_n\phi$	$\Delta_s\phi$
I	− 8′ 22″.78	− 8′ 22″.93	+ 8′ 22″.05	− 8′ 23″.81
II	− 5 36 .29	− 5 36 .44	+ 5 36 .04	− 5 36 .84
III	− 2 51 .98	− 2 52 .13	+ 2 52 .03	− 2 52 .23
IV	− 0 0 .82	− 0 0 .97	+ 0 0 .97	− 0 0 .97
V	+ 2 43 .03	+ 2 42 .88	− 2 42 .97	+ 2 42 .79
VI	+ 5 31 .29	+ 5 31 .14	− 5 31 .52	+ 5 30 .76
VII	+ 8 9 .79	+ 8 9 .64	− 8 10 .48	+ 8 8 .80

When $\Delta_n\varphi$ and $\Delta_s\varphi$ have been thus determined from a considerable number of observations, their mean values may be used to reduce the observations of each night separately.

We may now also find the thread intervals themselves by the formula (177), which gives

I	II	III	IV	V	VI	VII
$+32^s.37$	$+21^s.65$	$+11^s.08$	$+0^s.06$	$-10^s.48$	$-21^s.31$	$-31^s.51$

which are the distances from the collimation axis. The equatorial intervals of the side threads from the middle thread are, therefore,

	I	II	III	V	VI	VII
i,	$+32^s.31$	$+21^s.59$	$+11^s.02$	$-10^s.54$	$-21^s.37$	$-31^s.57$

which agree with those given on p. 249 as well as can be expected when but four observations on each thread have been taken.

185. *To find the latitude from the observed transits of a star over the prime vertical when the instrument is reversed between the east and west transits, the intervals of the threads being unknown.*—Let

τ, τ' = the hour angles of the star on the same thread at the east and west transits;

then, c denoting the distance of the thread from the collimation axis, we have

$$-\sin c = \sin n \sin \delta - \cos n \cos \delta \cos (\tau - \lambda)$$
$$\sin c = \sin n \sin \delta - \cos n \cos \delta \cos (\tau' - \lambda)$$

the sum of which gives

$$\cot n = \tan \delta \sec \tfrac{1}{2} (\tau' - \tau) \sec [\tfrac{1}{2} (\tau' + \tau) - \lambda]$$

But by (167) we have

$$\cot n \cos \lambda = \tan (\varphi - \beta)$$

and therefore

$$\tan (\varphi - \beta) = \tan \delta \sec \tfrac{1}{2} (\tau' - \tau) \sec [\tfrac{1}{2} (\tau' + \tau) - \lambda] \cos \lambda$$

in which β = inclination of the rotation axis; and in this case, if b and b' are the inclinations in the two positions, we take $\beta = \frac{1}{2}(b + b')$.

If now, to avoid all further consideration of the clock rate, we suppose all the observed times to be reduced to some assumed epoch (T) at which the clock correction is ΔT, and put

T, T' = the clock times on the given thread at the east and west transits, respectively, reduced for rate to the assumed epoch (T),

T_0, T_0' = the same for the middle thread,

we have

$$\tau = T + \Delta T - \alpha \qquad \tau' = T' + \Delta T - \alpha$$

and, since the middle thread is very near the collimation axis,

$$\begin{aligned} \lambda &= \tfrac{1}{2}(T_0' + T_0) + \Delta T - \alpha \\ \tfrac{1}{2}(\tau' - \tau) &= \tfrac{1}{2}(T' - T) = \tfrac{1}{2}\,\text{elapsed sid. time,} \\ \tfrac{1}{2}(\tau' + \tau) - \lambda &= \tfrac{1}{2}(T' + T) - \tfrac{1}{2}(T_0' + T_0) \end{aligned}$$

Hence, if we adopt the following more simple notation,

2ϑ = the elapsed sidereal time between the east and west observations on the same thread $= T' - T$,

t = the mean of the observed times on that thread $= \tfrac{1}{2}(T' + T)$,

t_0 = the mean of the observed times on the middle thread $= \tfrac{1}{2}(T_0' + T_0)$,

and put

$$\gamma = t - t_0 \qquad \lambda = t_0 + \Delta T - \alpha$$

we shall have

$$\left.\begin{aligned} \tan\varphi' &= \tan\delta \sec\vartheta \sec\gamma \cos\lambda \\ \varphi &= \varphi' + \tfrac{1}{2}(b + b') \end{aligned}\right\} \qquad (178)$$

This method of observation and reduction has the same advantage as that of Professor Peirce, in not requiring a knowledge of the thread intervals; and it further enables the observer to reduce each observation independently of all others, and thus to obtain a definite result from one night's work.

Example.—Let us apply this method to the observations taken at Cronstadt, given on p. 252.

For the star γ *Cassiopeæ* we have but three threads to reduce, since thread I. was omitted at the west and thread V. at the east transit. For the others, we proceed as follows:

$$\begin{array}{rrrl} t_0 = & 0^h\ 47^m & 0^s.5 & \\ \Delta T = & + & 30\,.2 & \\ \hline \text{Sid. time} = & 0\ \ 47 & 30\,.7 & \\ \alpha = & 0\ \ 47 & 21\,.5 & \\ \hline \lambda = & 0\ \ \ 0 & 9\,.2 & = 2'\ 18'' \end{array} \qquad \begin{array}{rl} \log\tan\delta & 0.2362409 \\ \log\cos\lambda & 9.9999999 \\ \hline \log\tan\delta\cos\lambda & 0.2362408 \end{array}$$

Neglecting the chronometer rate, which is insensible in these intervals, we have

	II	III	IV
t	$0^h\ 39^m\ 24^s.$	$0^h\ 47^m\ 0^s.5$	$0^h\ 53^m\ 29^s.$
$t \backsim t_0 = \gamma$	0 7 36 .5	0 0 0 .	0 6 28 .5
ϑ	0 21 38 .	0 22 54 .5	0 21 57 .
log sec γ	0.0002393	0.0000000	0.0001733
log sec ϑ	0.0019377	0.0021732	0.0019949
log tan φ'	0.2384178	0.2384140	0.2384090
φ'	59° 59′ 30″.6	59° 59′ 29″.8	59° 59′ 28″.8

$$\begin{aligned} \text{Mean } \varphi' &= 59^\circ\ 59'\ 29''.73 \\ \text{“} \quad b &= \underline{\qquad + \ 1\ .69} \\ \varphi &= 59 \quad 59 \quad 31\ .42 \end{aligned}$$

For δ *Cassiopeæ* we find, in like manner, $\lambda = 1'\ 27''$, log tan δ cos λ $= 0.2284381$; and from the several threads,

	I	II	III	IV	V
γ	$0^h\ 6^m\ 12^s.$	$0^h\ 3^m\ 20^s.5$	$0^h\ 0^m\ 0^s.$	$0^h\ 2^m\ 53^s.5$	$0^h\ 6^m\ 1^s.$
ϑ	0 48 32 .	0 48 49 .5	0 48 55 .	0 48 50 .5	0 48 33 .
φ'	59° 59′ 28″.8	59° 59′ 30″.1	59° 59′ 28″.3	59° 59′ 29″.1	59° 59′ 28″.2

$$\begin{aligned} \text{Mean } \varphi' &= 59^\circ\ 59'\ 28''.90 \\ \text{“} \quad b &= \underline{\qquad + \ 2\ .08} \\ \varphi &= 59 \quad 59 \quad 30.\ 98 \end{aligned}$$

The mean result by the two stars is, then, $\varphi = 59^\circ\ 59'\ 31''.20$, which differs only $0''.03$ from the result found on p. 254, where the thread intervals were used.

186. *To find the latitude from the observed transits of a star over the prime vertical, east and west of the meridian, when the instrument is reversed, at each transit, between the observations of the star on opposite sides of the prime vertical.* (STRUVE's method.)

When the star passes near the zenith, the intervals between its transits over the threads become sufficiently great to allow the observer to reverse the instrument between the observations on two threads. He may then, *first*, observe the star at the east transit on all the threads on one side of the middle thread or prime vertical, and, reversing the axis, *secondly*, observe the star on the same threads on the opposite side of the prime vertical; then, allowing the axis to remain in the last position, *thirdly*, observe the star at the west transit on the same threads, and then, reversing the axis, *fourthly*, observe the star on the same threads on the same side of the prime vertical as at first. By this mode of observation the same thread is alternately a north and a south thread at precisely the same distance from the collimation axis at each of the four observations made upon it. Now, in the equation (166) we have $\tau - \lambda = \frac{1}{2}$ elapsed sidereal time between the east and west transits over the same thread in the same position of the axis: so that, if we put

$t = \frac{1}{2}$ elapsed time between the two observations on a thread in one position of the axis,
$t' =$ ditto for the same thread in the reverse position of the axis,

we have, c being the distance of this thread from the axis,

$$-\sin c = \sin n \sin \delta - \cos n \cos \delta \cos t$$
$$\sin c = \sin n \sin \delta - \cos n \cos \delta \cos t'$$

the sum of which gives

$$\cot n = \tan \delta \sec \tfrac{1}{2}(t + t') \sec \tfrac{1}{2}(t - t')$$

But by (167) we have

$$\cot n \cos \lambda = \tan (\varphi - \beta)$$

in which for β we must here employ the mean of the level determinations in the two positions, or $\beta = \frac{1}{2}(b + b')$. Hence, denoting $\varphi - \beta$ by φ', we find

$$\left.\begin{aligned} \tan \varphi' &= \tan \delta \sec \tfrac{1}{2}(t + t') \sec \tfrac{1}{2}(t - t') \cos \lambda \\ \varphi &= \varphi' + \beta \end{aligned}\right\} \quad (179)$$

where λ will be the same for all the threads, and may be found with sufficient accuracy from any single thread by taking the difference between the right ascension of the star and the mean of the two sidereal times of observation on that thread.*

Each thread thus gives a value of the latitude free from all the instrumental errors. The clock errors, however, have nearly the same effect as in all the other methods: error in the clock *rate* affects t and t'; error in the clock *correction* affects λ.

When there is time, the middle thread may also be observed, once as a north thread and once as a south thread, and the latitude will be found from it, according to the method of the preceding article, by the formula

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* Or we may neglect the factor $\cos \lambda$, and apply a correction to the final mean latitude, as in Art. 188.

$$\tan \varphi' = \tan \delta \sec t \cos \lambda$$

where t will be one-half the elapsed sidereal time between the observations on the middle thread. In taking the mean, the value of the latitude found from the middle thread should have but one-half the weight of the value on any other thread, since it depends on two observations instead of four.

This method is not much used in the field, as portable instruments, usually not very firmly mounted, and never provided with reversing apparatus, cannot be quickly reversed without risk of disturbing the azimuth.

EXAMPLE.*—In the following observation, the axis was reversed immediately after the star had crossed the middle thread at the east transit, and was then left in the same position until after the star had crossed the middle thread at the west transit, when it was again reversed, and, consequently, restored to its first position.

Cronstadt, August 16, 1843.

δ Cassiopeæ.		East transit.	West transit.
		$b = +1''.7$	$b = +1''.2$
	Thread.	Chronometer.	Chronometer.
Circle S.	I	$0^h\ 20^m\ 18^s.5$	$2^h\ \ 9^m\ 50^s.5$
	II	0 22 56 .	2 7 16 .
	III	0 26 9 .	....
Circle N.	III	....	2 4 0 .
	II	0 29 38 .	2 0 32 .
	I	0 32 45 .	1 57 24 .
		$b = -2''.7$	$b = -1''.6$

The chronometer correction at $1^h\ 15^m$ was $+ 40^s.1$; its daily rate, $+ 1^s.74$ on sidereal time. The star's place was

$$\alpha = 1^h\ 15^m\ 40^s.71 \qquad \delta = 59°\ 25'\ 7''.75$$

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* SAWITSCH, *Pract. Astron.*, Vol. I. p. 377.

We find from the middle thread $\lambda = 3^s.9$, $\cos\lambda = 1$. The computation for the several threads may be arranged as follows:

	I	II	III
Diff. obs'd. times *S*.	$1^h\ 49^m\ 32^s.0$	$1^h\ 44^m\ 20^s.0$	$1^h\ 37^m\ 51^s.0$
Chron. Rate	$+\ 0.1$	$+\ 0.1$	$+\ 0.1$
Diff. obs'd. times *N*.	1 24 39 .0	1 30 54 .0	
Chron. Rate	$+\ 0.1$	$+\ 0.1$	
$2t$	1 49 32 .1	1 44 20 .1	1 37 51 .1
$2t'$	1 24 39 .1	1 30 54 .1	
$\frac{1}{2}(t+t')$	0 48 32 .8	0 48 48 .55	0 48 55 .55
$\frac{1}{2}(t-t')$	0 6 13 .3	0 3 21 .5	
$\log\sec\frac{1}{2}(t+t')$	0.0098171	9243	9722
$\log\sec\frac{1}{2}(t-t')$	0.0001600	0466	
$\log\tan\delta\cos\lambda$	0.2284455	4455	4455
$\log\tan\varphi'$	0.2384226	4164	4177
φ'	$59^\circ\ 59'\ 31''.61$	$30''.33$	$30''.60$
β	$-\ 0\ .35$	$-\ 0\ .35$	$-\ 0\ .35$
φ	59 59 31 .26	29 .98	30 .25

Giving the value found from the middle thread but one-half the weight of either of the other two, the mean is $\varphi = 59^\circ\ 59'\ 30''.55$.

187. *To find the latitude from stars observed at only one of their transits over the prime vertical.*—Notwithstanding the simplicity of the preceding methods, it is not always possible to apply them in the field. If the observer has but a short time to remain at a station, he may fail to find a sufficient number of bright stars which pass near his zenith, and, if he uses those which pass at greater zenith distances, much time is lost in waiting. But if he can use stars observed at only one of their transits, he may in two or three hours obtain sufficient data for a very accurate determination of his latitude. The following method is based upon that originally given by BESSEL,* with some modifications, which appear to me to facilitate its application.

If in the general equation (166), where c denotes the distance of a thread from the collimation axis, we substitute $i + c$ for this distance, denoting now by i the distance of the thread from the mean thread, and by c the distance of the mean thread from the axis, we have

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* *Astron. Nach.*, Vol. VI. Nos. 181 and 182.

$$i + c = -\sin n \sin \delta + \cos n \cos \delta \cos (\tau - \lambda)$$

in which τ is the hour angle of the star, and n and λ are determined by the conditions (167).

Each thread gives an equation of this form. The mean of these equations may be found by the aid of our Tables VIII. and VIII.A., according to the method already explained in Art. 173. Thus, T being the mean of the observed times on the several threads, I the interval obtained by subtracting each observed time from this mean, $\varkappa$ and $\log k$ the mean of the several values of these quantities taken from Table VIII. with the argument I, we have

$$T_1 = T + \varkappa$$

and, since here τ is the west hour angle,

$$\tau_1 = T_1 + \Delta T - \alpha$$

Then, i_0 denoting the mean of the equatorial distances of the threads from the mean thread, we have

$$c + i_0 = -\sin n \sin \delta + \frac{\cos n \cos \delta \cos (\tau_1 - \lambda)}{k}$$

or, putting

$$\gamma \cos \delta_1 = \frac{1}{k} \cos \delta$$

$$\gamma \sin \delta_1 = \sin \delta$$

the mean equation is

$$\frac{c + i_0}{\gamma} = -\sin n \sin \delta_1 + \cos n \cos \delta_1 \cos (\tau_1 - \lambda)$$

Developing $\cos(\tau_1 - \lambda)$, and substituting the values of $\sin n$, $\cos n \cos \lambda$, $\cos n \sin \lambda$, from (167),

$$\frac{c + i_0}{\gamma} = -h \cos(\varphi - \beta) \sin \delta_1 + h \sin(\varphi - \beta) \cos \delta_1 \cos \tau_1 + \sin a \cos b \cos \delta_1 \sin \tau_1$$

in which h and β are determined by the conditions

$$h \sin \beta = \sin b$$
$$h \cos \beta = \cos b \cos a$$

But, since we can always put $\cos b = 1$, these conditions give

$b = \beta \cos a$, and $h = \cos a$; and even if a were as great as 1° and $b = 20''$, we should have $b = \beta - 0''.003$: so that we may always put $b = \beta$.

We shall here assume that the instrument can be readily brought within 20′ of the prime vertical, and then we may safely take $h = \cos a = 1$, and substitute a for its sine. Hence we have

$$\frac{c + i_0}{\gamma} = -\cos(\varphi - b)\sin\delta_1 + \sin(\varphi - b)\cos\delta_1\cos\tau_1 + a\cos\delta_1\sin\tau_1$$

Let φ_1 and z be determined by the conditions

$$\cos z \sin \varphi_1 = \sin \delta_1$$
$$\cos z \cos \varphi_1 = \cos \delta_1 \cos \tau_1$$
$$\sin z = \cos \delta_1 \sin \tau_1$$

then

$$\frac{c + i_0}{\gamma} = \sin(\varphi - \varphi_1 - b)\cos z + a \sin z$$

where $\varphi - \varphi_1 - b$ must be of the same order as a and $c + i_0$, and therefore may be substituted for its sine. Again, since in this method of finding the latitude no observation will be regarded as having any value unless some threads on each side of the mean thread have been observed, i_0 will always be so small that no error will arise in practice by putting $\gamma = 1$.* Our equation is, therefore,

$$c + i_0 = (\varphi - \varphi_1 - b)\cos z + a \sin z$$

Now let

$\varphi_0 =$ the assumed latitude,
$a_0 =$ the assumed azimuth of the instrument,
$\Delta\varphi$, $\Delta a =$ the required corrections of these quantities;

then, substituting $\varphi_0 + \Delta\varphi$ and $a_0 + \Delta a$ for φ and a, dividing the equation by $\cos z$, and denoting the known terms by f, *i.e.* putting

$$f = \varphi_1 + b - \varphi_0 - a_0 \tan z + i_0 \sec z \qquad (180)$$

re have

$$c \sec z - \Delta a \tan z - \Delta\varphi + f = 0 \qquad (181)$$

hich is the equation of condition furnished by each star. From l the equations thus formed, the most probable values of c, Δa, id $\Delta\varphi$ will be found by the method of least squares.

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* Should an extreme case occur where the true value of γ was required, it could adily be found by the equations $\gamma \cos \delta_1 = \frac{1}{k} \cos \delta$, $\gamma \sin \delta_1 = \sin \delta$.

The values of φ_1 and z will be most readily found by the formulæ

$$\left.\begin{aligned} \tan\varphi_1 &= \tan\delta_1 \sec\tau_1 = k\tan\delta\sec\tau_1 \\ \tan z &= \tan\tau_1 \cos\varphi_1 \end{aligned}\right\} \quad (182)$$

and it must be observed that $\tan z$ will be negative when $\tan\tau_1$ is negative, that is, when the star is east of the meridian. The sign of the term $c\sec z$ must also be changed when the axis of the instrument is reversed.

EXAMPLE.—The following observations (among others) were taken by BESSEL with a very small portable transit instrument, for the express purpose of demonstrating the advantages of this method.*

Munich, 1827, June 27.

Circle North.	I	II	III	IV	V	Level.
π Lyræ E.	48^m $6^s.4$	46^m $54^s.4$	$11^h 45^m 43^s.2$	44^m $31^s.2$	43^m $16^s.8$	$+4^d.875$
υ Herculis W.	9 36 .4	11 38 .4	12 13 36 .8	15 34 .8	17 35 .6	+0 .403
γ Cygni E.	29 38 .0	28 47 .2	12 27 55 .2	27 2 .6	26 8 .0	+0 .117
Circle South.						
φ Herculis W.	44 47 .2	43 19 .2	12 41 49 .2	40 17 .2	38 37 .6	−1 .966
66 *Cygni* E.	48 40 .8	50 5 .6	12 51 31 .2	52 59 .6	54 32 .8	−1 .876

These observations were taken in the garden of Dr. STEINHEIL's house, where the assumed latitude was 48° 8′ 40″.

The chronometer was a pocket *mean time* chronometer of KESSEL. Its correction to sidereal time at 12^h (by chron.) was $\Delta T = +\ 5^h\ 1^m\ 3^s.31$,† and its rate on sidereal time was $+\ 9^s.19$ per hour.

The equatorial intervals of the threads from the mean of all, expressed in seconds of arc, were as follows, for *circle north;*

I	II	III	IV	V
+ 598″.08	+ 303″.09	+ 6″.19	− 294″.91	− 612″.46

The value of one division of the level was 4″.49. The pivots were of unequal thickness, the correction for which had been previously found to be − 1″.89 for circle north.

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* *Astron. Nach.*, Vol. IX. p. 415.

† See the example on p. 234.

The apparent places of the stars on the given date were as follows:

	α	δ
π Lyræ	$18^h\ 50^m\ 7^s.74$	$43°\ 43'\ 27''.72$
υ Herculis	15 57 27 .55	46 31 23 .21
γ Cygni	20 16 4 .59	39 42 32 .96
φ Herculis	16 3 21 .85	45 23 40 .03
66 Cygni	19 35 33 .81	45 7 14 .89

We shall illustrate the use of our formulæ by giving the reduction of the observations of *π Lyræ* in full. We have, employing the mean time columns of Table VIII.,

π Lyræ	T	I	$\varkappa$	log k
I	$11^h\ 48^m\ 6^s.4$	$-\ 2^m\ 24^s.0$	$-\ 0^s.04$	0.0000239
II	46 54 .4	− 1 12 .0	0 .00	60
III	45 43 .2	− 0 .8	0 .00	0
IV	44 31 .2	+ 1 11 .2	0 .00	59
V	43 16 .8	+ 2 25 .6	+ 0 .04	244
Means	11 45 42 .40		0 .00	0.0000120

Hence we have

$$\begin{array}{rl} T_1 = T + \varkappa = & 11^h\ 45^m\ 42^s.40 \\ \Delta T = & \underline{\ \ 5\ \ \ 1\ \ \ 1.12} \\ T_1 + \Delta T = & 16\ \ 46\ \ 43.52 \\ \alpha = & \underline{18\ \ 50\ \ \ 7.74} \\ \tau_1 = & -2\ \ \ 3\ \ 24.22 = -\ 30°\ 51'\ 3''.3 \end{array}$$

log sec τ_1	0.0662574	log tan τ_1	n9.77621
log tan δ	9.9806553	log cos φ_1	9.82476
log k	0.0000120	log tan z	n9.60097
log tan φ_1	0.0469247	log sec z	0.03208

We shall assume $\varphi_0 = 48°\ 8'\ 40''$, $a_0 = 7'\ 52''$, as in the computation given by **Bessel**;[*] and hence we have

---

[*] These quantities are, of course, arbitrary; but it simplifies the equations of condition to make them as nearly correct as possible. An approximate value of the azimuth may be found from any star by the formula $a_0 = (\phi_1 - \phi_0) \cot z$.

$$
\begin{aligned}
\varphi_1 &= 48^\circ\ 5'\ 21''.64 \\
b &= \qquad +20\ .00 \\
-a_0 \tan z &= \underline{\quad +3\ \ 8\ .33} \\
&\ \ 48\ \ 8\ \ 49\ .97 \\
f &= \qquad +\ 9''.97
\end{aligned}
$$

The equation of condition from π *Lyræ* is, therefore,

$$1.0767\,c + 0.3990\,\Delta a - \Delta\varphi + 9''.97 = 0$$

In the same manner, the equations for the other stars are found to be

$$
\begin{aligned}
1.0269\,c - 0.2336\,\Delta a - \Delta\varphi + 10''.83 &= 0 \\
1.1645\,c + 0.5967\,\Delta a - \Delta\varphi + 15\ .93 &= 0 \\
-1.0468\,c - 0.3094\,\Delta a - \Delta\varphi - 17\ .01 &= 0 \\
-1.0504\,c + 0.3214\,\Delta a - \Delta\varphi - 12\ .62 &= 0
\end{aligned}
$$

From these five equations we find the normal equations,

$$
\begin{aligned}
5.7688\,c + 0.8708\,\Delta a - 1.1709\,\Delta\varphi + 71''.46 &= 0 \\
0.8708\,c + 0.7688\,\Delta a - 0.7741\,\Delta\varphi + 12\ .16 &= 0 \\
-1.1709\,c - 0.7741\,\Delta a + 5.0000\,\Delta\varphi - \ 7\ .10 &= 0
\end{aligned}
$$

whence

$$c = -12''.19 \qquad \Delta a = -4''.09$$
$$\Delta\varphi = -\ 2''.06 \text{ with the weight } 4.203$$

Substituting these values in the equations of condition, we find the residuals as follows:

v	vv
$-2''.72$	7.40
$+1\ .33$	1.77
$+1\ .36$	1.85
$-0\ .92$	0.85
$+0\ .93$	0.86
$[vv] =$	12.73

The number of observations being $m = 5$, and the number of unknown quantities $\mu = 3$, the mean error ε of a single observation is

$$\varepsilon = \sqrt{\left(\frac{[vv]}{m-\mu}\right)} = 2''.52$$

and the mean error of $\Delta\varphi$ is

$$\varepsilon_0 = \frac{2''.52}{\sqrt{4.203}} = 1''.23$$

Hence we have, finally,

$$\varphi = 48^\circ\ 8'\ 37''.94 \text{ with mean error } \pm\ 1''.23$$

The true latitude, found by referring the position of the instrument to the Observatory of Munich, was 48° 8′ 39″.50. Thus, five observations, taken within about one hour with a very small instrument, sufficed to determine the latitude within 1″.5. From the observations of two other evenings combined with the above, the latitude found by BESSEL was 48° 8′ 40″.08, which was only 0″.58 in error.

DETERMINATION OF THE DECLINATIONS OF STARS BY THEIR TRANSITS OVER THE PRIME VERTICAL.

188. The transit of a star over the prime vertical has been used in the preceding articles to determine the latitude of the place of observation when the star's declination is known. Conversely, if the latitude is otherwise known, the observation may be used to determine the star's declination. The modifications of the formulæ given in Arts. 177, &c., necessary for this purpose, are obvious.

When the star passes very near to the zenith, the errors in the time of transit have comparatively small effect upon the computed declination; for, by differentiating the equation

$$\tan \delta = \tan \varphi \cos t$$

we find

$$d\delta = -\tfrac{1}{2} \sin 2\delta \tan t\,.\,dt$$

so that the effect of a given error dt in the hour angle upon the computed declination diminishes with the hour angle itself.

But an error in the assumed latitude φ is not eliminated, though in certain cases it will have less effect than in others; for we have

$$d\delta = d\varphi \cdot \frac{\sin 2\delta}{\sin 2\varphi}$$

The several values of the declination of the same star determined on different dates will, therefore, be affected by the constant error depending upon the error in the latitude, but the differences in these values will nevertheless be accurately found. Hence, the most important use of such observations is not so much to determine the absolute declination of a star as the changes of its declination resulting from aberration, nutation, and parallax.

189. In order to eliminate the instrumental errors in the most complete manner, STRUVE proposed the system of observation given in Art. 186; and, in order to facilitate the application of this system, he gave a new form to the instrument constructed under his direction for the Pulkowa Observatory,—a form which has since been adopted in other observatories.

Plate VI. exhibits the principal features of the Pulkowa prime vertical transit instrument,* made by REPSOLD. The telescope TT is at the end of the horizontal axis DE, which rests in Vs at VV. The pier PP is of a single piece of stone. The apparatus for reversing the instrument is permanently secured within the pier, as shown in the plate, the vertical rod R and its arms aa being raised by the crank f by means of the bevelled wheels c, and thus lifting the telescope out of the Vs. When the telescope is lifted sufficiently to clear the Vs, it is revolved 180° (the exact semi-revolution being determined by a stop d), and is then again lowered into the Vs. The time required in this operation is but 16 seconds; and if the astronomer has commenced an observation with the tube north, he can continue the observation with the instrument reversed, tube south, after 1 minute and 20 seconds, this time being sufficient for the observer to rise, unclamp the instrument, reverse it, and resume his position for the observation. Thus, even with an instrument of large dimensions, the system of observation given in Art. 186 is easily carried out.

The pressure on the Vs is in part removed by the counterpoises WW acting at NN.

The pressure on the two Vs is equalized by placing at D a weight equal to that of the telescope.

The level LL may remain upon the axis during reversal.

The finder F is similar to that described in Art. 120.

The reticule at the focus m contains 15 vertical threads and two horizontal threads, as shown in Fig. 2. All the transits over the vertical threads should be made to occur exactly midway between these two horizontal threads, the telescope being made to follow the star's change of altitude by a fine motion screw (not shown in the plate), the handle of which is within reach of the observer's hand. The equatorial interval between the extreme vertical threads is 15′ 15″ or 61^s of time.

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* *Description de l'observatoire astronomique central de Poulkova* (St. Petersburg, 1845), p. 167.

There is also a movable micrometer thread parallel to the transit threads.

The field is illuminated by light thrown through the horizontal axis and reflected by a mirror at *E* towards the reticule.

190. **Example.**—The following observation was taken by Struve with the instrument above described.*

1842. January 15. *o Draconis.*

	East Vertical.—5°.6 R.		West Vertical.—5°.4 R.	
	Tube S.		*Tube S.*	
Level.	+ 40^d.35	— 35^d.8	+ 40.5	— 35.35
	40 .4	35 .8	40.55	35.35
	40 .4	35 .8	40.5	35.4
	40 .4	35 .8	40.45	35.4
Threads.				
I	17^h 54^m 30^s.7		19^h 42^m 51^s.4	
II	55 8 .65		42 13 .65	
III	55 44 .4		41 38 .0	
IV	56 22 .25		40 59 .85	
V	57 0 .6		40 21 .7	
VI	57 40 .9		39 41 .4	
VII	17 58 19 .5		19 39 2 .7	
	Tube N.		*Tube S.*	
VII	18^h 1^m 4^s.0		19^h 36^m 17^s.85	
VI	1 45 .5		35 37 .0	
V	2 29 .8		34 52 .35	
IV	3 12 .7		34 9 .3	
III	3 57 .6		33 24 .7	
II	4 39 .8		32 42 .1	
I	18 5 26 .35		19 31 55 .6	
Level.	+ 37^d.2	— 39^d.0	+ 37^d.25	— 38^d.7
	37 .2	39 .0	37 .25	38 .7
	37 .2	39 .0	37 .3	38 .7
	37 .15	39 .1	37 .25	38 .7

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* *Astronomische Nachrichten*, Vol. XX. p. 209.

The value of one division of the level was 1''.002. The latitude, $\varphi = 59° \, 46' \, 18''.00$. The correction of the interval between the east and west transits for the rate of the clock was $+ \, 0^s.09$. The temperature of the air is recorded at the time of the observation (in degrees of Réaumur), as the value of a division of the level depends in some degree upon it.

According to formula (179), the declination will be found from these observations by the formula

$$\tan \delta = \tan \varphi' \cos \tfrac{1}{2}(t + t') \cos \tfrac{1}{2}(t - t')$$

where, β being the mean inclination of the axis, we have $\varphi' = \varphi - \beta$, $t = \frac{1}{2}$ elapsed time between the observations on the same thread for "tube south," $t' =$ the same for "tube north." We omit the factor $\cos \lambda$, because a fixed instrument can always be adjusted so accurately that we can put $\cos \lambda = 1$.

But, instead of computing δ directly by this formula, we may find an approximate value by using the constant value of φ in the second member, and then apply a correction for the inclination β. Thus, we find*

$$\left.\begin{aligned} \tan \delta' &= \tan \varphi \cos \tfrac{1}{2}(t + t') \cos \tfrac{1}{2}(t - t') \\ \Delta\delta &= \beta \frac{\sin 2\delta'}{\sin 2\varphi} \\ \delta &= \delta' + \Delta\delta \end{aligned}\right\} \quad (183)$$

in which we make $\Delta\delta$ additive by supposing β to be positive when the *south* end of the axis is too high.

The distance c of any thread from the collimation axis may be found from the two equations

$$\begin{aligned} -\sin c &= \cos \varphi \sin \delta - \sin \varphi \cos \delta \cos t \\ \sin c &= \cos \varphi \sin \delta - \sin \varphi \cos \delta \cos t' \end{aligned}$$

the difference of which gives

$$\sin c = - \sin \varphi \cos \delta \sin \tfrac{1}{2}(t + t') \sin \tfrac{1}{2}(t - t') \qquad (184)$$

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* We have $\dfrac{\tan \delta}{\tan \delta'} = \dfrac{\tan \phi'}{\tan \phi}$, whence we readily deduce

$$\sin(\delta - \delta') = \sin(\phi' - \phi) \frac{\sin(\delta + \delta')}{\sin(\phi + \phi')}$$

which gives the formula for $\Delta\delta$ used in the text, when its sign is changed for the reason given.

The computation of the preceding observation may be arranged in the following form:

	I	II	III	IV	V	VI	VII
W.—E. $2t$	1ʰ 46ᵐ 20ˢ.79	47ᵐ 5ˢ.09	45ᵐ 53ˢ.69	44ᵐ 37ˢ.69	43ᵐ 21ˢ.19	42ᵐ 0ˢ.59	40ᵐ 43ˢ.29
W.—E. $2t'$	1 26 29 .34	28 2 .39	29 27 .19	30 56 .69	32 22 .04	33 51 .50	35 13 .94
$\frac{1}{2}(t+t')$	0 48 42 .53	48 46 .87	48 50 .22	48 53 .60	48 55 .96	48 58 .05	48 59 .31
$\frac{1}{2}(t-t')$	0 5 27 .86	4 45 .67	4 6 .62	3 25 .25	2 44 .64	2 2 .25	1 22 .34
log cos $\frac{1}{2}(t'+t)$	9.9901167	0871	0642	0411	0249	0106	0020
log cos $\frac{1}{2}(t'-t)$	9.9996765	9063	9301	9516	9689	9828	9922
log tan ϕ	0.2345728	5728	5728	5728	5728	5728	5728
log tan δ'	0.2215660	5662	5671	5655	5666	5662	5670
δ'	59° 11′ 39″.00	39″.04	39″.23	38″.90	39″.12	39″.04	39″.21

$$\beta = +\,0''.806 \qquad \begin{aligned} \text{Mean } \delta' &= 59^\circ\ 11'\ 39''.077 \\ \Delta\delta &= \underline{\qquad +\ \ 0\ .815} \\ \delta &= 59\ \ 11\ \ 39\ .892 \end{aligned}$$

By comparing the mean value of δ' with the several values found from the different threads, we find the probable error of a single determination by one thread in the four positions is in this case only 0″.08. This observation, however, was taken when the atmosphere was unusually steady. From a discussion of the observations of 29 days on this star, STRUVE finds the probable error of a single determination by one thread to be 0″.125, and that of the mean of seven threads, consequently, only 0″.047. To this is to be added the probable error of the level determination, which, from the above example, is evidently exceedingly small. STRUVE concludes that, under the most favorable conditions of the atmosphere, the declination is determined by this method with a probable error of not more than 0″.05, and in average circumstances with a probable error under 0″.1.

191. If we wish to compute the time of the transit of the star over the meridian of the instrument from these observations with the utmost rigor, we must take into account the difference of level at the east and west transits over the prime vertical. The effect of a difference of level is the same as that of a difference of latitude: hence, differentiating the equation

$$\cos\tau = \tan\delta\cot\varphi$$

in which τ is the hour angle at the west transit, we have

$$15\,\Delta\tau = \frac{\Delta\varphi\tan\delta}{\sin^2\varphi\sin t} = \frac{\Delta\varphi\sin\delta}{\sin\varphi\,\sqrt{[\sin(\varphi+\delta)\sin(\varphi-\delta)]}}$$

The mean of the times of transit over the east and west vertical, or T_φ, will be increased by $\frac{1}{2}\Delta\tau$. Putting then $\beta' - \beta$ for $\Delta\varphi$, the correction of the time T_0 will be expressed by the formula

$$\Delta T_0 = \frac{(\beta - \beta')\sin\delta}{30\sin\varphi\sqrt{[\sin(\varphi+\delta)\sin(\varphi-\delta)]}} \tag{185}$$

Thus, in the preceding observations, we have at the east transit $\beta = +\,0''.689$, and at the west transit $\beta' = +\,0''.924$, and

$$\beta - \beta' = -\,0''.235 \qquad \begin{array}{rl} (T_0) = & 18^h\ 48^m\ 41^s.09 \\ \Delta T_0 = & -\qquad\quad 0\ .08 \\ \hline \text{Corrected } T_0 = & 18\ \ 48\ \ 41\ .01 \end{array}$$

We can now find the exact azimuth of the instrument. The clock correction at $18^h\ 48^m$ was $+\ 8^s.31$, and the apparent right ascension of *o Draconis* was $18^h\ 48^m\ 50^s.17$: hence

$$\begin{array}{rl} \text{Sid. time} = & 18^h\ 48^m\ 49^s.32 \\ \alpha = & 18\ \ 48\ \ 50\ .17 \\ \hline \lambda = & -\qquad\ \ 0\ .85 = -\,12''.75 \text{ in arc,} \end{array}$$

where λ is the angle which the meridian of the instrument makes with the true meridian. Hence, a being the azimuth of the rotation axis, we have, by the formula $a = \lambda\sin\varphi$,

$$a = -\,11''.0$$

Finally, if we wish to determine the effect of the azimuth upon the observed declination, we have the formula

$$\tan\delta = \frac{\tan\delta_1}{\cos\lambda}$$

in which δ_1 is the declination deduced by assuming $\cos\lambda = 1$, and δ is the true declination. From this we readily deduce

$$\delta - \delta_1 = (\tfrac{1}{2}\lambda)^2 \sin 1'' \sin 2\delta \tag{186}$$

and hence, in the above example,

$$\delta - \delta_1 = 0''.00017$$

which is altogether insignificant.

192. The extreme precision of the method is evident from the above example. Nevertheless, there remains yet a doubt as to

the perfect accuracy of the declination deduced, arising from the possibility of a change of azimuth between the east and west transits. It is evident from the formula

$$\sin c = -\sin n \sin \delta + \cos n \cos \delta \cos (\tau - \lambda)$$

that an increase of λ by the quantity $\Delta\lambda$ has the same effect as an equal decrease of the hour angle τ, and a change of $-\Delta\lambda$ in τ produces a change of $-\frac{1}{2}\Delta\lambda$ in the hour angles used in computing δ. To find the effect of this upon the computed δ, we have, by differentiating the equation

$$\cos \tau = \tan \delta \cot \varphi$$

with reference to τ and δ,

$$\Delta\delta = -\Delta\tau \cos^2 \delta \tan \varphi \sin \tau$$

or, putting $\frac{1}{2}\Delta\lambda$ for $-\Delta\tau$, and eliminating τ,

$$\Delta\delta = \tfrac{1}{2}\Delta\lambda \,.\, \frac{\cos \delta \sqrt{[\sin (\varphi + \delta) \sin (\varphi - \delta)]}}{\cos \varphi}$$

$$= \Delta a \,.\, \frac{\cos \delta \sqrt{[\sin (\varphi + \delta) \sin (\varphi - \delta)]}}{\sin 2\varphi} \qquad (187)$$

The following table, computed by this formula, is given by STRUVE to exhibit the effect of a change of azimuth $\Delta a = 1''$, for different values of $\varphi - \delta$.

$\varphi - \delta$	$\Delta\delta$
0° 0′	0″.000
0 20	0 .042
0 40	0 .060
1 0	0 .074
2 0	0 .108
3 0	0 .136
4 0	0 .162

The values of $\Delta\delta$ here increase very nearly as $\sqrt{\varphi - \delta}$. For ι *Draconis*, the correction would be $\Delta\delta = 0''.055$. STRUVE investigated the probability of a change of azimuth occurring in his instrument. He found that the fluctuations of the azimuth during a whole year had not probably exceeded one second of arc on either side of its mean value, and that even the extreme changes of temperature from winter to summer had not produced any sensible effect upon it. Hence he concludes that since the temperatures at the east and west transits of a star on the same day never differed by more than 2° R. or $4\frac{1}{2}°$ Fahr., and generally but a fraction of a degree, the variations of the azimuth could not have produced any error which amounted to even $0''.01$. It is important to observe that, during the period referred to, the screws for adjusting the azimuth were not touched.

193. *Micrometer observations in the prime vertical.*—When a star passes within a few minutes of the zenith, its *lateral* motion (across the threads) becomes so slow that the observation of the transit over the side threads would occupy too much time. The star may indeed be within the limits of the extreme threads during the whole time from its east to its west transit. In such cases, the movable micrometer thread takes the place of the fixed threads. This may be used in two ways: either by setting the micrometer successively upon round numbers, identical before and after reversing, in which case the observations are reduced precisely as those made on fixed threads; or by setting at pleasure and as often as the time permits, in which case the observations are reduced as follows.

The micrometer reading for the case when the movable thread is in the collimation axis is known approximately: let its assumed value be denoted by M, and its true value by $M + c$. Let us suppose that for "tube south" the micrometer readings increase as the thread is moved towards the north; then, if m is the reading at an observed transit, the thread is at the distance $m - (M + c)$ *north* of the collimation axis, and this distance is to be substituted for c in our fundamental equation (166). In this equation, we shall also put $\lambda = 0$, $n = 90° - \varphi$, on the supposition that the azimuth and inclination of the axis are each zero, since the resulting declination may be corrected by the methods above explained. We have then

$$\begin{aligned}\sin(m - M - c) &= -\cos\varphi\sin\delta + \sin\varphi\cos\delta\cos\tau \\ &= \sin(\varphi - \delta) - 2\sin\varphi\cos\delta\sin^2\tfrac{1}{2}\tau\end{aligned}$$

or, since in the case here considered $\varphi - \delta$ is but a few minutes,

$$m - M - c = \varphi - \delta - \frac{2 \sin \varphi \cos \delta \sin^2 \frac{1}{2} \tau}{\sin 1''}$$

For convenience in computation, let us put

$$e = M - m$$
$$z = \varphi - \delta$$
$$R = \frac{2 \sin^2 \frac{1}{2} \tau}{\sin 1''} \sin \varphi \cos \delta$$

in which $\sin \varphi \cos \delta$ will be constant, and $\log \frac{2 \sin^2 \frac{1}{2} \tau}{\sin 1''}$ may be taken directly from our Table VI.; then the equation becomes

$$z + c = R - e \tag{188}$$

in which e is given by the observation for each thread, and R is to be computed for the several values of τ found from the observed sidereal times and the star's right ascension.

This equation applies to the case of "tube south." When we have "tube north," the equation becomes

$$- m + M + c = \varphi - \delta - \frac{2 \sin \varphi \cos \delta \sin^2 \frac{1}{2} \tau}{\sin 1''}$$

so that, putting in this case

$$e' = m - M$$

we have

$$z - c = R - e' \tag{189}$$

The instrument is reversed but once. The first series of observations is taken before the meridian passage, and the second after it. We thus find from the means of the observations the values of $z + c$ and $z - c$, whence both z and c. The uncorrected declination is then

$$\delta = \varphi - z$$

to which we apply the correction for the level, as in Art. 190, and, if necessary, also the correction for the azimuth according to (186).

It is evident that this method may be applied even to stars whose declinations are somewhat greater than the latitude.

Example.—The following observations are given by Struve from among those taken with the Pulkowa instrument:[*]

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[*] *Astr. Nach.*, Vol. XX. p. 217.

1842, January 15. *ν Ursæ Majoris.*

	East Vertical.		(— 6°.5 R.) West Vertical.	
	Tube S.		*Tube N.*	
Level.	$+ 40^d.25$	$-37^d.3$	$+ 38^d.0$	$-39^d.7$
	40 .3	37 .35	38 .0	39 .7
	40 .3	37 .35	38 .0	39 .7
	40 .3	37 .35	38 .0	39 .7
	Transits.	Microm.	Transits.	Microm.
	$9^h\ 30^m\ 29^s.$	$9^r.315$	$9^h\ 48^m\ 42^s.5$	$14^r.771$
	30 56 .5	9 .550	48 14	14 .527
	31 24 .5	9 .775	47 46	14 .276
	32 0	10 .083	47 17	14 .068
	32 28	10 .298	46 44	13 .825
	32 54	10 .470	46 9	13 .597
	33 29	10 .691	45 35	13 .361
	34 4	10 .879	45 11	13 .232
	34 37	11 .062	44 40	13 .077
	9 35 11	11 .226	9 44 12	12 .942
Level.	$+ 40^d.3$	$-37^d.25$	$+ 38^d.0$	$-39^d.7$
	40 .35	37 .3	38 .0	39 .7
	40 .35	37 .25	38 .0	39 .7
	40 .25	37 .3	38 .0	39 .7

$$\beta = + 0^d.828 = + 0''.324$$

In these observations, in order to avoid any possible err lost motion in the micrometer screw, the thread is always advance of the star by a final *positive* motion of the screw, tl by that motion which increases the readings.

The value of a revolution of the micrometer screw was by the formula

$$r = 28''.682 + 0''.000292\ (9.6 - T)$$

in which T is the temperature indicated by the Réaumur mometer; and, since in this example $T = -6°.5$, we empl

$$r = 28''.6867 \qquad \log r = 1.45768$$

The apparent position of the star on January 15, 1842, according to ARGELANDER'S Catalogue,

$$\alpha = 9^h\ 39^m\ 46^s.1 \qquad \delta = 59°\ 46'\ 24''.$$

The clock was slow $8^s.3$, and hence the clock time of the culmination was $9^h\ 39^m\ 37^s.8$, for which we may, for simpl take $9^h\ 39^m\ 38^s$, since a small error in this quantity will not the final value of z when the hour angles on the opposite sides of the meridian are so nearly equal as in the present case.

With the value $\varphi = 59° 46' 18''$, we find log sin φ cos $\delta = 9.63846$. The assumed value of $M = 12^r.000$; and hence the observations may be reduced as follows:

Tube S.

τ	$\log \frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$	R	$m - M$	$\log(m - M)$	c	$R - c = z + c$	Diff. from mean.
$-9^m\ 9^s$	2.21581	71″.49	2ʳ.685	0.42894	77″.02	− 5″.53	− 0″.09
8 41.5	2.17118	64 .51	2.450	0.38917	70 .28	5 .77	− 0 .33
8 13.5	2.12325	57 .77	2.225	0.34733	63 .83	6 .06	− 0 .62
7 38	2.05842	49 .76	1.917	0.28262	54 .99	5 .23	+ 0 .21
7 10	2.00363	43 .86	1.702	0.23096	48 .82	4 .96	+ 0 .48
6 44	1.94946	38 .72	1.530	0.18469	43 .89	5 .17	+ 0 .27
6 9	1.87075	32 .30	1.309	0.11694	37 .55	5 .25	+ 0 .19
5 34	1.78420	26 .46	1.121	0.04961	32 .16	5 .70	− 0 .26
5 1	1.69385	21 .49	0.938	9.97220	26 .91	5 .42	+ 0 .02
4 27	1.58974	16 .91	0.774	9.88874	22 .20	5 .29	+ 0 .15
					Mean	− 5 .438	

Tube N.

			$M - m$			$z - c$	
$+4^m\ 34^s.$	1.61222	17″.81	0ʳ.942	9.97405	27″.02	− 9″.21	− 0″.03
5 2	1.69678	21 .64	1.077	0.03222	30 .90	9 .26	− 0 .08
5 33	1.78160	26 .31	1.232	0.09061	35 .34	9 .03	+ 0 .15
5 57	1.84204	30 .23	1.361	0.13386	39 .04	8 .81	+ 0 .37
6 31	1.92105	36 .27	1.597	0.20330	45 .81	9 .54	− 0 .36
7 6	1.99551	43 .05	1.825	0.26126	52 .35	9 .30	− 0 .12
7 39	2.06031	49 .98	2.068	0.31555	59 .32	9 .34	− 0 .16
8 8	2.11352	56 .49	2.276	0.35717	65 .29	8 .80	+ 0 .38
8 36	2.16198	63 .16	2.527	0.40261	72 .49	9 .33	− 0 .15
9 4.5	2.20867	70 .33	2.771	0.44264	79 .49	9 .16	+ 0 .02
					Mean	− 9 .178	

Hence we have

$$\begin{array}{rrl} \text{Tube } S. & z + c = & -\ 5''.438 \\ \text{“ } N. & z - c = & -\ 9\ .178 \\ & z = & -\ 7\ .308 \qquad c = +\ 1''.870 \\ & \varphi = & 59° \ 46' \ 18''.000 \\ & \delta = \varphi - z = & 59 \ \ 46 \ \ 25\ .308 \\ \text{Corr. for incl. of the axis} & = & \qquad\quad +\ 0\ .324 \\ & \delta = & 59 \ \ 46 \ \ 25\ .632 \end{array}$$

From the differences in the last column of this computation, we find the probable error of a single observation to be 0″.194, produced by the error of observation and the error of the micrometer. This agrees well with the probable error found for ι *Draconis*, which was 0″.08 for four observations on one thread.

The probable error of four observations of *υ Ursæ Majoris* is $0''.194 \div 2 = 0''.097$, which is somewhat greater than $0''.08$, apparently because it involves the additional error of the micrometer.

The probable error of the mean value of z or of the value of δ found by the preceding micrometer observations is $0''.194 \div \sqrt{20} = 0''.043$. The results obtained by the micrometer have, therefore, very nearly the same degree of precision as those obtained by the fixed threads, when each method is skilfully applied.

The extreme precision of the observations with this instrument in the hands of STRUVE is strikingly exhibited in the accordance of the values of the aberration constant determined from the changes of declination of seven stars, which have already been cited in Vol. I. Art. 440.

CHAPTER VI.

THE MERIDIAN CIRCLE.

194. THE *Meridian Circle*, or *Transit Circle*, is a combination of a transit instrument and a graduated vertical circle. This circle is firmly attached at right angles to the horizontal axis, and is read by verniers or microscopes (see Arts. 18 and 21), which are in some cases attached to the piers, and in others to a frame which rests upon the axis itself.

By means of this combination, the instrument serves to determine both co-ordinates of a star's position,—the right ascension from the time of its transit, and the declination from the zenith distance measured with the circle; or, if the star's place is given, it serves to determine either the local time or the latitude of the place of observation.

For the measurement of declinations, it takes the place of the *Mural Circle*, which consists of a circle mounted upon one side of a pier, the circle being secured to the end of a horizontal axis which enters the pier. As the latter instrument cannot be reversed, and its axis is not symmetrically supported, it is not suited to the accurate determination of right ascensions, and is to be regarded as designed solely for the measurement of declinations. Even for this purpose the meridian circle is preferable, as it admits of reversal; and there is always an advantage in combining determinations made in reverse positions of an instrument, whereby unknown errors may be either wholly or in part eliminated. I shall, therefore, not treat specially of the mural circle. It is not probable that any more instruments of that form will hereafter be constructed; and the method of using those that exist will readily be understood by any one who has mastered the meridian circle.

195. Plates VII., VIII., and IX. represent a meridian circle of Repsold, belonging to the U. S. Naval Academy, and mounted at Annapolis in 1852. It is almost identical in form with the meridian circles constructed by the same artist for Struve and Bessel at the Pulkowa and Königsberg Observatories.

It has two circles, *CC* and *C′C′*, of the same size, but only one of these, *CC*, is graduated finely; this is read by four microscopes, two of which are seen at *RR*. The microscopes are carried upon a square frame which is centred upon the rotation axis itself: the form of this frame is shown in Plate IX., where the instrument is represented upon the reversing car. The horizontal sides of the frame carry two spirit levels *l*, *l*, by which any change of inclination of the frame with respect to the horizon may be detected.

The second circle *C′C′*, constructed of the same size as the first for the sake of symmetry, is graduated more coarsely, is read at either of two points, and is used only as a finder.

The counterpoises *WW* act at *XX*, points nearly equidistant between the telescopes and the Vs, and very near to the circles; an arrangement which prevents the possibility of any appreciable flexure in the horizontal axis, at the same time that the pressure on the Vs is reduced to a very small quantity.

The inclination of the rotation axis is measured with a *hanging level LL*.

An arm *FG*, turning upon a joint at *F*, receives, when horizontal, an arm which is connected with a collar upon the rotation axis. By turning a screw, the head of which is at *G*, the telescope is clamped in the collar, and then a screw (not seen in the drawing) acting horizontally near *G* gives fine motion to the telescope by acting upon the vertical arm.

Another arm *fg*, nearly similar in its form and arrangement to *FG*, receives a vertical arm attached to the microscope frame. Screws acting horizontally at *g* upon the vertical arm serve to adjust the frame.

These arms are shown in Plate VIII. as they appear when thrown down and out of use while the instrument is being reversed. In this plate is also seen the arrangement of the vertical arms and the friction rollers by which the counterpoises act upon the horizontal axis, together with the form of the Vs.

The field is illuminated by light thrown into the interior of the telescope through tubes at *AA* and reflected towards the reticule by a mirror in the central cube. The quantity of light is regulated by revolving discs with eccentric apertures at the extremities of the tubes nearest to the Vs. These discs are revolved by means of a cord to which hangs a small weight *S*.

The reticule at *m* contains seven transit threads and three micrometer threads at right angles to the transit threads. These three threads have a common motion, their distance from each other being constant. This distance being known, an observation on either of the extreme threads can be reduced to the middle thread. The micrometer thus arranged is intended for the measurement of small differences of declination, and also for the measurement of absolute declinations when used in conjunction with the graduated circle, as will be fully explained hereafter.

The graduated circle of this instrument is nearly 30 inches in diameter, and reads directly to 2″ by the graduations on the micrometer heads of the reading microscopes; and by estimating the fraction of a graduation of the micrometer head, the reading is carried down to 0″.2. This is a sufficiently great degree of accuracy of reading to correspond to the dimensions and optical power of this instrument; but in larger instruments the reading is sometimes carried down to 0″.05, or even less.

The discussion of the errors of the circle of this instrument is given in Arts. 28, 32, and 33.*

---

* The errors of the circle may not be constant, since they may fluctuate with the temperature of its various parts. We may, however, assume that the errors at different temperatures will be the same, provided the expansion of the circle for an increase of temperature is uniform throughout all its parts. For the greatest precision, therefore, we should endeavor to secure this condition of *uniform temperature*,

A mercury collimator should be placed permanently beneath the floor directly under the centre of the instrument, covered by a movable trap-door.

I proceed to consider the methods of observing with the meridian circle. Its application as a transit instrument will be sufficiently clear from the preceding chapter. It is necessary to treat here only of the use of the circle and micrometer in the measurement of nadir distance, zenith distance, polar distance, or altitude of a star, from which either the declination of the star or the latitude is found.

196. *Nadir point.*—The first of the methods of using the instrument which I shall treat of is that in which all observations with the circle are referred to the nadir. Let us first suppose the instrument to be perfectly adjusted in the meridian, and the observation of a star to be made at the instant of its transit. The nadir point is obtained by directing the telescope vertically towards the mercury collimator. To take the simplest case, let us suppose the sight line to be determined by a fixed horizontal thread (at right angles to the transit threads). Let this thread be brought into coincidence with its reflected image. The sight line is then vertical, and the reading of the circle (by which we always understand the mean of all the microscopes added to the degrees and minutes under the first microscope, or microscope *A*) represents the nadir point of the circle. Let this reading be denoted by C_0. The telescope being then directed towards a star, and the fixed horizontal thread being made to bisect the star at the instant ot the transit over the middle vertical thread, let the circle reading be C'. Then the apparent nadir distance of the star, which I shall denote by N', will be

$$N' = C' - C_0$$

---

and, for this purpose, it is advisable to make the piers sufficiently high and broad to protect the whole circle; for, since the temperature of the piers will often differ from that of the circle, the radiation from them will tend to produce unequal temperatures in the different parts of the circle, unless the latter is equally exposed to this radiation throughout. But even this arrangement will fail of its object if the temperature of the piers is not uniform; and therefore they must be protected against fluctuations of temperature as much as possible; for example, by first coating them with oil or some other preparation to exclude moisture, then wrapping them in cloth, and finally encasing them in wood, as proposed by Dr. GOULD for the meridian circle of the Dudley Observatory.

and this distance is usually reckoned from 0° to 360° from the nadir, through either the south point or the north point, according to the direction in which the graduations increase. This direction is different in the two positions of the rotation axis. Supposing the position of the axis to be indicated by that of the circle itself, let us assume that the nadir distance is reckoned through the *south* point for *circle east*, and through the north point for *circle west*. If we denote the apparent zenith distance of the star *south* of the zenith by z', we shall then have

$$z' = \pm (180° - N') \quad \left\{ \begin{array}{l} + \text{ for circle east} \\ - \text{ for circle west} \end{array} \right\}$$

In obtaining the circle readings C_0 and C', the correction for *error of runs*, when such error exists, must be applied as explained in Art. 22. But, with the aid of the telescope micrometer, we can avoid the error of runs, as follows. In observing the nadir point, set the circle so that an exact division is under or nearly under the zero of one of the reading microscopes, that is, so that all the microscopes will read nearly $0''$: their mean will not require any sensible correction for runs. But the fixed thread will then not be in coincidence with its image. Measure the distance of the fixed thread from its image by the micrometer. One-half this distance, being applied to the circle reading, will give the reading for absolute coincidence. In like manner, in observing the star, set the circle again upon an exact division, and bisect the star with the micrometer thread; the distance of the micrometer thread from the fixed thread, being applied to the circle reading, will give the required reading C'.

But, when the micrometer is employed, it is altogether preferable to dispense with the fixed thread and to depend solely upon the movable one. Thus, to determine the nadir point, having brought the circle division which is nearest to the nadir point reading under microscope A, let the mean reading obtained from all the microscopes be called C_0. Bring the micrometer thread into coincidence with its image, and let the micrometer reading be M_0, which we shall suppose to be converted into arc by multiplying by the value of a revolution found according to Art. 46 or 47. It is now evident that when the telescope is directed upon a star, if the micrometer reading remains M_0 while the thread bisects the star and the circle reading is C', the nadir distance is $C' - C_0$, precisely as if the micrometer thread were

. But the reading C' will, in general, involve an error of to avoid which, set the circle as before upon a neighboring division, and let the reading be still called C'; then bisect ar with the micrometer thread, and let the reading be M'; adir distance of the star will be

$$N' = (C' - C_0) + (M' - M_0) \qquad (190)$$

actice, this method will be found much simpler than it at appears. The finder should always be adjusted so that minutes in its reading correspond to whole minutes of the ipal circle. Then, in all observations of the nadir point, t the finder to the same exact division; and, in observing ar, we compute its approximate nadir distance to the nearest te, and set the finder upon this minute.

the above formula, we suppose the micrometer readings to ase with the circle readings.

AMPLE.—On May 4, 1856, the telescope of the Meridian e of the Naval Academy was directed to the nadir by setting nder upon 0° 0′, and the mean of the four microscopes gave ircle reading

$$C_0 = 359^\circ\ 59'\ 54''.70\ (\text{or } -0^\circ\ 0'\ 5''.30)$$

micrometer thread was then brought alternately north and of its own image in the collimator, so as to form each time are with the middle transit thread and its image (as in Art. and the micrometer readings were as follows:

Image N.	S.	Means.
$5^r\ 33^d.4$	$40^d.8$	$5^r\ 37^d.10$
32 .9	40 .4	36 .65
33 .0	40 .3	36 .65
33 .5	40 .5	37 .00

$$M_0 = 5^r\ 36^d.85$$

t M_0 was the reading when the micrometer thread was in idence with its image.

telescope was then directed to *Polaris* at its upper culmi- by setting the finder at 229° 32′ (the latitude being 38° 59′, the declination 88° 32', and the refraction 1', approximately;) and at the time of the star's transit, the micrometer thread bisecting the star, there were found

$$\begin{array}{ll} \text{Circle reading } C' = 229^\circ\ 32'\ 7''.47 \\ \text{Microm. `` } \quad M' = 5^r\ 50^d.6 \end{array}$$

The value of one division of the micrometer was 0''.927. Hence

$$\begin{array}{rrl} & C' - C_0 = & 229^\circ\ 32'\ 12''.77 \\ M' - M_0 = +\,13^d.75 = & & \underline{\quad\quad\ +\,12\ .75} \\ & (N') = & 229\ \ 32\ \ 25\ .52 \end{array}$$

This is the apparent nadir distance upon the supposition that the position of the reading microscopes (which rest on the axis of the telescope*) remained absolutely fixed while the instrument revolved from the nadir to the star. To determine this, the spirit level was applied to the microscope frame. At the nadir reading, the inclination of the frame was $i_0 = -\,1''.23$, and at the observation of the star it was $i' = -\,1''.54$; and hence we have

$$\begin{array}{rl} (N') = & 229^\circ\ 32'\ 25''.52 \\ i' - i_0 = & \underline{\quad\quad\quad -\,0\ .31} \\ N' = & 229\ \ 32\ \ 25\ .21 \end{array}$$

In this observation, the circle was east, and the nadir distance was reckoned through the south point.

197. Since C_0 and M_0 will be applied in reducing all the observations made on the same day, or so long as these quantities are regarded as constant, it will be convenient to combine them once for all. We may either convert the micrometer reading into seconds of arc and add it to the circle reading, which will give the circle reading when $M_0 = 0$; or convert the seconds of the circle reading into divisions of the micrometer and add it to the micrometer reading, which will give the micrometer reading when $C_0 = 0$. Thus, if we take the latter method in the preceding example, we have $C_0 = -\,5''.30 = -\,5^d.72$ of the micrometer. We then take $(M) = C_0 + M_0 = 5^r\ 36^d.85 - 5^d.72 =$

---

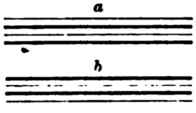
* As this construction involves the necessity of an additional observation, and thus introduces another source of error, it appears to be preferable to attach the reading microscopes permanently to the piers, provided the piers are well guarded against changes of temperature which might alter the relative positions of the microscopes.

$5^r\ 31^d.13$, which we may call the *micrometer zero;* and in any observation of a star when the circle reading is C' and micrometer reading M', the nadir distance will be simply $(N') = C' + M' - (M)$. In this example, therefore, we should have

$$\begin{array}{rrrrr} & C' = & 229^\circ & 32' & 7''.47 \\ M' - (M) = + 19^d.47 = & & & + 18 & .05 \\ \hline & (N') = & 229 & 32 & 25\ .52 \end{array}$$

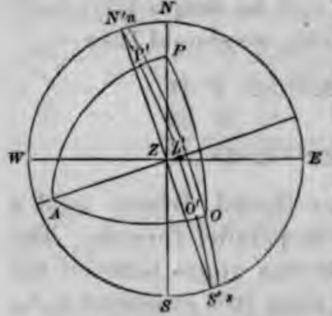
198. Instead of a single micrometer thread, **Bessel** used a double one, consisting of two very close parallel threads. The sight line is then a line which bisects the angle between the threads, and a star is always observed when it is estimated to be midway between them. It was the opinion of **Bessel** that even greater accuracy was attainable in this way than in bisecting a star by a single thread. Although there may be some doubt of this being true for all observers, still the method has advantages in determining the nadir point. The sight line determined by the middle point between the threads will be vertical when each thread is in coincidence with the image of the other thread. But, as we cannot depend upon such directly observed coincidences, the micrometer reading for coincidence is found by taking the mean of two observations, at one of which the image of one of the threads is placed midway between the threads, and at the other the image of the other thread is so placed. Thus, at one observation we make the observation *a*, Fig. 47, and at the other the observation *b*, and take the mean of the corresponding readings.

Fig. 47.



199. *Reduction to the meridian.*—In the above method of observation, the determination of the nadir point is made very precise by repeating the readings of the circle and micrometer, but the reading for the star depends upon a single observation. In order to give both measures at least equal precision, we must make several bisections of the star by the micrometer thread during the passage of the star across the field. But, since the star in general describes a small circle in the field, all the measures on either side of the meridian will require a correction. In investigating this correction, I shall suppose that the instrument is not precisely in the meridian, in order to see what effect its errors have upon the observed declination.

Fig. 48.



In Fig. 48, constructed as in Art. 123, let O be the position of the star. The great circle described by the telescope is $N'Z'S'$, and Z' is the zenith of the instrument. The arc AO drawn from the pole of the great circle $N'Z'S'$ to the star intersects this circle in O', and OO' represents the micrometer thread which bisects the star, since this thread is also perpendicular to the plane of the instrument, and $O'O = c$ is the distance of the star from the collimation axis. If the telescope were directed to the pole, the thread would coincide with PP', P' being the point in which the great circle AP intersects $N'Z'S'$. Hence, P' is the apparent pole of the instrument, and the apparent polar distance of the star, as given by the instrument, is $P'O' = 90° - \delta'$ (denoting the instrumental declination by δ'). But, since the triangle $P'AO'$ is right angled at P' and O', the angle $P'AO'$ is measured by $P'O'$. We have, therefore, in the triangle PAO (with the notation of Art. 123), the sides $PA = 90° - n$, $AO = 90° + c$, $PO = 90° - \delta$, with the angle $APO = 90° + \tau - m$, and the angle $PAO = 90° - \delta'$. Hence, by Sph. Trig.,

$$\left.\begin{aligned} \sin\delta &= -\sin n \sin c + \cos n \cos c \sin\delta' \\ \cos\delta \sin(\tau - m) &= \cos n \sin c + \sin n \cos c \sin\delta' \\ \cos\delta \cos(\tau - m) &= \cos c \cos\delta' \end{aligned}\right\} \quad (191)$$

in which δ is the corrected declination,[*] τ is the east hour angle of the star, and m and n are the instrumental constants as determined by transit observations (Art. 151). But, since n is exceedingly small (seldom more than $0^s.5 = 7''.5$) and c not more than $15'$ even when the star is observed near one of the extreme transit threads, the product $\sin c \sin n$ will be insensible, and we may always put $\cos n = 1$. The first and third of these equations, therefore, become

$$\sin\delta = \cos c \sin\delta'$$

$$\cos\delta \cos(\tau - m) = \cos c \cos\delta'$$

whence

$$\tan\delta = \cos(\tau - m) \tan\delta' \quad (192)$$

---

[*] That is, δ is the apparent declination (affected by refraction and parallax) as it would be given by an observation in the meridian with a perfectly adjusted instrument.

from which it appears that the only correction for the error of the instrument with respect to the meridian is the subtraction of the constant m from the hour angle. The value of δ will be found more conveniently by developing it in series by Pl. Trig. Art. 254; we find

$$\delta = \delta' + \frac{q \sin 2\delta'}{\sin 1''} + \frac{q^2 \sin 4\delta'}{2 \sin 1''} + \&c.$$

in which

$$q = -\frac{\sin^2 \frac{1}{2}(\tau - m)}{1 - \sin^2 \frac{1}{2}(\tau - m)} = -\tan^2 \tfrac{1}{2}(\tau - m)$$

As it is more convenient to employ $\sin^2 \frac{1}{2}(\tau - m)$ instead of $\tan^2 \frac{1}{2}(\tau - m)$, because tables of the former quantity are in common use (see Tables V. and VI.), we develop q in the form

$$\begin{aligned} q &= -\sin^2 \tfrac{1}{2}(\tau - m)\,[1 - \sin^2 \tfrac{1}{2}(\tau - m)]^{-1} \\ &= -\sin^2 \tfrac{1}{2}(\tau - m) - \sin^4 \tfrac{1}{2}(\tau - m) - \&c. \end{aligned}$$

and, substituting this value, we find

$$\delta = \delta' - \frac{\sin^2 \frac{1}{2}(\tau - m)}{\sin 1''} \sin 2\delta' - \frac{2 \sin^4 \frac{1}{2}(\tau - m)}{\sin 1''} \sin 2\delta' \sin^2 \delta' \quad (193)$$

where the last term is usually insensible, and the term $\frac{\sin^2 \frac{1}{2}(\tau - m)}{\sin 1''} \sin 2\delta'$ is called the reduction to the meridian.* In computing this term, we may use δ for δ'. The correction is always subtractive from the instrumental declination. If, however, we wish to apply it to the observed nadir distance N', we must observe the sign of N' in (190). For circle east, the reduction will be additive to N', and for circle west, subtractive from N'.

Example.—In the observation of *Polaris* on May 4, 1856, p. 287, the star was not only observed at the time of its transit, but it was bisected by the micrometer thread a number of times during its passage over the field, the clock being noted at each bisection, as in the following table, which contains also the reduction of the observations:

---

* The last term of the series becomes a maximum for a given value of $\tau - m$ when $\delta = 60°$, in which case the value of the term is $\frac{\sin^4 \frac{1}{2}(\tau - m)}{\sin 1''} \cdot \frac{3}{4}\sqrt{3}$, which amounts to $0''.01$ only when $\tau - m = 6^m\,23^s$. For $\delta = 88°\,30'$, the term amounts to $0''.01$ only when $\tau - m = 12^m$.

T	M'	$M' - M_0 = M''$		$\alpha' - T = \tau - m$	R	$M'' + R$	Diff. from mean.
$1^h\ 1^m\ 51^s$.	$5^r\ 50^d\ 5$	$+ 13^d.65$ =	$+ 12''.65$	$+ 2^m\ 52^s$.	$- 0''.41$	$+ 12''.24$	$- 0''.20$
2 17	50 .9	14 .05	13 .02	2 26	0 .30	12 .72	+ 0 .28
2 49	50 .8	13 .95	12 .93	1 54	0 .18	12 .75	+ 0 .31
3 16	50 .5	13 .65	12 .65	1 27	0 .11	12 .54	+ 0 .10
3 35	50 .2	13 .35	12 .38	1 8	0 .06	12 .32	− 0 .12
4 0	50 .4	13 .55	12 .56	0 43	0 .03	12 .53	+ 0 .09
4 30	50 .8	13 .95	12 .93	+0 13	0 .00	12 .93	+ 0 .49
4 57	50 .4	13 .55	12 .56	−0 14	0 .00	12 .56	+ 0 .12
6 11	49 .4	12 .55	11 .63	1 28	0 .11	11 .52	− 0 .92
6 37	50 .4	13 .55	12 .56	1 54	0 .18	12 .38	− 0 .06
7 0	49 .8	12 .95	12 .00	2 17	0 .26	11 .74	− 0 .70
7 24	51 .2	14 .35	13 .30	2 41	0 .36	12 .94	+ 0 .50
7 55	50 .9	14 .05	13 .02	−3 12	− 0 .51	+ 12 .51	+ 0 .07
					Mean	+ 12 .44	

The column T contains the observed clock times; M' the micrometer reading at each bisection of the star; $M' - M_0$ is found from the observation of the nadir, which gave $M_0 = 5^r\ 36^d.85$, and M'' is the value of $M' - M_0$ in arc, the value of a division being $0''.927$. To find $\tau - m$, we observe that the hour angle τ is found by the formula

$$\tau = \alpha - (T + \Delta T)$$

α being the right ascension of the star and ΔT the clock correction, and hence

$$\tau - m = \alpha - \Delta T - m - T$$

or, putting

$$\alpha' = \alpha - \Delta T - m$$

we have

$$\tau - m = \alpha' - T$$

In the present example, the value of m was $+ 0^s.42$, and ΔT was $+ 1^m\ 2^s.85$. The apparent place of the star, from the American Ephemeris, was

$$\alpha = 1^h\ 5^m\ 46^s.29 \qquad \delta = 88^\circ\ 32'\ 26''.00$$

Hence, $\alpha' = 1^h\ 4^m\ 43^s.0$, the difference between which and each T is given in the column $\tau - m$.

The reduction to the meridian, here denoted by R, is conveniently computed by the aid of Table VI., under the form

$$R = -\frac{2\sin^2\frac{1}{2}(\tau - m)}{\sin 1''}\cos\delta\sin\delta \qquad (194)$$

This reduction is here to be applied to the observed nadir dise with the same sign as to the declination, for the finder was , and the nadir distance, being reckoned through the south ; over the zenith, increases with the declination. The two tities M'' and R being applied to the difference of the circle ngs for the nadir point and the star, we have the apparent distance of the star in the meridian. The sum $M'' + R$ d then be the same for each observation, and we have here l its value for each in order to determine the probable error servation. From the "differences from the mean" in the olumn, we find that the probable error of a single observa- vas $0''.28$, which includes the error in bisecting the star by hread, the error arising from unsteadiness of the star, and s of the micrometer.

e meridian nadir distance of the star from the mean of all bservations is then found as follows:

$$
\begin{array}{rr}
\text{(From page 288) } C' - C_0 = & 229^\circ\ 32'\ 12''.77 \\
M'' + R = & +\ 12\ .44 \\
\text{Corr. for incl. of microscopes } = i' - i_0 = & -\ \ 0\ .31 \\
\hline
N' = & 229\ \ 32\ \ 24\ .90
\end{array}
$$

observation was taken to determine the latitude, and, in r to find the refraction, the barometer and thermometer observed both before and after the observation, as follows:

	At $1^h\,0^m$.	At $1^h\,12^m$.	Means.
Barometer	$30^{in}.176$	$30^{in}.210$	$30^{in}.193$
Attached Therm.	56°.	56°.5	56°.3
External "	54 .9	54 .6	54 .75

e, using BESSEL's Refraction Table, we find

$$
\begin{array}{rr}
-z' = & 49^\circ\ 32'\ 24''.90 \\
\text{Refraction} = & 1\ \ 8\ .05 \\
\hline
-z = & 49\ \ 33\ \ 32\ .95 \\
\delta = & 88\ \ 32\ \ 26\ .00 \\
\hline
\varphi = & 38\ \ 58\ \ 53\ .05
\end{array}
$$

). *Horizontal point.—Observation of a star by reflection.*—The d method of using the instrument is that in which the ent altitude of a star is determined by taking half the ar distance between the star and its image reflected in a basin of mercury. The direct observation of the star is usually made before the meridian transit, and that of the reflected image after the transit, or *vice versa,* and each is reduced to the meridian The difference of the two reduced circle readings (*plus* the difference of the micrometer readings if the observations are made on the movable thread) is twice the meridian altitude. The half sum of these readings is the reading when the sight line is horizontal, and represents the *horizontal point* of the circle.*

In observing equatorial stars by this method, the circle is set approximately for the direct observation, and the microscopes read off before the star comes into the field. Then one or more bisections of the star are made, with the micrometer thread, before the star arrives at the middle transit thread. The telescope is then quickly turned towards the mercury and clamped at the approximate position of the reflected image, several bisections are made with the micrometer, and finally the circle is again read off. That no time may be lost in setting the circle upon the reflected image, a spirit-level finder attached to the tube of the telescope is previously set to the approximate depression of the image; the telescope is then revolved until the bubble plays.

In the case of stars near the pole, the circle may be read off a number of times during the transit, as in the following example from BESSEL.

EXAMPLE.—The following observations of α *Ursæ Minoris* were taken by BESSEL with the Repsold meridian circle of the Königsberg Observatory in 1842, April 22. The star, or its reflected image, was brought in the middle between the two close threads of the micrometer by moving the telescope by the tangent screw, the micrometer thread being used as fixed, and the circle was read off after each observation. Five direct observations are preceded and followed by three reflection observations.

---

* The determination of the horizontal point by reflection observations should be used, in conjunction with the other methods given in the text, for the sake of verification. Indeed, it is desirable that *all* the instrumental constants should be found by at least two independent methods. The construction of the instrument so that this shall always be possible presents difficulties, which, however, have been successfully overcome by DR. B. A. GOULD in the large meridian circle constructed under his direction for the Dudley Observatory.

α Ursæ Minoris.—Upper Culmination.

Clock.	$\tau - m$	Circle.	R	Meridian.
$0^h\ 45^m\ 54^s$	$17^m\ 20^s$	$146°\ 15'\ 11''.0$	$+15''.8$	$146°\ 15'\ 26''.8$
49 1	14 13	16 .9	+10 .6	27 .5
51 6	12 8	20 .2	+ 7 .7	27 .9
54 9	9 5	33 44 44 .0	− 4 .3	33 44 39 .7
58 53	4 21	41 .5	− 1 .0	40 .5
1 2 54	0 20	40 .5	0 .0	40 .5
7 28	4 14	42 .8	− 0 .9	41 .9
12 6	8 52	45 .6	− 4 .1	41 .5
18 25	15 11	146 15 15 .4	+12 .1	146 15 27 .5
21 27	18 13	10 .4	+17 .4	27 .8
23 46	20 32	5 .4	+22 .1	27 .5

Mean. Direct		33 44 40 .82
" Reflect.		146 15 27 .50
App. merid. zen. dist.		33 44 36 .66
Barom. $29^{in}.808$ Att. Therm. 47°.1 F. Ext. " 49 .0 " } Refraction		+ 38 .76
Correction of the circle graduation	+ 0″.470	
Corr. for distance of mercury	+ 0 .018	+ 0 .49
Star's polar distance		1 31 53 .53
Complement of latitude		35 17 9 .44
$\varphi =$		54 42 50 .56

In computing $\tau - m$ by the form $\alpha' - T$, we have assumed $\alpha' = 1^h\ 3^m\ 14^s$. The circle readings are the means obtained from the readings of four microscopes.

The reduction to the meridian R is computed for the reflection observations by the same formulæ as for direct ones, only changing its sign.

The correction of the circle graduation was derived by BESSEL from a special investigation of the errors of those divisions which come into use in the observation of *Polaris* by direct and reflection observations at its upper culmination. For a given zenith distance z, the four divisions that come into use in the direct observation by the use of the four microscopes are z, $90° + z$, $180° + z$, $270° + z$; and in the reflection observation, $360° - z$, $90° - z$, $180° - z$, and $270° - z$. The correction $0''.470$ is here the mean of the corrections of these eight divisions for $z = 33°\ 44'$, the sign of the correction for the reflection observations being changed.*

---

* See BESSEL, in *Astron. Nach.*, Nos. 481 and 482.

The correction for the distance of the mercury from the instrument is simply the difference of the latitude of the mercury basin and the centre of the telescope. For in this method we really measure the angle between the direct and reflected rays which is formed at the surface of the mercury, and, consequently, the latitude determined is that of the mercury. The basin was here north of the instrument, and the deduced latitude would require a subtractive correction, or the zenith distance an additive one.

To find the horizontal point of the circle corrected for the division errors, we have, according to Bessel, for $z = 33° 44'$ in the direct observation, the correction $+ 0''.156$, and for the supplement of this the correction $- 0''.784$, the half difference of which is the correction $+ 0''.470$ used above, and the half sum $- 0''.314$ is the correction of the horizontal point found by taking the mean of the circle readings in the direct and reflected observations. Thus, we have

Mean of circle readings $= 90° \; 0' \; 4''.16$
Corr. of graduations $= \qquad - 0 \; .31$
Horizontal point $= 90 \;\; 0 \;\; 3 \; .85$

The *zenith point* of the circle is, therefore, 0° 0′ 3″.85. So long as the state of the instrument is unchanged, this is the constant correction of all zenith distances observed, additive or subtractive, according as the object is south or north of the zenith.

201. The nadir, horizontal, and zenith points of the circle are all determined when any one of them is determined,* and therefore we ought to be able to combine the results obtained by the mercury collimator and by reflection observations of stars. Nevertheless, observers have sometimes found discrepancies between the two methods which appeared to be greater than could fairly be ascribed to errors of observation. Among the sources of error which may produce such discrepancies, we may here mention the *personal equation* in bisecting a star by a micrometer thread. Prof. J. H. C. Coffin† has demonstrated the existence of such an equation, more or less constant, between different observers, by comparing the declinations of the same

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* Provided the errors of division and of flexure have been duly eliminated.

† *Astronomical Journal*, Vol. III. p. 121.

star obtained by the different observers using the mural circle of the Washington Observatory during the years 1845 to 1849 inclusive, the declinations having all been reduced to the same epoch. He also found a constant difference between the declinations of zenith stars observed by himself when they were observed as southern stars—*i.e.* with the body fronting south—and when they were observed as northern stars, and this under conditions which excluded the hypothesis of a parallax resulting from a maladjustment of focus. This difference amounted to nearly $0''.5$.

A really constant error in bisecting a star will affect the zenith distances of all stars alike, but will have opposite effects upon the deduced declinations of stars north and south of the zenith. It will also have opposite effects upon the declination of the same star deduced from direct observations and by reflection; and hence the discordance between the results of these two kinds of observations will be twice that error. It will also cause the zenith points determined from north and south stars to differ by twice the error of bisection.

Professor COFFIN also suggests that the discrepancies referred to may possibly be produced, in part at least, by a habit of making the bisection constantly before or constantly after the instant for which it is recorded, in which case the error will vary with the declination. Thus, if the observation is recorded as made at the time the star passes the middle thread, and the observer always makes the bisection at a *constant time* before or after the transit, the error will be simply the reduction to the meridian for this time, and, consequently, proportional to $\sin 2\delta$; but if he observes at the *constant distance* c from the middle thread, the error in the time being $c \sec \delta$, the corresponding error in the declination will be proportional to $c^2 \sec^2 \delta \sin 2\delta$, that is, proportional to $\tan \delta$.

Inclination of the micrometer thread is another source of error, which should always be attended to and removed by adjustment if possible, or by computing the correction for it. It is evident that the error in the observed declination will be proportional to the distance of the point at which the observation is made from the middle thread. The inclination will be determined by bisecting a star at two extreme points on the right and left of the field. The difference of the two observations, when both have been reduced to the meridian, will give the required correction for inclination. A star near the pole will be preferable for this purpose, as a number of bisections may be made at each extremity of the field.

202. EXAMPLE.—As an example involving all the various corrections, I extract the following from the Greenwich Observations:

Zenith distances observed with the Transit Circle.—Greenwich, April 16, 1852.

Object.	Pointer.	Microscopes of Circle.						Telescope micrometer.	$N.$
		A	B	C	D	E	F		
η *Bootis* (Reflected)	147° 20′	$2^r.173$	$2^r.130$	$2^r.270$	$1^r.802$	$2^r.085$	$2^r.150$	$19^r.110$	1
η *Bootis* (Direct)	32 0	0 .942	0 .901	1 .038	0 .612	0 .820	0 .903	20 .163	7
Nadir point	179 40	0 .753	0 .712	0. 818	0 .420	0 .636	0 .743	21 .364	

At the observation of η *Bootis* there were also observed

Barom. $29^{in}.86$, Att. Therm. 33°.2, Ext. Therm. 36°.8.

The pointer, which is used in setting the circle for an observation, gives the degrees and next preceding 5′ of the circle reading.

One revolution of a circle microscope is called a "nominal minute," and the mean value of $4^r.902$ corresponds to 5′, so that the nominal minutes are reduced to true minutes of arc by increasing them by their $\frac{1}{50}$ part. Since the mean of the microscopes is to be found by dividing their sum by 6, and the decimal part of the quotient is then to be converted into nominal seconds by multiplying by 60, the nominal seconds in the mean are obtained at once by simply adding the decimals of the several microscope readings (making the integers the same in all) and removing the decimal point one place. Thus, in the first observation, making 2 the common integer, the sum of the decimals is .610, and hence the mean is 2′ 6″.10 (nominal), which increased by its $\frac{1}{50}$ or $\frac{2}{100}$ part is 2′ 8″.62 of arc. This requires a further correction for variation of the value of a microscope revolution from its mean value, that is, for error of runs (Art. 22). The correction for runs on the given date was + 0″.576 for 100 nominal seconds, and, therefore, the correction of the first observation is $+ 0''.576 \times 1.261 = + 0''.73$.

There is next to be applied the correction for error of gradua- n and of flexure. These are combined in a table given in e introduction to the observations, from which their values, as ed in the following reduction, are taken with the argument Pointer reading."

The value of one revolution of the telescope micrometer was ″.626, and the reading multiplied by this number is always ditive to the circle reading.

The distance of the star from the meridian is expressed by number in the last column of the above table, here denoted N, which is the number of the transit thread at which the ection is made. The middle thread is assumed to be in the ridian;* and, since the average distance of two adjacent eads was 207″.31, the number of the middle thread being 4, distance of the star from the meridian is represented by

$$c = 207''.31\ (N - 4)$$

e formula for reduction to the meridian is put under the ap- roximate form

$$R = \tfrac{1}{4}\,\tau^2 \sin 1'' \sin 2\delta = \tfrac{1}{2}\,\tau^2 \sin 1'' \sin \delta \cos \delta$$

d τ is also found approximately by the formula $\tau = c \sec \delta$: nce, according to this (rather inaccurate) method, we have

$$R = \tfrac{1}{2}\,c^2 \sin 1'' \tan \delta$$

hich for the Greenwich instrument gives

$$R = 0''.1042 \tan \delta \times (N - 4)^2$$

given in the explanations of the observations.

The micrometer thread was inclined so that an observation at e of the side threads required the correction $- 0''.775 \times$ $N - 4)$.

The complete reduction of the above observations is, there- e, as follows. In computing the reduction R we have as- med $\delta = 19° 8'$.

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*I am here stating the method employed at the Greenwich Observatory, not re- ending it. For stars near the pole it is not sufficiently accurate, as will be d by reducing some of the observations of α and λ *Ursæ Minoris* by our com- te formula (198). A difference of 0″.2 or 0″.8 occurs in some cases.

	η Bootis (R)	*η Bootis* (D)	Nadir Pt
Mean of microscopes	+ 2′ 6″.10	+ 0′ 52″.16	+ 0′ 4
Reduction to arc = $\frac{1}{50}$	+ 2 .52	+ 1 .04	+ 0
Correction for runs	+ 0 .73	+ 0 .30	+ 0
Division error	+ 1 .51	+ 1 .24	+ 0
Telescope micrometer	+ 9 26 .15	+ 9 57 .35	+ 10 32
Reduction to meridian	— 0 .32	+ 0 .32	
Corr. for inclination of thread	+ 2 .33	— 2 .33	
Pointer	147° 20′	32° 0′	179° 40′
Corrected merid. circle reading	147 31 39 .02	32 10 50 .08	179 51 15

Hence, by η *Bootis*, we have

App. zenith dist. (R)	32° 28′ 20 .98
" " " (D)	32 10 50 .08
Mean app. zen. dist.	32 19 35 .53
Refraction	+ 38 .01
$z =$	32 20 13 .54
$\varphi =$	51 28 38 .20
$\delta =$	19 8 24 .66

The half difference of the apparent zenith distances (R)
(D) is evidently the zenith point correction, and is here + 8′ 45′
additive to all circle readings. According to the nadir p
observation, it is + 8′ 44″.33. The practice at the Greenv
Observatory, however, is to employ for a number of consecu
days a mean value of the zenith point correction obtained f
all the values determined during the period. Thus, the m
value employed from April 12 to April 24, 1852, a period
cluding the above observations, was + 8′ 45″.16. The prac
recommended by **Bessel** of employing the nadir point readi
determined at the time of the observation is preferable.

203. The zero points of the circle may also be determined
reversing the axis, if the microscopes rest on the axis and,
sequently, are reversed with it. Let a collimating telescop
placed anywhere in the meridian with its axis directed tow
the rotation axis of the meridian circle, and let it be provi
with a cross thread in its focus. Direct the telescope upon
collimator, and bring the micrometer thread upon the intersec
of the cross thread. Let C be the circle reading corrected

inclination of the microscope frame, micrometer reading, &c. ow reverse the rotation axis, and make a similar observation on the collimator. Let C' be the corrected reading. Then it evident that $\frac{1}{2}(C - C')$ is the true zenith distance of the colli- tor (supposing the readings to commence at the zenith), while $C + C'$) is the true reading when the telescope is vertical, and presents the *zenith point.* This method may occasionally be ed for the purpose of comparison with the methods already en; but it is too troublesome for constant use. Moreover, servations depending on the spirit level are not so reliable as se made from the surface of mercury, which, when at rest, st be *perfectly* horizontal.

Another method, suggested by the ever-inventive BESSEL fore the introduction of the mercury collimator, however), is o dependent on the spirit level, but admits of greater accuracy an the above, because a level of larger dimensions may be used. e level is applied to the collimating telescope, which is placed the horizontal plane of the axis of the meridian circle. When e bubble is in any given position, the sight line of the colli- ator makes a given angle with the vertical. If, then, the colli- ator with its level is first placed south and then north of the rcle, and the bubble of the level brought to the same reading each case, the zenith distance of the cross thread observed by e circle must be the same, but on opposite sides of the zenith. e mean of the two circle readings will therefore be the zenith int reading. Instead of bringing the level of the collimator the same reading, it will be preferable to observe the inclina- n in each position north and south, by reversing the level in e usual manner; then the difference of the inclinations will e applied as a correction to the mean of the circle readings to tain the true zenith point. This method has the advantage f not requiring a reversal of the axis of the meridian circle. Plate III. Fig. 2 represents a collimator with its spirit level, as quired in this method. Two piers, one north and one south f the circle, are each provided with Vs, which receive the col- mating telescope alternately.

Finally, to complete the enumeration of methods depending n the spirit level, the collimating telescope may be placed ver- ically over or under the telescope of the meridian circle. The vel is then attached to the collimator at right angles to its tical axis. Two observations are made upon the cross thread of the collimator as before, the collimating telescope bein (between the two observations) revolved 180° about the vertic line. The mean of the circle readings, corrected for differenc in the inclination of the collimator as shown by the level, wil be the zenith or nadir point reading.

204. *Flexure.*—Notwithstanding the conical form which i given to the telescope tubes of large instruments, their weigh produces a sensible flexure, which may change the position of the optical axis of the telescope with respect to the zero points of the circle. It is important, therefore, to investigate the amount of this flexure. The following is BESSEL's method.

Two collimators, such as that represented in Plate III. Fig. 2, are mounted in the horizontal plane of the axis of the circle, one north and the other south. The cross threads of the collimators admit of adjustment (by a micrometer screw, for example), so that they may be brought to coincide with each other, the meridian circle being raised upon the reversing apparatus during this adjustment. The two intersections of the cross threads of the collimators now represent two infinitely distant points whose angular distance is exactly 180°. The meridian circle being replaced, observe this angular distance in the usual manner. It is evident that the errors of division of the circle will not enter, since the same two divisions come under the opposite reading microscopes in the two observations in reverse positions. The difference of the two circle readings will, therefore, be exactly 180° if there is no flexure. But if the difference is less than 180° by a quantity x, then $\frac{1}{2}x$ is the correction for flexure in the horizontal position of the telescope. In this way, AIRY found that when the Greenwich transit circle was directed upon th south collimator, the circle reading was 89° 46′ 15″.52, an when upon the north collimator, 269° 46′ 16″.35; the differenc 180° 0′ 0″.83 is the apparent distance of the two opposite poin measured through the nadir, and hence one-half of 0″.83, or 0″.4 is the effect of flexure in increasing apparent nadir distances o in diminishing apparent zenith distances.

In different positions of the telescope, the mechanical effect o each particle of metal, supposing it to act simply as a weigh attached to a lever, will vary as the sine of the zenith distance so that if f is the horizontal flexure, $f \sin z$ expresses the flexu in general. It is not quite certain, however, that the flexu ys follows this simple law; and to determine the law experi- ally, we should have the means of mounting a pair of col- tors in a line making any angle with the vertical.

ie flexure determined by the above method is properly d the *astronomical* flexure, as it gives the deviation of the al axis, which becomes a direct correction of our astro- ical measures. It is evident, however, that it does not ess the absolute flexure of the tube. If when the tube is zontal both ends drop the same distance, the optical line rmined by the centre of the objective and the micrometer ad will merely be moved parallel to itself, and no flexure appear from the circle readings; for the collimators do not rmine merely a single fixed line in space, but rather a m of parallel lines, or simply a fixed direction.

he effect of the flexure upon an observation is, then, zero ie absolute flexures of the two halves of the telescope are l; and when these are unequal, the effect is proportional to r difference. This leads directly to the method of elimi- ng flexure, first suggested by the elder Repsold in 1823 or '24, nterchanging the objective and ocular of the telescope. Let uppose that at any given zenith distance the centre of the ctive drops the linear distance a, and the horizontal thread ie focus drops the distance a', so that a and a' represent the lute flexures of the two halves of the tube. Then, if the le length of the tube is denoted by $2r$, the angles of depres- of the two portions may be expressed by $\frac{a}{r}$ and $\frac{a'}{r}$ respect- y. If then γ is the angle which the sight line now makes h the direction it would have had if no flexure had taken ce, we have $\gamma = \frac{a - a'}{2r}$; that is, the astronomical flexure is portional to the absolute flexure. Now let the objective and lar be interchanged, and the telescope revolved 180°, so as to again directed upon a point at the same zenith distance as fore. *The absolute flexures being the same as before*, that of the ject end is now a', and that of the eye end is a: so that the tronomical flexure is now $\frac{a' - a}{2r} = -\gamma$. Hence the mean of ro observations of the same star made with the objective and ular reversed will be free from the effect of flexure. More- ver, the half difference of the measured zenith distances will e the astronomical flexure. It is here assumed that the absolute flexures of the two halves remain the same when the o jective and ocular are interchanged. For a discussion by HANSE of the conditions necessary in the construction of the telescope i order to satisfy this condition (if possible), see *Astr. Nach.*, Vo XVII. p. 70.*

As to the effect of gravity upon the form of the circle, se BESSEL's paper, *Astr. Nach.*, Vol. XXV.

205. *Observations of the declination of the moon with the meridia circle.*—In these observations, the micrometer thread is usuall brought into contact with the full limb, and a correction i applied to the deduced declination of the limb for the moon' parallax and semidiameter. When the observation is not mad in the meridian, the reduction to the meridian (194) is also to b applied, together with a correction for the moon's proper motio The most precise formula for making these reductions is tha given by BESSEL, which is deduced as follows.

In Fig. 46, p. 290, let O now represent the apparent positio of the moon's centre, and suppose the observed point of th moon's limb to be designated by M (not given in the figure). Conceive an arc to be drawn from A tangent to the moon's limb. The point of contact M, and the points A and O, form a triangle, right angled at M, of which the side MO is the moon's apparent semidiameter $= s'$, the side $AO = 90° + c$, and the angle at A may be denoted by d. We have then

$$\sin s' = \sin d \cos c$$

Let

$\delta_1 =$ the observed declination of the limb, corrected for refraction,

$\delta' =$ the apparent declination of the moon's centre;

then in the triangle AOP we have the sides $AO = 90° +$ $PA = 90° - n$, $PO = 90° - \delta'$, and the angles $PAO = \delta_1 \mp$ $APO = 90° + (\tau - m)$; whence, as in Art. 199

$$\begin{aligned} \sin \delta' &= -\sin n \sin c + \cos n \cos c \sin (\delta_1 \mp d) \\ \cos \delta' \sin (\tau - m) &= \cos n \sin c + \sin n \cos c \sin (\delta_1 \mp d) \\ \cos \delta' \cos (\tau - m) &= \cos c \cos (\delta_1 \mp d) \end{aligned}$$

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* See also Dr. GOULD's remarks on the meridian circle of the Dudley Observator Proceedings of the Am. Association for the Adv. of Science, 10th meeting, p. 116.

But, as before, we shall neglect the insensible term $\sin n \sin c$, and put $\cos n = 1$, and then the first and third of these equations will suffice to determine δ'. Moreover, since in the case of the moon τ will not exceed 1^m, the neglect of m will cause no sensible error in $\cos(\tau - m)$. Hence we take

$$\begin{aligned}\sin \delta' &= \cos c \sin (\delta_1 \mp d) \\ \cos \delta' \cos \tau &= \cos c \cos (\delta_1 \mp d)\end{aligned}$$

or, developing the second members,

$$\begin{aligned}\sin \delta' &= \cos c \cos d \sin \delta_1 \mp \sin s' \cos \delta_1 \\ \cos \delta' \cos \tau &= \cos c \cos d \cos \delta_1 \pm \sin s' \sin \delta_1\end{aligned}$$

whence, by eliminating $\cos c \cos d$, we find

$$\mp \sin s' = \sin \delta' \cos \delta_1 - \cos \delta' \sin \delta_1 \cos \tau \tag{195}$$

If now we put

δ = the moon's geocentric declination,
s = " " semidiameter,
π = " eq. hor. parallax,
φ' = the geocentric or reduced latitude of the place of observation,
ρ = the earth's radius for the latitude φ,
Δ, Δ' = the moon's distance from the centre of the earth and from the place of observation, respectively, the equatorial radius of the earth being unity,

we have, by the formulæ of Art. 98, Vol. I.,

$$\begin{aligned}\Delta' \sin \delta' &= \Delta \sin \delta - \rho \sin \varphi' \\ \Delta' \cos \delta' &= \Delta \cos \delta - \rho \cos \varphi' \cos \tau\end{aligned}$$

this last being equivalent to the more rigorous one in (133) of Vol. I., when the moon is near the meridian; and by Art. 128, Vol. I., we also have

$$\Delta' \sin s' = \Delta \sin s$$

Substituting these expressions in (195), after multiplying it by Δ', we find

$$\begin{aligned}\mp \Delta \sin s = \Delta \sin (\delta - \delta_1) &+ 2 \Delta \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau \\ &- \rho \sin (\varphi' - \delta_1) - \rho \cos \varphi' \sin \delta_1 \sin^2 \tau\end{aligned}$$

Dividing by $\varDelta = \frac{1}{\sin \pi}$, this becomes

$$\begin{aligned} \mp \sin s = {} & \sin (\delta - \delta_1) + 2 \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau \\ & - \rho \sin \pi \sin (\varphi' - \delta_1) - \rho \sin \pi \cos \varphi' \sin \delta_1 \sin^2 \tau \end{aligned}$$

where the last term is evidently insensible. If then we put

$$\sin p = \rho \sin \pi \sin (\varphi' - \delta_1) \tag{196}$$

we have

$$\sin (\delta - \delta_1) = \sin p \mp \sin s - 2 \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau$$

The last term (which is the reduction to the meridian) will seldom exceed 1″, and may be put under the form

$$\sin R = \left(\frac{15}{2}\right)^2 \sin^2 1'' . \sin 2\delta . \tau^2$$

The quantity τ is here the true hour angle of the moon, to find which, let

μ_1 = the sidereal time of the observation,
μ = " " moon's transit,
λ = the increase of the moon's right ascension in one sidereal second;

then

$$\tau = (1 - \lambda)(\mu - \mu_1)$$

and hence

$$R = \frac{225}{4} \sin 1'' \sin 2\delta (1 - \lambda)^2 (\mu - \mu_1)^2 \tag{197}$$

The first two terms of the value of $\sin (\delta - \delta_1)$ differ but little from $\sin (p \mp s)$. To find their exact value, we have

$$\begin{aligned} \sin p \mp \sin s & = \sin (p \mp s) + \sin p (1 - \cos s) \mp \sin s (1 - \cos p) \\ & = \sin (p \mp s) + 2 \sin p \sin^2 \tfrac{1}{2} s \mp 2 \sin s \sin^2 \tfrac{1}{2} p \end{aligned}$$

The last two terms of this will seldom amount to a tenth of a second, and therefore the formula may be regarded as perfectly accurate under the form

$$\sin p \mp \sin s = \sin (p \mp s) \mp \tfrac{1}{2} (p \mp s) \sin 1'' \sin p \sin s$$

Now, since $\delta - \delta_1$ and $p \mp s$ differ by so small a quantity, the ratio of the sine to the arc will be the same for both of them: hence we shall have, with the utmost precision,

$$\delta = \delta_1 + p \mp s \mp \tfrac{1}{2} (p \mp s) \sin p \sin s - R \tag{198}$$

as given by BESSEL.* The upper or lower sign is to be used according as the north or the south limb is observed.

The declination thus found is reduced to the time μ_1 of the observation. But if we wish its value at the time of the meridian passage, we must add to it the correction $(\mu - \mu_1)\lambda'$, in which λ' is the increase of the declination in one sidereal second, or

$$\lambda' = \frac{\Delta\delta}{60.1643}$$

where $\Delta\delta$ = the increase of declination in one minute of mean time, as now given in the American Ephemeris. The value of $1 - \lambda$ is found as in Art. 154: namely, taking $\Delta\alpha$ = the increase of the moon's right ascension in one minute of mean time, we have

$$\lambda = \frac{\Delta\alpha}{60.1643}$$

so that, putting

$$1 - \lambda = \frac{1}{B}$$

we shall have

$$\log(1 - \lambda) = \text{ar. co.} \log B$$

and log B may be taken from the table on page 179.

In practice, it will generally be most convenient to apply the several reductions directly to the observed zenith distance, as in the following example.

EXAMPLE.—The declination of the moon was observed with the meridian circle of the Washington Observatory, 1850, September 17. The nadir point was first observed as follows:

	Circle Microscopes.					
	A	B	C	D	Means.	
Nadir point at $20^{h}.5$	0″.9	1″.9	2″.2	1″.4	1″.60	Micrometer thread in coincidence with its image: mean of 10 readings = $38^{r}.934$.
	0 .7	1 .4	2 .0	1 .6	1 .42	
Means	0 .80	1 .65	2 .10	1 .50	1 .51	

The value of one revolution of the micrometer = 34″.356, or

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* *Tabulæ Regiomontanæ*, Introd. p. LV.

$1'' = 0^r.0291$; and hence, by the method of Art. 197, the micr meter zero (or reading of the micrometer when the circle readin was 0° 0′ 0″) was

$$(M) = 38^r.934 + 0^r.0291 \times 1.51 = 38^r.978$$

The observation of the moon was as follows, S. L. denotin south limb:

	Circle Microscopes.					Clock $= \mu_1$	Micro-meter $= M$.
	A	B	C	D	Mean.		
Moon, S. L.	55° 52′ 45″.7	42″.8	45″.2	46″.1	44″.95	$21^h 17^m 21^s$	$39^r.956$
	Barom. 30^{in}.114 Att. Therm. 64°. Ext. Therm. 52°.8					32	39 .904
						43	39 .875

The circle was *west*, in which position the readings are zenith distances towards the south. The correction for runs was $-0''.75$ for 3′, and since the excess of the reading over a multiple of 3′ is 1′ 44″.95, the proportional correction for runs is $-0''.43$.

The clock time of transit of the moon's centre over the meridian was $\mu = 21^h\ 17^m\ 16^s.80$.

The latitude of the observatory is $\varphi = 38°\ 53'\ 39''.25$, and therefore $\varphi - \varphi' = 11'\ 14''.54$, $\log \rho = 9.9994302$. The longitude is $5^h\ 8^m\ 12^s$ west of Greenwich.

For the date of the observation, we take from the Nautical Almanac

$\delta = -\ 16°\ 1'.7$

$\Delta\delta = +\ 6''.377$ in 1^m mean time, $\pi = 54'\ 9''.64$

$\Delta\alpha = 2^s.0150$ " " " " $s = 14'\ 45''.49$

whence $\log(1 - \lambda) = 9.98521$ and $\lambda' = +\ 0''.1060$

The correction for the micrometer, or $M - (M)$, converted into seconds, is additive to the circle reading. The reduction to the meridian, or R, found by (197), is also algebraically additive to the circle reading, attention being paid to the sign of δ; and the correction for change of declination to be added to the circle reading will be $-(\mu - \mu_1)\,\lambda'$. Since the sum of these three corrections should be the same for each micrometer observation, the precision of the observations will be shown by computing this sum for each. Thus, we find

$\mu-\mu_1$	$M-(M)$	R	$-(\mu-\mu_1)\lambda'$	Sums.
− 4°.2	33″.60	− 0″.00	+ 0″.44	34″.04
− 15 .2	31 .82	− .03	+ 1 .61	33 .40
− 26 .2	30 .82	− .09	+ 2 .78	33 .51
			Mean =	33 .65

Hence we have

	Circle reading =	55° 52′ 44″.95
	Corr. for runs =	− 0 .43
	Mean corr. for microm., &c. =	+ 33 .65
	Apparent zenith distance =	55 53 18 .17
	By Table II. Refraction =	+ 1 25 .60
$\varphi'-\delta_1=\varphi-\delta_1-(\varphi-\varphi')$	$\varphi-\delta_1=$	55 54 43 .77
$=55° 43' 29''$	$-(p+s)=$	− 59 27 .24
By (196), $p=44' 41''.75$	$-\frac{1}{2}(p+s)\sin p\sin s=$	− 0 .10
	$\varphi-\delta=$	54 55 16 .48
	$\varphi=$	38 53 39 .25
	$\delta=$	− 16 1 37 .18

206. *Observations of the declination of a planet, or the sun.*—The larger planets are observed in the same manner as the moon, that is, by making the micrometer thread tangent to the limb, and when the planet is treated as a spherical body the observation is also reduced in the same manner.

In the case of the sun, both limbs may be observed. The reduction to the meridian may be facilitated by a table giving the logarithm of the factor

$$b=\frac{225}{4}\sin 1''(1-\lambda)^2\sin 2\delta$$

for each day of the fictitious year (Vol. I. Art. 406), such as BESSEL's Table XII. of the *Tabulæ Regiomontanæ.* This table also gives for each day of the year the value of

a = increase of the sun's declination in 100 sidereal seconds,

so that the reduction of the observed declination to the meridian, including the correction for the change of declination in the interval τ, is

$$\frac{a\tau}{100}+b\tau\tau$$

The correction for parallax may be put under the form

$$p = \frac{8''.57116}{r} \rho \sin(\varphi' - \delta)$$

in which r = sun's distance from the earth, the mean distance being unity, and in each observatory this quantity may be computed for the latitude, and for each day of the year, and also inserted in the table. In order to embrace every thing necessary for the complete reduction of the observed declination, the table contains also the sun's semidiameter for each day of the fictitious year.

207. *Correction of the observed declination of a planet's or the moon's limb for spheroidal figure and defective illumination.*—Let us consider the most general case of a spheroidal planet partially illuminated. The correction to reduce the observed declination of the limb to that of the centre is equal to the perpendicular distance from the centre to the micrometer thread, which is tangent to the limb and perpendicular to the meridian. The formulæ for computing this perpendicular in general are (Vol. I. p. 580)

$$\tan \vartheta' = \frac{\tan \vartheta}{c} \qquad\qquad \sin \chi = \sin \vartheta' \sin V$$

$$s'' = \frac{s \sin \vartheta \cos \chi}{\sin \vartheta'}$$

in which s'' is the required perpendicular, ϑ the angle which it makes with the axis of the planet (reckoning from the north point of the disc towards the east), c is a constant depending upon the eccentricity of the planet's meridian, V the angular distance of the earth and sun as seen from the planet, and s is the equatorial radius of the disc, or greatest apparent semidiameter at the time of the observation. The perpendicular here coincides with the declination circle, and consequently we have at once $\vartheta = -p$, or $180° - p$, according as the north or the south limb is observed; p denoting, as in the article referred to, the position angle of the axis of the planet. From the discussion in Vol I. Art. 354, it follows that (putting $-p$ for ϑ) the north limb will be full (and, consequently, the south limb gibbous) when $\sin p$ and $\sin V$ have the same sign. We shall, therefore, here change the sign of $\sin \chi$, and take

$$\left.\begin{array}{l} \tan p' = \dfrac{\tan p}{c}, \qquad \sin \chi = \sin p' \sin V \\ s'' = \dfrac{s_0}{r'} \cdot \dfrac{\sin p}{\sin p'} \cos \chi \end{array}\right\} \qquad (199)$$

in which $s_0 =$ the greatest apparent semidiameter at the mean distance of the sun from the earth, and $r' =$ the planet's geocentric distance. We then have the rule: *the north or the south limb is the full limb according as sin χ is positive or negative.* The formulæ for computing p, V, and c are given in Vol. I. Arts. 348 et seq., and s_0 is given on p. 578.

The gibbosity of Saturn, however, is wholly insensible, and even that of Jupiter at the north and south points of the limb cannot exceed 0″.05, which is so much less than the usual errors of declination observations that it may be disregarded. Hence, for Saturn and Jupiter the correction will depend only upon the figure of the planet, and will be computed by the equations

$$\tan p' = \frac{\tan p}{c} \qquad s'' = \frac{s_0}{r'} \cdot \frac{\sin p}{\sin p'} = \frac{cs_0}{r'} \cdot \frac{\cos p}{\cos p'}$$

in which for Jupiter we take $\log c = 9.9672$, and for Saturn $c = \sqrt{(1 - ee \cos^2 l)} = \sqrt{(1 - [9.2706] \cos^2 l)}$, l and p being taken directly from the tables for Saturn's Ring given in the Ephemeris.

A further simplification may be permitted in the case of Saturn; for, on account of the small values of p, the ratio $\dfrac{\cos p}{\cos p'}$ will be very nearly unity, and if we take $s'' = \dfrac{cs_0}{r'}$ we shall have the true value of s'' within less than 0″.05.

It is hardly necessary to remark that when we neglect the gibbosity of Jupiter or Saturn, the mean of the observed declinations of the north and south limbs gives at once the declination of the centre.

For Mars, Venus, and Mercury the correction will be only for defective illumination; but in this case we can avoid the separate computation of p and V, as follows. Substituting in the equation for $\sin \chi$ (199) the values of $\sin p$ and $\sin V$ given in Vol. I. p. 577, and moreover observing that, since these bodies are regarded as spherical, we have $c = 1$, and, consequently, $p' = p$, there results

$$\sin \chi = \frac{R}{R'} \left[\cos \delta' \sin D - \sin \delta' \cos D \cos (\alpha' - A)\right] \qquad (200)$$

in which

α', δ' = the planet's right ascension and declination,
A, D = the sun's " " "
R, R' = the earth's and the planet's distances from the sun;

and a positive value of $\sin\chi$ will here also indicate that the north limb is full and the south limb gibbous, and a negative value the reverse. Adapting this formula for logarithms, we have, therefore,

$$\left.\begin{aligned}\tan F &= \tan D \sec(\alpha' - A)\\ \sin\chi &= \frac{R}{R'}\cdot\frac{\sin(F-\delta')\sin D}{\sin F}\end{aligned}\right\} \qquad (201)$$

or, more conveniently, perhaps,

$$\left.\begin{aligned}\tan E &= \tan\delta' \cos(\alpha' - A)\\ \sin\chi &= \frac{R}{R'}\cdot\frac{\sin(D-E)\cos\delta'}{\cos E}\end{aligned}\right\} \qquad (201^*)$$

E being taken less than 90°, with the sign of its tangent. Then we find the reduction to the centre of the planet by the formula

$$s'' = \frac{s_0}{r'}\cos\chi \qquad (202)$$

If the declination of a *cusp* of Venus or Mercury has been observed, we must find p by the formula (Vol. I. p. 577)

$$\tan p = \cot(\alpha' - A)\sin(F - \delta')\sec F \qquad (203)$$

in which F has the same value as above, and then the reduction to the centre of the planet will be

$$s'' = \frac{s_0}{r'}\cos p$$

For the moon, when the gibbous limb has been observed, the formulæ (201) may be used for computing χ; but on account of the small difference of R and R', we may put their quotient $=1$. Since the declination of the gibbous limb will not be observed except when the moon is nearly full, it will be best to reduce the observations as if the observed limb were full, according to Art. 205, and then to apply a small correction for gibbosity.

This correction will be $\Delta s = s - s\cos\chi = s \operatorname{versin}\chi$. Hence the formulæ for the moon will be

$$\left.\begin{aligned} \tan E &= \tan\delta'\cos(\alpha' - A) \\ \sin\chi &= \frac{\sin(D - E)\cos\delta'}{\cos E} \\ \Delta s &= s \operatorname{versin}\chi \end{aligned}\right\} \qquad (204)$$

Example 1.—The apparent declination of the southern cusp of Venus, at its transit over the meridian of Greenwich, July 16, 1852, observed with the transit circle, was

$$\delta' = 15^\circ\ 0'\ 45''.60$$

From the Nautical Almanac, we have

$$\begin{array}{ll} \alpha' = 8^h\ 11^m\ 1^s.46 & \log r' = 9.4675 \\ A = 7\ \ 43\ \ 42.80 & D\ \ = 21^\circ\ 19'\ 8'' \end{array}$$

and from Vol. I. p. 578,

$$s_0 = 8''.55$$

Hence, by (203), we find log tan $p = 0.0031$, and, consequently,

$$s'' = \frac{s_0}{r'}\cos p = 20''.53$$

and the apparent declination of the planet's centre was, therefore,

$$\delta = 15^\circ\ 1'\ 6''.13$$

Example 2.—The apparent declinations of Jupiter's north and south limbs, observed at Greenwich, March 18, 1852, were—

$$\begin{array}{ll} \text{N.L.} & \delta' = -17^\circ\ 21'\ 57''.36 \\ \text{S.L.} & \delta' = -17\ \ 22\ \ 37\ .61 \end{array}$$

To illustrate the complete formulæ, let us take the gibbosity of the planet into account. For this purpose, we take from the Nautical Almanac

$$\begin{array}{lll} \alpha' = \ 230^\circ\ 56'.4 & A = 224^\circ\ 25'.0 & \\ \delta' = -\ 17\ \ 22.2 & \varepsilon = \ 23\ \ 27.5 & \log r' = 0.6783 \end{array}$$

and from Vol. I. p. 574,

$$N = 357^\circ\ 56'.5 \qquad i = 25^\circ\ 25'.8$$

Hence, by the formulæ (619), Vol. I,

$$F = 201° \; 23'.5 \qquad \lambda = 234° \; 52' \; 3$$

$$V = \Lambda - \lambda = -\,10° \; 27'.7$$

$$F' = -\,20° \; 47'.5 \qquad \log \tan p = 9.4281$$

Then, by (199), taking $\log c = 9.9672$, we have

$$\log \sin \chi = n8.7025$$

from which it follows that the south limb was full. Hence, taking $s_0 = 99''.70$, we find

$$\text{For full limb} \quad (s'') = \frac{s_0}{r'} \cdot \frac{\sin p}{\sin p'} = 19''.50$$

$$\text{For gibbous limb} \quad s'' = (s'') \cos \chi = 19\;.47$$

The declination of the centre was, therefore, according to these observations,

$$\text{From N. L.} \quad \delta = -\,17° \; 22' \; 16''.83$$

$$\text{" S. L.} \qquad \text{" " } 18\;.11$$

Considering the difference of these results, which is by no means as great as often occurs in the Greenwich observations of Jupiter, it appears that the practice there followed of always applying the *polar* semidiameter (which is the one given in the Nautical Almanac) is quite accurate enough *for these observations*. Our more exact method will not be without application, however, in cases where greater refinement both in observation and reduction are attained.

EXAMPLE 3.—At Greenwich, Feb. 6, 1852, the declination of the moon's centre deduced from an observation of the north limb, on the assumption that this limb was full, was

$$\delta' = +\,13° \; 17' \; 0''.58$$

For the time of the moon's transit on this date, we have

$$\alpha' = 158° \; 18'.6 \qquad A = \;\; 319° \; 56'.1$$

$$s = \;\; 16' \; 31'' \qquad D = -\;\; 15 \;\; 36\;.3$$

whence, by (204),

$$\chi = -\,2° \; 58'$$

h shows that the north limb was gibbous. The correction

$$\Delta s = s \text{ versin } \chi = 1''.33$$

he true declination was, therefore,

$$\delta = +13^\circ\ 17'\ 1''.91$$

CHAPTER VII.

THE ALTITUDE AND AZIMUTH INSTRUMENT.

3. **This** instrument may be regarded as a transit instrument
ined with both a vertical and a horizontal circle, by means
hich both the altitude and the azimuth of a star may
bserved at the instant of its transit through the vertical
described by the telescope. This combination is not often
for the higher purposes of astronomical research, as every
ional movement introduced into an instrument diminishes
ability and increases the risk of error. However, at Green-
, a regular series of extra-meridian observations of the moon
rried on with such an instrument, for the sake of comparison
meridian observations. The instrument has there received
ame of the *altazimuth*. In other places, it has been called
stronomical theodolite; and, in fact, the general theory of the
ument, which will be given hereafter, will be found to be
tly applicable to the common theodolite employed in geo-
measurement.

ill another name is the *universal instrument*, so called on
unt of its numerous applications; but this name is usually
n only to the portable instruments of this class. The small
ersal instruments of Ertel are well known.

9. Sometimes the horizontal circle is reduced to small
ensions, and designed simply as a finder, or to set the instru-

ment approximately at a given azimuth; while the vertical circl is made of unusually large dimensions, and is intended for th most refined astronomical measurement. The instrument i then known simply as a *vertical circle*. Such is the Ertel Vertica Circle of the Pulkowa Observatory, the telescope of which ha a focal length of 77 inches, and its vertical circle a diameter of 43 inches.*

This instrument is permanently mounted upon a solid granite pier *G*, Plates X. and XI., which is insulated from the walls and floor of the building. It stands upon a tripod which is adjusted by foot screws. The three feet are so placed that two of them are in the east and west line: hence, but one of these two is seen in Plate X., which is a projection of the instrument upon the plane of the meridian, while all three are seen in Plate XI., which is a projection upon the plane of the prime vertical. The meridional foot screw ω carries a small circle γ graduated into 360°, the index of which is attached to the foot. One revolution of this circle changes the inclination of the instrument in the plane of the meridian 318″: consequently, one division corresponds to 0″.88.

The centre of the instrument is held in place by the support *a* attached to the pier.

The vertical stand consists of a hollow cone of brass, in which turns the steel axis *b*. The lower extremity of this axis is convex and smoothly finished, and is supported by a system of three counterpoises *c*, suspended upon levers which relieve the pressure upon the bearing points of the vertical axis, and thus diminish the friction. At the top of the conical stand is a 13 inch azimuth circle, the verniers of which are attached to the axis. This is provided with a clamp and tangent screw which is moved by the rod *d* in giving the upper portion of the instrument a small motion in azimuth.

The upper extremity of the vertical steel axis carries the strong oblong bar *e*, which may be called the bed of the instrument. On this bed rests the adjustable frame *vfgv*, which supports the horizontal axis *i* in the Vs at *vv*. This axis should be perpendicular to the vertical axis, and its adjustment in this respect is effected by means of two opposing screws at *h*.

The axis *i* has two equal cylindrical pivots of steel at *vv*. It is hollow, to admit light from the lamp $\varkappa$, which is reflected upon the threads of the reticule of the telescope by a mirror in the interior of the tube at *u*. The telescope and principal vertical circle *o* are firmly and invariably attached to one extremity of this axis. At the opposite end of the axis is a smaller vertical circle *m*, which serves as a finder. From the centre of this finding circle radiate four conical arms terminating in ivory balls *n*. The telescope is swept in the vertical plane solely by means of these balls, never by touching the telescope or principal vertical circle. When the telescope is approximately pointed and clamped, fine vertical motion is given to the tangent screw by the rod *k*. The instrument is swept in azimuth by means of an ivory ball at *l*, the fine azimuthal motion being given by the rod *d*.

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* See *Description de l'obser. cent.*, &c., p. 130.

The circle is read off by four microscopes attached to a square frame *a*, which is fixed to the frame *vfgv*. The level β attached to this frame indicates its inclination with respect to the horizon. The circle is divided to $2'$, and the microscopes read directly to single seconds, and by estimation to $0''.1$, or even less. The probable error of reading of a single microscope is given by PETERS as only $0''.090$ in observations by day, and $0''.098$ in observations by night.

The friction of the horizontal axis in the Vs is diminished by the single counterpoise *p*, which, by means of a lever, the fulcrum of which is at *q*, supports the principal part of the weight of the telescope, vertical circles, and horizontal axis, by exerting an upward pressure at *r*. The point *r* being at suitable distances from the two Vs respectively (nearer to the principal circle than to the finder), the friction in both Vs is equally relieved; while the whole weight of the movable portion of the instrument is transferred to a point *q*, very near to the vertical axis of rotation.

The striding level *s* rests upon the pivots of the horizontal axis, and, by reversal in the usual manner, serves to measure the inclination of this axis to the horizon.

The reticule at *t* is composed of three horizontal threads, two of which are close parallel threads (the clear space between them being only $6''$), which serve for the observation of objects which present sensible discs, or of those which are too faint to be observed by bisection (see Art. 198). The third thread is $18''$ from the others, and is used in observing stars by bisection. The unequal distances prevent mistakes in the choice of threads. These horizontal threads are crossed by two vertical ones, the distance of which is 1′ of arc. The middle point betwe determines the optical centre of the instrument, and al vations are made as nearly as possible at this point.

The extreme accuracy attainable in the observation o distances with this instrument may be inferred from the ing values of the *zenith point Z* (see Art. 219) of the ci cited by STRUVE, from observations by PETERS upon *Pola* upper and lower culminations:

1843.	Upper transit. Z	Diff. from mean.		Lower transit. Z	I
April 13	0° 0′ 33″.13	— 0″.32	April 14	0° 0′ 33″.64	-
14	33 .26	— 0 .19	16	33 .32	-
17	33 .82	+ 0 .37	20	33 .45	-
19	33 .27	— 0 .18	21	33 .94	-
20	33 .75	+ 0 .30	22	33 .48	-
22	33 .17	— 0 .28	24	33 .50	-
24	33 .45	0 .00	25	33 .94	-
25	33 .68	+ 0 .23	26	33 .98	-
26	33 .29	— 0 .16	27	33 .82	-
27	33 .68	+ 0 .23	28	34 .12	-
Mean	0 0 33 .45		**Mean**	0 0 33 .72	

Hence, assuming that the zenith point of the circle was cc the probable error of an observed value of Z was, for series, $= 0''.22$. This error, however, is the combined e error of observation and variability of Z. But the pı error of observation was obtained from the discrepancies b the several values of the latitude deduced from these same vations, and was $= 0''.17$: so that the probable erro arising from variation in the instrument was $= \sqrt{[}$ $- (0''.17)^2] = 0''.14$. The means for the two transits di $0''.27$, which results from the use of different divisions circle and different parts of the micrometers. To compar justly, it would be necessary first to eliminate especia division errors.

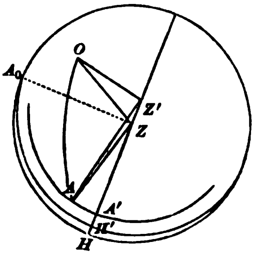
In order to eliminate the effects of flexure, the objecti ocular are made interchangeable (see Art. 204).

The dimensions of the various parts of the instrument aken from the plates, which are accurately drawn upon a scale of $\frac{1}{10}$.*

210. The portable universal instruments are usually so arranged that the vertical circle may be removed altogether from the instrument when horizontal angles only are to be measured. One of these instruments is represented in Plate XII. In Fig. 1, the instrument is arranged for measuring horizontal angles exclusively. In Fig. 2, the telescope of Fig. 1 is replaced by another which is connected with a vertical circle and (unlike the azimuth telescope) is at the end of the horizontal axis. The weight of the telescope and vertical circle is counterpoised by a weight at the opposite end of the axis. The focal length of the telescope in instruments of this kind seldom exceeds 24 inches.

The following discussion of the theory of these instruments will apply to any of the forms above mentioned, as I shall consider their two applications—to azimuths and to altitudes—independently of each other.

Fig. 49.



211. *Azimuths.*—Let A_0H, Fig. 49, represent the true horizon, Z the zenith. Let us suppose the vertical axis of the instrument to be inclined to the true vertical line, so that when produced it meets the celestial sphere in Z'. Let A_0H' be the great circle of which Z' is the pole. The plane of this circle is that of the graduated horizontal circle of the instrument. Let us suppose, further, that the horizontal rotation axis, which should be at right angles to the vertical axis, and, consequently, parallel to the horizontal circle, makes a small angle with this circle. As the instrument revolves about its vertical axis, this rotation axis will describe a conical surface, and the prolongation of this axis to the celestial sphere will describe a small circle AA' parallel to A_0H'. Let A be the point in which this axis produced through the circle end meets the sphere at the time of an observation, and O the position of a star observed on any given vertical thread

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* For all the particulars of the use of this instrument in the determination of the declination of a circumpolar star, consult the memoir of Dr. C. A. F. PETERS, *Astron. Nach.*, Vol. XXII., *Resultate aus Beobachtungen des Polarsterns am Ertelschen Verticalkreise der Pulkowaer Sternwarte.*

in the field. As the telescope revolves upon the horizontal ax its axis of collimation describes a great circle of which A is t pole, and the given thread describes a small circle parallel this great circle. Let

c = the distance of the thread from the collimation axis, positive when the thread is on the same side of the collimation axis as the vertical circle,
b = the elevation of A above the horizon as given by the spirit level applied to the horizontal axis, positive when the circle end of this axis is too high,
i = the inclination of the vertical axis to the true vertical line,
i' = the inclination of the horizontal axis to the azimuth circle,
$a = AZH$,
$a' = AZ'H$,
A = the azimuth of the star O, reckoned from A_0 as the origin,
z = the zenith distance of the star;

then, in the triangle AZZ', we have $AZ = 90° - b$, $ZZ' =$ $AZ' = 90° - i'$, $AZZ' = 180° - a$, $AZ'Z = a'$, and hence, Sph. Trig.,

$$\begin{aligned} \sin b &= \cos a' \cos i' \sin i + \sin i' \cos i \\ \cos b \cos a &= \cos a' \cos i' \cos i - \sin i' \sin i \\ \cos b \sin a &= \sin a' \cos i' \end{aligned}$$

But, i, i', and b being always so small that we can neglect the squares, these equations may be reduced to the following

$$\left.\begin{aligned} a &= a' \\ b &= i \cos a' + i' = i \cos a + i' \end{aligned}\right\} \quad (20$$

In the triangle AZO, we have the angle $AZO = A_0ZO + A_0Z$ $= A + 90° - a$, and the sides $AO = 90° + c$, $AZ = 90° -$ $ZO = z$; and hence

$$-\sin c = \sin b \cos z - \cos b \sin z \sin(A - a)$$

or, since c and b are small,

$$\sin(A - a) = \frac{b}{\tan z} + \frac{c}{\sin z}$$

Hence $\sin(A - a)$ is also a small quantity, and the angle $A -$

is either nearly 0° or nearly 180°. When the vertical circle at the extremity of the horizontal axis is to the left of the observer, as supposed in the above diagram, it is evident that A and a are nearly equal, and $A - a$ is nearly 0°. But if the instrument be revolved about its vertical axis, the azimuth circle remaining fixed, and the telescope be again directed to the same point O, the vertical circle will be on the right of the observer, and the angle a will be increased by 180°. In this case, therefore, $180° - (A - a)$ will be a small quantity. Putting, then, $A - a$ or $180° - (A - a)$ for $\sin (A - a)$, we have

$$\begin{aligned} A &= a + b \cot z + c \operatorname{cosec} z && \text{[Circle L.]} \\ A &= a + 180° - b \cot z - c \operatorname{cosec} z && \text{[Circle R.]} \end{aligned}$$

Now, a is not read directly from the azimuth circle; but if we put $A' =$ the actual reading and $A_0 =$ the reading when the point A in the diagram is at A' (in which case the telescope, when horizontal, is directed towards the point A_0), we have

$$\begin{aligned} a = a' &= A' - A_0 && \text{[Circle L.]} \\ a + 180° &= A' - A_0 && \text{[Circle R.]} \end{aligned}$$

and, therefore,

$$A = A' - A_0 \pm b \cot z \pm c \operatorname{cosec} z$$

We have supposed the azimuths to be reckoned from the point A_0; but it is indifferent what point of the circle is taken as the origin when the instrument is used only to determine *differences* of azimuth, since the constant A_0 of the above equation will disappear in taking the difference of two values of A. For *absolute* azimuths, let us denote the azimuth of the point A_0 from the south point of the horizon by A_1; then the azimuth of the star, also reckoned from the south point, will be equal to the above value increased by A_1. If, therefore, we add A_1 to the second member, and then write ΔA for the constant $A_1 - A_0$, we shall have

$$A = A' + \Delta A \pm b \cot z \pm c \operatorname{cosec} z \quad \begin{bmatrix} + \text{Circle L.} \\ - \text{Circle R.} \end{bmatrix} \qquad (206)$$

where A now denotes the absolute azimuth of the star, and ΔA is the index correction of the circle, or reduction of the readings to absolute azimuths. The readings for circle right differing by 180° from those for circle left, we shall always assume that the former have been increased or diminished by 180°, when two observations in different positions of the instrument are compared. We must now determine the quantities c, b, and ΔA.

212. To find c and b.—The most convenient method of finding c with a fixed instrument is to employ a collimating telescope placed on a level with the horizontal axis, such as that of Plate III. Fig. 2. The cross thread of the collimator is observed as an infinitely distant point or star, whose zenith distance is 90°; and hence $\cot z = 0$, $\operatorname{cosec} z = 1$. Observing it both with circle left and circle right, let A' and A'' be the readings of the azimuth circle (the latter reading being changed 180°); then we have

$$A = A' + \Delta A + c$$
$$A = A'' + \Delta A - c$$

whence

$$c = \tfrac{1}{2}(A'' - A') \tag{207}$$

which will give c with its proper sign for *circle left.*

If, however, the collimator is below the level of the horizontal axis, so that the telescope must be depressed to observe it, we shall have

$$A = A' + \Delta A + b \cot z + c \operatorname{cosec} z$$
$$A = A'' + \Delta A - b \cot z - c \operatorname{cosec} z$$

in which z = the zenith distance of the collimator = 90° + depression of the telescope, as given by the vertical circle; and then

$$c = \tfrac{1}{2}(A'' - A') \sin z - b \cos z \tag{208}$$

and b must be observed with the striding level applied to the axis, as in the case of the transit instrument.

When the telescope is furnished with a micrometer, the value of c can be found with still greater accuracy, by means of two collimators, as in Art. 145.

213. In some cases the spirit level cannot be reversed upon the axis, but is permanently attached to it or to the frame which supports it. It is then reversed only when the instrument is reversed, and it becomes necessary to know the level zero, or that reading of the level which corresponds to a truly horizontal position of the axis. Let this reading be denoted by l_0, and let l be the reading at any observation; then we have

$$b = l - l_0$$

where l is the mean of the readings of the two ends of the bubble, the readings towards the circle end being always reckoned as positive. Then to find l_0 we have recourse to the observation of two stars, one near the zenith and the other near the horizon, or of the same star at different times. Let A' and A'' be the circle readings, z' and z'' the zenith distances of the high star for circle left and circle right, respectively; l', l'' the level readings; then, A_1 and A_2 being the true azimuths, we have

$$A_1 = A' + \Delta A + (l' - l_0) \cot z' + c \operatorname{cosec} z'$$
$$A_2 = A'' - \Delta A - (l'' - l_0) \cot z'' - c \operatorname{cosec} z''$$

The difference between A_1 and A_2 may be accurately computed from the known place of the star, and a small error in its assumed place will not sensibly affect this difference. If the star is near the meridian (which will be advisable), the change in azimuth will be sensibly proportional to the interval of time between the two observations: so that if T' and T'' are the sidereal clock times, and δA the change of azimuth in one second, we shall have

$$A_2 - A_1 = \delta A\,(T'' - T') \tag{209}$$

in which $T'' - T'$ is in seconds; and δA may be found by the differential formula

$$\delta A = \frac{dA}{dT} = \frac{15'' \cos \delta \cos q}{\sin z}$$

where δ = the star's declination, and the parallactic angle q is found by Art. 15 of Vol. I. The difference of the above equations will then give us the equation

$$- ml_0 + nc = p \tag{210}$$

where, to abbreviate, we denote the known quantities as follows:

$$\left.\begin{array}{ll} m = \cot z' + \cot z'' & n = \operatorname{cosec} z' + \operatorname{cosec} z'' \\ p = A'' - A' - (A_2 - A_1) - l' \cot z' - l'' \cot z'' & \end{array}\right\} \tag{211}$$

In like manner, the low star gives a similar equation,

$$- m' l_0 + n'c = p' \tag{212}$$

and from the two equations the unknown quantities l_0 and c are found by the usual method of elimination. If a greater number of stars have been observed, the equations may be combined by the method of least squares. Where there is a collimator, it may always be used as the low star of this method.

214. To determine the index correction ΔA, observe any known star in either position of the instrument; then, having computed its true azimuth A (Vol. I. Art. 14), we have

$$\Delta A = A - (A' \pm b \cot z \pm c \operatorname{cosec} z) \tag{213}$$

215. With a portable instrument, such as is described in Art. 210, the use of a collimator is impracticable, since the telescope is at the extremity of the axis, and, therefore, cannot be directed towards the collimator in both positions. We must then employ stars, as in the preceding article; but, as in portable instruments the inclination b is usually found directly by the striding level, a single star observed in both positions of the instrument will suffice. If we take the *pole star* when near the meridian, we can suppose z to have the same value for both observations, and we shall have the two equations

$$A_1 = A' + \Delta A + b' \cot z + c \operatorname{cosec} z$$
$$A_2 = A'' + \Delta A - b'' \cot z - c \operatorname{cosec} z$$

whence

$$c = \tfrac{1}{2}\,[A'' - A' - (A_2 - A_1)] \sin z - \tfrac{1}{2}\,(b' + b'') \cos z \tag{214}$$

and it will then be expedient to determine ΔA at the same time from either A_1 or A_2.

216. If instead of a single vertical thread there are several such threads, the *horizontal transit* of the star is observed over each by the clock, as in ordinary transit observations, the reading of the horizontal circle remaining constant. If the star is not too far from the equator, the intervals of time between the transits over the threads may be assumed to be proportional to the distances of the threads, and then the mean of the times will be the time of the star's transit over the mean thread. The collimation constant c, determined from stars as in the preceding articles, will then be that of the mean thread.

If some of the threads have failed to be observed, let f_1, f_2, &c. be the distances of the threads from the mean thread, positive for threads on the same side of the mean as the vertical circle;

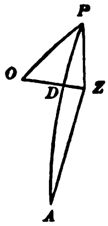
and let f_0 be the mean of the distances of the threads observed, and T_0 the mean of the observed times. Then $f_0 + c$ is the distance of the mean of the observed threads from the collimation axis; and the azimuth at the time T_0 is found by the formula (206), substituting $f_0 + c$ for c.

217. If, however, we wish to proceed rigorously, we can reduce each thread to the mean thread by the complete formula (138),

$$\sin I = \frac{\sin f}{\cos \delta \cos n \cos t} + 2 \tan t \sin^2 \tfrac{1}{2} I$$

where I is the interval of time in which the star describes the distance f, and $t = \tau - m$, τ being the *east* hour angle of the star, and m and n being determined by (78). But we can simplify this formula for our present purpose as follows. Let A, Fig. 50, be the point in which the horizontal axis of the instrument meets the sphere when produced through the circle end (as in Fig. 49); Z the zenith; P the pole; O the star when in the collimation axis of the telescope. Since the small inclination of the horizontal and vertical axes will not sensibly affect the thread intervals, we can here regard A as the pole of the vertical circle ZO, and the triangle OPD may be regarded as right angled at D. In this triangle we have, according to the definitions of m, n, and τ in Art. 123, the angle $OPD = OPZ - APZ = -\tau - (90° - m) = -90° - t$, and the side $PD = AP - 90° = (90° - n) - 90° = -n$. We have also $OP = 90° - \delta$, and the parallactic angle $POD = q$. Hence

Fig. 50.



$$\begin{aligned} \cos n \cos t &= -\cos q \\ \tan t &= \tan q \sin \delta \end{aligned}$$

and our formula becomes

$$\sin I = -\frac{\sin f}{\cos \delta \cos q} + 2 \sin \delta \tan q \sin^2 \tfrac{1}{2} I$$

This applies for circle left. For circle right it is only necessary to change the sign of the first term, so that the complete formula is

$$\sin I = \mp \frac{\sin f}{\cos \delta \cos q} + 2 \sin \delta \tan q \sin^2 \tfrac{1}{2} I \qquad (215)$$

in which we take $\begin{bmatrix}\text{upper}\\ \text{lower}\end{bmatrix}$ sign for $\begin{bmatrix}\text{circle L.}\\ \text{circle R.}\end{bmatrix}$, and I will be the correction algebraically additive to the observed time on a thread to reduce it to the mean thread. The angle q is found by the formula

$$\sin q = \frac{\sin A \cos \varphi}{\cos \delta} \tag{216}$$

where q will have a negative value for a negative value of $\sin A$, that is, for a star east of the meridian.

It is evident that, except for stars of considerable declination, the last term of (215) will be inappreciable, and that we may usually take

$$I = \mp \frac{f}{\cos \delta \cos q} \tag{217}$$

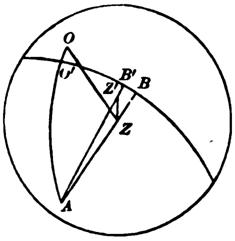
which amounts to assuming that I is proportional to f, as in the preceding article.

218. To find the equatorial values f of the thread intervals, we observe the transit of a slow moving star near the meridian, and from the observed intervals I we deduce

$$\sin f = \mp \sin I \cos \delta \cos q$$

219. *Zenith distances.*—Let Z, Fig. 51, be the zenith; Z' and A the points in which the vertical and horizontal axes meet the celestial sphere; $BB'O'$ the great circle of which A is the pole, and, consequently, the circle which represents the vertical circle of the instrument. This circle is also that which is described by the collimation axis of the telescope. Let the star O be observed on a horizontal thread OO', which is perpendicular to the great circle BO' and coincides with the arc AO' produced. The point B', in which AZ' produced meets the circle BB', represents the extremity of that diameter of the alidade circle which is in the plane of the vertical axis of the instrument. The arc $B'O'$, or the angle $B'AO'$ which it measures, is then the zenith distance, as given directly by the circle when the circle readings for B' and O' are given. Let the reading of the circle, when the thread is at B', be denoted by ζ_0, and the reading on the star by ζ, and put $B'O'$ or $B'AO' = z_1$; then, for circle left,

Fig. 51.



$$z_1 = \zeta_0 - \zeta$$

the graduations of the circle being supposed to increase from right to left. Now, for different azimuths the relative position of B and B' is different; and they coincide only when the point A is in the plane of the circle ZZ'. Their relative position at any time is given by the level attached to the alidade circle; for let l_0 be the reading of the level when B and B' coincide, and l the reading in any other case; then, denoting BB' by Δz_1, we have

$$\Delta z_1 = l_0 - l$$

where we take the left-hand end of the level as the positive end, the observer facing the circle, and l is half the algebraic sum of the readings of the ends of the bubble.

Let us now denote the arc BO' by z'; then we have

$$z' = z_1 + \Delta z_1$$

and in the triangle AOZ we have the required true zenith distance $ZO = z$, the angle $OAZ = z'$; and, in accordance with the notation before employed, $AO = 90° + OO' = 90° + c$, $AZ = 90° - b$. Hence

$$\cos z = - \sin c \sin b + \cos c \cos b \cos z'$$

Substituting $\cos z' = \cos^2 \tfrac{1}{2} z' - \sin^2 \tfrac{1}{2} z'$, we obtain

$$\begin{aligned} \cos z &= - \sin c \sin b \left(\cos^2 \tfrac{1}{2} z' + \sin^2 \tfrac{1}{2} z'\right) \\ &\qquad + \cos c \cos b \left(\cos^2 \tfrac{1}{2} z' - \sin^2 \tfrac{1}{2} z'\right) \\ &= \cos (c + b) \cos^2 \tfrac{1}{2} z' - \cos (c - b) \sin^2 \tfrac{1}{2} z' \\ \cos z' - \cos z &= 2 \sin \tfrac{1}{2} (z + z') \sin \tfrac{1}{2} (z - z') \\ &= 2 \sin^2 \tfrac{1}{2} (c + b) \cos^2 \tfrac{1}{2} z' - 2 \sin^2 \tfrac{1}{2} (c - b) \sin^2 \tfrac{1}{2} z' \end{aligned}$$

The second member involving only the squares of the small quantities c and b, the correction $z - z'$ is very small, so that for the factor $\sin \tfrac{1}{2}(z + z')$ we may take $\sin z' = 2 \sin \tfrac{1}{2} z' \cos \tfrac{1}{2} z'$. Hence, substituting the arcs for the sines of the quantities $\tfrac{1}{2}(z - z')$, $\tfrac{1}{2}(c + b)$, $\tfrac{1}{2}(c - b)$, we find

$$z - z' = \left(\frac{c + b}{2}\right)^2 \sin 1'' \cot \tfrac{1}{2} z' - \left(\frac{c - b}{2}\right)^2 \sin 1'' \tan \tfrac{1}{2} z' = \epsilon \qquad (218)$$

and ϵ will denote the correction for collimation and the inclination of the horizontal axis. Substituting the value of z' above given, we find as the value of the zenith distance given by the observation *circle left*,

$$z = \zeta_0 - \zeta + l_0 - l + \epsilon$$

In this equation the constants ζ_0 and l_0 are unknown; but now revolve the instrument 180° in azimuth, and observe zenith distance of the same point, we shall have

$$z_1 = \zeta' - \zeta_0 \qquad \Delta z_1 = -(l_0 - l')$$

where ζ' and l' denote the new readings of circle and level; a hence, for *circle right*,

$$z = \zeta' - \zeta_0 - l_0 + l' + \varepsilon'$$

in which ε' is computed by the formula

$$\varepsilon' = \left(\frac{c' + b'}{2}\right)^2 \sin 1'' \cot \tfrac{1}{2} z' - \left(\frac{c' - b'}{2}\right)^2 \sin 1'' \tan \tfrac{1}{2} z'$$

c' and b' being the collimation and the inclination of the h zontal axis in this second observation. The mean of the values of z is

$$z = \tfrac{1}{2}(\zeta' - \zeta) + \tfrac{1}{2}(l' - l) + \tfrac{1}{2}(\varepsilon' + \varepsilon) \qquad (2$$

Their difference gives the constant quantity

$$\zeta_0 + l_0 = \tfrac{1}{2}(\zeta' + \zeta) + \tfrac{1}{2}(l' + l) + \tfrac{1}{2}(\varepsilon' - \varepsilon) \qquad (2$$

If the observed point is moving, as in the case of a star, the va of z obtained by (219) is the zenith distance at the mean ti between the two observations; and, in general, if a series of ze distances is taken, one half in each position of the circle, an ζ denotes the mean of all the readings of the circle in the position, ζ' the mean of all the readings in the second posit l and l' the corresponding means of the readings of the ci level, the value of z given by (219) will be the zenith distanc the mean of all the observed times, *provided* always that the se is not extended so far as to introduce second differences of change of zenith distance. The correction for second differen when necessary, may be found by Vol. I. Art. 151.

The corrections ε and ε' are, however, usually rendered in sible in practice by observing the star only in the middle of field, or as near the middle vertical thread as possible, whic effected by giving the instrument a slow motion in azimuth w the star passes obliquely across the field, and thus keeping middle thread constantly upon the star until it is bisected by horizontal thread.

220. The equation (220) gives the constant $\zeta_0 + l_0$ only when the observed point is fixed. The cross thread of a collimating telescope, or a distant terrestrial object, may be used as such a fixed point; and, making the observations in the two positions of the circle only in the middle of the field, we shall have $\varepsilon' - \varepsilon = 0$: so that if we denote this constant by Z we shall have

$$Z = \tfrac{1}{2}(\zeta + \zeta') + \tfrac{1}{2}(l + l') \qquad (221)$$

With this constant thus determined, a single observation of a star, in either position of the instrument, will suffice to determine its zenith distance, since we shall then have

$$\left.\begin{array}{ll} z = Z - (\zeta + l) & \text{for circle L.} \\ z = (\zeta' + l') - Z & \text{" \quad " \quad R.} \end{array}\right\} \qquad (222)$$

The constant Z expresses the *zenith point of the instrument*, since in any position of the instrument it is equal to the corrected circle reading when the observed object is in the zenith.

If we wish to deduce Z from the two observations of a star, at the times T and T', we must compute the difference between the zenith distances for the interval $T' - T$, which, when the interval is small, may be done by the differential formula

$$\Delta z = (T' - T)\frac{dz}{dt} = (T' - T)\cos\varphi\sin A$$

in which $T' - T$ is supposed to be reduced to seconds of arc; and then we shall have

$$Z = \tfrac{1}{2}(\zeta + \zeta') + \tfrac{1}{2}(l + l') - \tfrac{1}{2}\Delta z$$

It should be remarked that when ζ' is numerically less than ζ we should increase it by 360°, both in finding z and Z.

When the two observations, in opposite positions of the axis, are made very near to the meridian, it will be advisable to reduce each to the meridian by applying the correction for circum-meridian altitudes, Vol I. equation (289) or (290).

EXAMPLE.—To determine the zenith point of an ERTEL universal instrument, the telescope was directed towards a distant terrestrial object, and the horizontal thread was brought into coincidence with a sharply defined point in the object, twice in each position of the vertical circle. The readings of the circle and level were as below. The graduations of the level proceed continuously from the right to the left end of the tube, so that the values of l are simply the arithmetical means of the readings of the two ends of the bubble. The value of one division $= 2''.0$.

	Circle readings.	Level readings.		l
Circle L.	180° 2′ 30″.	40.2	14.6	27.4
	180 2 35	40.4	14.5	27.45
Circle R.	359 56 20	38.2	12.8	25.5
	359 56 30	38.5	12.9	25.7

Hence, taking the means, we have

$$\begin{array}{ll} \zeta = 180^\circ\ 2'\ 32''.5 & l = 27.43 \\ \zeta' = 359\ \ 56\ \ 25\ . & l' = 25.60 \\ \hline \zeta_0 = 269\ \ 59\ \ 28\ .75 & l_0 = 26.52 = 53''.04 \\ l_0 = \qquad\quad +\ 53\ .04 & \\ \hline Z = 270\ \ 0\ \ 21\ .79 & \end{array}$$

A series of zenith distances of the sun's lower limb near the meridian was then taken, as follows:

	Circle reading.	Level reading.		Circle reading corrected for level.	Observed zenith distance.
Circle L.	229° 50′ 50″	38.4	12.7	229° 51′ 41″.1	40° 8′ 40″.7
	229 57 15	38.	12.3	229 58 5 .3	40 2 16 .5
	230 2 5	37.	11.5	230 2 53 .5	39 57 28 .3
	230 5 15	37.6	12.	230 6 4 .6	39 54 17 .2
	230 7 0	37.	11.4	230 7 48 .8	39 52 33 .0
Circle R.	309 52 15	33.4	7.9	309 52 56 .3	39 52 34 .5
	309 54 10	33.	7.4	309 54 50 .4	39 54 28 .6
	309 57 50	33.6	8.0	309 58 31 .6	39 58 9 .8
	310 2 40	33.8	8.3	310 3 22 .1	40 3 0 .3
	310 9 15	34.	8.8	310 9 57 .8	40 9 36 .0

Here we have, at the first observation,

$$\zeta = 229^\circ\ 50'\ 50'' \qquad l = +\overset{\text{div.}}{25.55} = +\ 51''.1$$

and hence the corrected circle reading is

$$\zeta + l = 229^\circ\ 51'\ 41''.1$$

The correction ϵ being neglected, as all the observations were made near the middle vertical thread, we obtain the observed zenith distance by subtracting this number from the above reading Z of the zenith point, whence $z = 40° 8' 40''.7$.

In like manner, the fifth observation gives $\zeta' + l' = 309° 52' 8''.3$, from which Z is subtracted to obtain the observed zenith distance. The results are given in the last column.

These observations have been employed in Vol. I. Art. 171, as circummeridian zenith distances for determining the latitude.

221. In the methods of observation above adopted, a knowledge of the deviations i and i' of the horizontal and vertical axes from their normal positions is not required: it is only necessary that they should be small. Their values, however, can be readily investigated. In the triangle AZZ', Fig. 51, we have the angle $AZ' = BB' = \Delta z_1 = l_0 - l$, as given by the level of the vertical circle; and this triangle gives, with the notation of Art. 211,

$$\sin \Delta z_1 = \frac{\sin i \sin a'}{\cos b}$$

or, taking a for a',

$$i \sin a = l_0 - l$$

At the same time, we have, from the level b of the horizontal axis,

$$i \cos a + i' = b$$

Now, revolving the instrument 180°, the angle a becomes $a + 180°$, and if the level reading of the vertical circle alidade is now l', and the inclination of the horizontal axis is b', we have

$$- i \sin a = l_0 - l'$$
$$- i \cos a + i' = b'$$

Hence, combining these equations with the former ones, we find

$$\left.\begin{array}{l} i \sin a = \frac{1}{2}(l' - l) \\ i \cos a = \frac{1}{2}(b - b') \end{array}\right\} \quad (223)$$

which determine i and a; and for i' we have

$$i' = \frac{1}{2}(b + b') \quad (224)$$

We can, also, find i and i' from the inclinations of the horizontal axis alone. Let the alidade of the azimuth circle be set at assumed reading A', and then also at $A' + 120°$ and $A' + 2$ and let b, b', b'', be the inclinations of the horizontal axis gi by the spirit level in the three positions. Then we have

$$\begin{aligned} i \cos a + i' &= b \\ i \cos (a + 120°) + i' &= b' \\ i \cos (a + 240°) + i' &= b'' \end{aligned}$$

the sum of which, since $\cos (a + 120°) + \cos (a + 240°) = -$ c gives

$$i' = \tfrac{1}{3} (b + b' + b'') \qquad ($$

This, subtracted from the 1st equation, gives

$$i \cos a = \frac{2b - b' - b''}{3} \qquad ($$

and the difference of the 2d and 3d equations gives

$$i \sin a = \frac{b'' - b'}{\sqrt{3}} \qquad ($$

which determine i and a. This method may be used for ins ments intended only for the measurement of horizontal ang In other instruments, both methods may be used, and accordance of the results will indicate the degree of perfect in the workmanship of the vertical pivots of the instrument.

222. If there are several horizontal threads, the *vertical tra* of the star over each may be observed, revolving the instrum slowly in azimuth, so as to make the transit occur in the mid of the field. The level of the alidade should be read b before and after the observation, and the mean taken as value of l at the mean of the times of observation. When star is not near the meridian, the zenith distance represented the mean of the threads may be assumed to correspond to mean of the observed clock times; but when near the merid a correction for second differences will be necessary.

In Vol. I. Art. 151, we have found that if T_1, T_2, T_3, &c. the several clock times, and T their mean, the corrected t corresponding to the mean of the zenith distances is

$$T_0 = T + \tfrac{1}{15} k m_0 \qquad ($$

in which, t being the hour angle, A the azimuth, and q the parallactic angle of the star,

$$k = \frac{\cos A \cos q}{\sin t}$$

and m_0 is the mean of the quantities

$$\frac{2 \sin^2 \tfrac{1}{2} (T_1 - T)}{\sin 1''}, \qquad \frac{2 \sin^2 \tfrac{1}{2} (T_2 - T)}{\sin 1''}, \text{ \&c.}$$

which can be taken from Table V.

For the moon, the correction will be

$$\tfrac{1}{15} (1 - \lambda)^2 k m_0 = \frac{k m_0}{15 B^2}$$

log B being found as in Art. 154.

If the transit is defective, that is, if only a portion of the threads have been used, it will be necessary to apply to the circle reading a correction which will be the difference between the mean of the threads observed and the mean of all the threads. Thus, f denoting the distance of any thread from the mean of all, and n the number of threads observed, the correction of the circle reading will be $\frac{1}{n} \Sigma f$. The value of f for each thread will be most readily found from complete vertical transits of stars which are not so near to the meridian as to require a correction for second differences, since we can then use the differential formula

$$f = 15\, I \times \frac{dz}{dt} = 15\, I \cos \varphi \sin A$$

in which I is the interval between the observed time on a thread and the mean of all the times.

To compute f with regard to second differences, see Vol. I. Art. 150.

223. *Correction of the observed azimuth and zenith distance of the limb of the moon or a planet for defective illumination.*—I shall here consider only the case where the defective limb of a spherical body has been observed. The formulæ for the more general case of a spheroidal planet may easily be deduced from those given in Vol. I. (occultations of a planet); but they are rarely if ever required. We can obtain the formulæ necessary for our present purpose from those given in Arts. 157 and 207 of t present volume. It is evident that in computing the appare outline of the disc of a planet as illuminated by the sun, a system of co-ordinates may be used, provided the places of t sun and planet are expressed in the same system. If, then, here substitute the zenith for the pole, and, consequently, tl horizon for the equator, we have only to substitute zenith di tance for polar distance and azimuth for right ascension, rather the negative of the azimuth, since the azimuth is reckon from left to right, while right ascension is reckoned from rig to left. Putting, therefore,

d = the sun's zenith distance,
a = " azimuth,
A = the planet's azimuth,
s = the planet's apparent semidiameter,
R, R' = the geocentric distances of the earth and planet, respectively,

we have, by (124), for computing the horizontal perpendicul from the centre of a planet upon the vertical thread in conta with the defective limb, the formulæ

$$\left.\begin{aligned} \sin\chi &= \frac{R}{R'}\sin d\sin(a-A) \\ s'' &= s\cos\chi \end{aligned}\right\} \quad (22$$

The value of $\sin\chi$ will be positive or negative according as tl 2d or the 1st limb is defective. The value of s may be foun from its mean value given in Vol. I. p. 578.

For the moon we can put $R = R'$.

Since we wish to deduce from the observed azimuth of th defective limb that of the true limb, the correction of the circl reading will evidently be

$$\delta A = \frac{s-s''}{\sin z} = \frac{s \text{ versin } \chi}{\sin z} \qquad (230$$

Again, for computing the vertical perpendicular from the centr of a planet upon the horizontal thread in contact with th defective limb, we deduce from (200), by changing the co-ordinates,

$$\sin\chi = \frac{R}{R'}\left[\sin z\cos d - \cos z\sin d\cos(a-A)\right] \qquad (231$$

r, by introducing an auxiliary,

$$\left.\begin{aligned} \tan E &= \tan d \cos(a - A) \\ \sin\chi &= \frac{R}{R'} \cdot \frac{\sin(z - E)\cos d}{\cos E} \end{aligned}\right\} \qquad (231^*)$$

and the correction to reduce the observed zenith distance to that of the true limb will be

$$\delta z = s \text{ versin } \chi \qquad (232)$$

A negative value of $\sin\chi$ will indicate that the upper limb is defective.

EXAMPLE 1.—The following observations of the azimuths of *Regulus* and of the moon's 1st limb were made at Greenwich with the "Alt-azimuth," May 3, 1852.

	Vertical circle.	Clock time of transit.	Circle reading $= A'$	Level $= l$	Clock corr.
☽ 1 L.	Left	$11^h\ 26^m\ 12^s.95$	140° 39′ 39″.71	− 19″.79	+ $11^s.46$
☽ 1 L.	Right	12 3 11 .30	328 45 10 .76	− 20 .14	11 .51
Regulus.	Right	12 31 55 .37	62 54 43 .04	− 21 .49	11 .55
Regulus.	Left	12 45 26 .33	246 34 47 .08	− 19 .28	11 .57

The clock time is the mean of the transits over six vertical threads. The clock correction is the reduction to sidereal time. The circle readings are the means of four microscopes. The level reading is the mean of the indications of six levels, permanently attached to the instrument, parallel to the horizontal axis. The level zero, found by the method of Art. 213, was

$$l_0 = -30''.16$$

The collimation constant for the mean of the threads was, for circle left,

$$c = +2''.68$$

The observations being taken for the purpose of determining the moon's azimuth, we shall first find the index correction of the circle from the known star *Regulus*. From the Nautical Almanac, we take

$$\begin{aligned} &\textit{Regulus}, \text{ R. A.} = \quad 10^h\ 0^m\ 29^s.32 \\ &\quad\text{"} \quad\ \text{Decl.} = +\,12^\circ\ 41'\ 16''.6 \end{aligned}$$

The hour angles of the star at the two observations are, therefore,

Circle R. $t = 2^h\ 31^m\ 37^s.60$
Circle L. $t = 2\ \ 45\ \ 8\ .58$

with which and the latitude $\varphi = 51°\ 28'\ 37''.84$ we find, by Vol. I. Art. 14, the stars's true azimuth and approximate zenith distance,

Circle R. $A = 52°\ 10\ 13''.10$ $z = 49°\ 22'$
Circle L. $A = 55\ \ 50\ \ 39\ .25$ $z = 51\ \ 4$

The zenith distances are apparent, *i.e.* affected by refraction. The instrumental corrections for the star are then as follows:

	$b = l - l_0$	$\pm b \cot z$	$\pm c \operatorname{cosec} z$
Circle R.	$+\ 8''.67$	$-\ 7''.45$	$-\ 3''.53$
Circle L.	$+10\ .88$	$+\ 8\ .79$	$+\ 3\ .45$

The corrected circle readings are, therefore (adding 180° to the reading for Circle R.),

	Corrected A'
Circle R.	242° 54′ 32″.06
Circle L.	246 34 59 .32

which, compared with the true azimuths A above found, give the index correction

	ΔA
Circle R.	169° 15′ 41″.04
Circle L.	169 15 39 .93

Mean $\Delta A = 169\ \ 15\ \ 40\ .48$

In the next place, to reduce the observations of the moon there were given the moon's apparent zenith distances (affected by parallax and refraction),

Circle L. ☽ $z = 77°\ 11'$
Circle R. ☽ $z = 73\ \ 17$

whence we find the instrumental corrections to be as follows:

	$b = l - l_0$	$\pm\, b \cot z$	$\pm\, c \operatorname{cosec} z$
Circle L.	$+\, 10''.37$	$+\, 2''.36$	$+\, 2''.75$
" R.	$+\, 10\ .02$	$-\, 3\ .01$	$-\, 2\ .80$

Applying these and the above found index correction, the true azimuths of the limb, as observed, were

Circle L. At $11^h\ 26^m\ 24^s.41$ Sid. time, $A = 309°\ 55'\ 25''.30$
" R. " $12\ \ 3\ \ 22\ .81$ " " $A = 318\ \ 0\ \ 45\ .43$

But the moon's limb was slightly gibbous; and we must yet apply the correction given by our formulæ (229) and (230). As the correction will not be sensibly different for the two observations, we may compute it for the middle instant between them, which corresponds to the mean solar time $8^h\ 57^m\ 16^s$. For this time, we find

$$\text{Sun's } \alpha = 2^h\ 44^m\ 15^s.74$$
$$\text{"}\quad \delta = +\, 15°\ 54'.6$$

from which we deduce the sun's azimuth and zenith distance

$$a = 136°\ 4'.9 \qquad d = 102°\ 8'.1$$

and hence, taking $A = 313°\ 58'.1$ (the mean value), we find

$$\log \sin \chi = n8.5570$$

Since $\sin \chi$ is negative, the first limb is defective. Then, since $s = 16'\ 36''.5$, and the mean value of $z = 75°\ 14'$,

$$dA = \frac{s \text{ versin } \chi}{\sin z} = 0''.67$$

which is to be added to the above values of A to obtain the azimuths of the true limb.

Example 2.—The following observations of the zenith distances of the collimator and of the moon's lower limb were made at Greenwich with the "Alt-azimuth," Sept. 21, 1852.

	Circle reading $= \zeta$	Level reading $= l$	$\zeta + l$
Collimator. Circle L.	$315°\ 47'\ 57''.53$	$74''.63$	$315°\ 49'\ 12''.16$
" R.	$160\ \ 23\ \ 30\ .34$	$82\ .46$	$160\ \ 24\ \ 52\ .80$
		$Z =$	$58\ \ 7\ \ 2\ .48$

The vertical transit of the moon was observed on six horizo threads, as follows:

	Thread.	Clock.	$T_n - T$	$m = \frac{2 \sin^2 \frac{1}{2} (T_n -}{\sin 1''}$
☽ L.L. Circle L.	I	$19^h\ 38^m\ 11^s.5$	$-3^m\ 43^s.4$	$27''.22$
	II	39 47 .0	-2 7 .9	8 .93
	III	41 16 .0	-0 38 .9	0 .83
	IV	42 42 .5	$+0$ 47 .6	1 .24
	V	44 5 .5	$+2$ 10 .6	9 .30
	VI	45 27 .0	$+3$ 32 .1	24 .53
	$T =$	19 41 54.92		$m_0 = 12$ .01

Clock corr. $= \quad +\ 7.90$

Sid. time $= 19\ 42\ \ 2.82$

Circle reading $\zeta = 341°\ 27'\ 12''.55$

Level " $l = \quad +\ 80\ .90$

$\zeta + l = 341\ \ 28\ \ 33\ .45$

$Z = 58\ \ 7\ \ 2\ .48$

$z = 76\ \ 38\ \ 29\ .03$

This zenith distance does not correspond precisely to the m time T, on account of the moon's proximity to the meridian. obtain the correction for second differences by our formula (2 we have found above the differences between the several cl times and T, and also the mean (m_0) of the corresponding val of m. Then, to compute the coefficient k, we have the appro mate azimuth of the moon at the time of observation,

$$A = +\ 8°\ 58'.8$$

and the moon's declination,

$$\delta = -\ 23°\ 34'.5$$

Hence, with $\varphi = 51°\ 28'.6$, by the formulæ

$$\sin q = \frac{\sin A}{\cos \delta} \cos \varphi \qquad \sin t = \frac{\sin A}{\cos \delta} \sin z$$

we find

$$\log \sin q = 9.0257, \qquad \log \sin t = 9.2194$$

and then

$$\log k = 0.7727$$

The change of the moon's right ascension in one minute of mean time was $2^s.40$; and hence, by the table in Art. 154,

$$\text{ar. co. log } B = \log(1 - \lambda) = 9.9823$$

We have, therefore, the correction

$$\tfrac{1}{15}(1 - \lambda)^2 k m_0 = + 4^s.37$$

which, being added to the sidereal time above found, gives $19^h\,42^m\,7^s.19$ as the sidereal time corresponding to the apparent zenith distance 76° 38′ 29″.03.

It should be observed that in the observation of the collimator one of the horizontal threads is made to bisect the cross thread of the collimator, and, therefore, in order to make the circle readings correspond to the mean of the threads, they must be increased by the distance of the horizontal thread employed from the mean. In the above observations the 4th thread was employed, the distance of which from the mean of the six threads was 1′ 0″.46. This quantity is included in the circle readings above given, so that they represent the readings that would have been obtained if the fictitious thread called the mean thread had actually been observed in coincidence with the threads of the collimator.

In conclusion, it is to be remarked that stars may be observed both directly and by reflection in a mercury horizon, in which case the difference of the readings of the vertical circle (corrected for any change in the alidade levels, &c.) will be twice the altitude. The combination of the reflected observations in both positions of the axis gives the nadir point of the instrument, precisely as the zenith point is obtained from the direct observations. The method of conducting such observations will be readily inferred from what has already been said under Meridian Circle, Art. 200.

[For an example of the use of a portable instrument in determining the longitude of a place by the moon's azimuth, see Vol. I. p. 380.]

CHAPTER VIII.

THE ZENITH TELESCOPE.

224. THE zenith telescope is a portable instrument specially adapted for the measurement of *small differences* of zenith distance. It is essentially the invention of Capt. ANDREW TALCOTT, of the U. S. Corps of Engineers (in 1834); but, having been exclusively adopted in the U. S. Coast Survey for the determination of latitudes, it has there received several improvements, which have given it a more general character than it possessed at first. As now constructed, it can be used at all zenith distances, and may be regarded as designed for the comparison of any two nearly equal zenith distances in any azimuths. The method of finding the latitude by this instrument, now known as *Talcott's Method*, is one of the most valuable improvements in practical astronomy of recent years, surpassing all previously known methods (not excepting that of BESSEL by prime vertical transits) both in simplicity and in accuracy.

Plate XIII. represents one of the zenith telescopes of the U. S. Coast Survey. The telescope is attached to one end of a horizontal axis Q, and is counterpoised by a weight O at the other end, which is so connected with the telescope by the curved lever P, P, P as to tend not only to equalize the pressure of the axis Q upon the two Vs, but to prevent the flexure of the axis. The Vs of the horizontal axis, one of which is seen at N, are connected with each other by the horizontal bar M, and thereby to the vertical column C. This column revolves about a vertical axis and carries a vernier and clamp c, by means of which it may be set at any reading of the horizontal circle BB. The vertical axis and horizontal circle are secured to a tripod, the feet of which, A, A, A, are levelling screws for adjusting the verticality of the axis. The striding level S is applied to the horizontal axis, as in the case of the transit instrument.

We now come to the distinctive features of the instrument, the spirit level L and the micrometer E. The level L is at right angles to the horizontal axis, and, consequently, in the plane of motion of the telescope, and is firmly connected with the bar *H*, which revolves upon a centre secured to the telescope: so that it may be placed at any angle with the optical axis of the telescope. In order to set the level at any given angle approximately, the bar *H* carries a vernier, which by the clamp *I* can be fixed at any reading of the vertical circle *K*, and this circle is permanently connected with the telescope. This circle, being graduated from 0° at its middle point to 90° in each direction, will, when properly adjusted, give the zenith distance of a star towards which the telescope is directed when the bubble of the level is in the middle of the tube; and it therefore serves as a finder by setting the vernier upon the given zenith distance of a star and then revolving the telescope until the bubble plays. When the telescope is thus approximately set, it is clamped by the screw *G*, which acts upon a circular collar around the horizontal axis, and then a fine motion in zenith distance can be given to the telescope by the tangent screw *F*. This fine motion is required only in bringing the bubble of the level nearly to the middle of the tube.

E is a filar micrometer with one or more movable threads carried by a single micrometer screw with a graduated head reading directly to hundredths of a revolution, and by estimation to thousandths. In the instruments in use, one revolution is usually less than 50″, and hence each observation is read off, by estimation, within less than 0″.05. There are usually added several fixed vertical threads, so that the instrument can be used as a transit instrument when required.

In the preliminary adjustment, when setting up the instrument, the test of the verticality of the axis *C* is that the reading of the striding level *S* is not changed while the instrument makes a complete revolution in azimuth. The perpendicularity of the horizontal and vertical axes *Q* and *C* is proved when, after having made *C* vertical, *Q* is horizontal; and the latter is proved by reversing the level *S* upon the axis.

The middle transit thread can be approximately adjusted by causing it to coincide with a very distant terrestrial point in two positions of the telescope for which the readings of the horizontal circle differ exactly 180°. This, however, is but an approximation; for there will be a parallax in the apparent position of any terrestrial point as observed in the two positions, since the absolute position of the centre of the telescope is changed by twice its constant distance from the vertical axis. We can easily compute the amount of this parallax in a given case and allow for it; for if d = the distance of the centre of the telescope from the vertical axis, D = the distance of the object, and p = the parallax, we have

$$p = \frac{d}{D \sin 1''}$$

but, as the horizontal circle is not designed for very accurate measures, it will not usually be worth while to use this method further than to make a first adjustment. A perfect adjustment can be directly effected by the use of two collimating telescopes (Transit Inst., Art. 145), for which we can temporarily use the telescopes of two theodolites or other field instruments at hand. When the instrument is used as a transit, the collimation constant can be determined from a number of stars observed in the two positions of the axis by the method of least squares, supposing two different azimuths but the same collimation in the two sets of equations of condition, as in the example, p. 202.

The verticality of the transit threads is proved by the methods used for the transit instrument.

In finding the latitude by meridian observations, the instrument is frequently revolved in azimuth 180° for the alternate observation of north and south stars, and, to save time in this operation, two stops, *b*, *b*, are provided, which can be clamped at any points of the limb of the horizontal circle, and, consequently, at such points that the telescope shall be in the meridian when the clamp *e* bears against either stop.

225. *Talcott's method of finding the latitude.*—Two stars are selected which culminate at nearly equal zenith distances, one north and the other south of the zenith. The difference of their zenith distances must be less than the breadth of the field of the telescope, and it is better to have it less than half this breadth, to avoid observations near the edge of the field. Their right ascensions should be nearly equal, so that their transits may occur within so short a period that the state of the instrument may be assumed to have remained unchanged; but a sufficient interval should be allowed for making the necessary observation of the level and micrometer and for reversing in azimuth. The stops

having been previously set (by means of some known star) so as to mark the meridian, the finding circle *K* is set to the mean zenith distance of the two stars, and the telescope is pointed so as to make the reading of the level *L* nearly zero. The telescope can now be directed upon either star by revolving the instrument about the vertical axis, and this axis is supposed to be so nearly vertical that the reading of the level will not be greatly changed, since for accurate determinations with a spirit level it is always important to make the inclinations which it is to measure as small as possible, and not to use the extreme divisions. The chronometer times of the transits of the stars have been previously computed from their right ascensions and the chronometer correction. The instrument being set for the star which culminates first, when the star comes into the field an assistant calls the seconds of the chronometer, and the observer bisects the star by the micrometer thread as nearly as possible at the computed time of transit; or, failing in doing this satisfactorily, he bisects it soon after, and records the actual time of the observation. He then reads the level and micrometer, revolves the instrument 180°, and observes the second star in the same manner.

Several bisections of the star might be made while it is passing through the field, and each could be reduced to the meridian; but in the Coast Survey a single deliberate meridian observation is regarded as preferable to several circummeridian observations.*

We must not fail to remark that, since the excellence of this method depends upon the invariability of the angle which the telescope and level make with each other, the observer must not touch the tangent screw *I* after having set for the proper zenith distance, until the observation of the two stars is completed. The same restriction does not apply to the tangent screw *F*, which moves the telescope and level together; and, in case the vertical axis is not very well adjusted, it may be necessary to use this screw, after turning to the second star, in order to bring the bubble of the level near the middle of the scale.

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* The single observation is preferable on the score of simplicity in the subsequent reductions, but it cannot be regarded as more accurate than the mean of several properly taken observations. The best reason for preferring the single observation is found in the present state of the star catalogues, for even the single observation with the zenith telescope is subject to a less probable error than the place of the star in most of the catalogues that have to be used. It is, therefore, preferable to simplify the individual observations and to multiply observations by taking different pairs of stars.

Now let m be the micrometer reading (reduced to arc) for the southern star. Let m_0 be the micrometer reading for any point of the field arbitrarily assumed as the micrometer zero; and let z_0 be the apparent zenith distance represented by m_0 when the level reading is zero. Let us also suppose that the micrometer readings increase as the zenith distances decrease. Then, if the level reading were zero, the apparent zenith distance of the star would be

$$z_0 + (m_0 - m)$$

Let l be the equivalent in arc of the level reading, positive when the reading of the north end of the level is the greater; let r be the refraction. Then the true zenith distance of the southern star is

$$z = z_0 + m_0 - m + l + r$$

The quantity $z_0 + m_0$ is constant so long as the relation of the level and telescope is not changed. We shall, therefore, have for the northern star

$$z' = z_0 + m_0 - m' - l' + r'$$

Hence we have

$$z - z' = m' - m + l' + l + r - r'$$

But, if δ and δ' are the declinations of the south and north stars, respectively, and φ the latitude, we have

$$\varphi = \delta + z$$
$$\varphi = \delta' - z'$$

and, therefore,

$$\left.\begin{aligned}\varphi &= \tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(z - z') \\ &= \tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(m' - m) + \tfrac{1}{2}(l' + l) + \tfrac{1}{2}(r - r')\end{aligned}\right\} \quad (233)$$

Thus, to the mean of the declinations we have to add three corrections, which I shall consider separately.

226. *The correction for refraction.*—The observations being usually restricted to zenith distances less than 25°, and the difference of zenith distance being necessarily less than the breadth of the field of the telescope, the difference of the refractions is

so small that the variations depending on the state of the barometer and thermometer are not sensible, and we may employ the equation

$$r - r' = (z - z')\frac{dr}{dz}$$

in which, if $z - z'$ is expressed in minutes, the differential quotient $\frac{dr}{dz}$ will denote the change of the mean refraction corresponding to a change of one minute of zenith distance. If we take BESSEL's formula for the refraction,

$$r = \alpha \tan z$$

in which α may be regarded as constant for small variations of z, we have

$$\frac{dr}{dz} = \frac{\alpha \sin 1'}{\cos^2 z}$$

by which we readily form the following table:

z	$\frac{dr}{dz}$
0°	0″.0168
5	.0169
10	.0173
15	.0180
20	.0190
25	.0205

The principal term in the value of $z - z'$ is $m' - m$, and we may in practice take ($m' - m$ being expressed in minutes)

$$\tfrac{1}{2}(r - r') = \tfrac{1}{2}(m' - m)\frac{dr}{dz} \qquad (234)$$

The correction for refraction then has the same sign as the correction for the micrometer.*

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* If we wish to consider the actual state of the air as given by the barometer and thermometer, we have only to multiply the values of $\frac{dr}{dz}$ by B and γ, whose logarithms are given in Table II.

227. *The correction for level.*—If we denote the readings of the north and south ends of the bubble by n and s, the inclinations observed at the observations of the south and north stars, respectively, expressed in divisions of the level, or, as I shall call them, the *level readings*, will be

$$L = \frac{n - s}{2} \qquad L' = \frac{n' - s'}{2}$$

and, putting $D =$ the value of a division of the level in seconds of arc, we shall have

$$l = LD \qquad l' = L'D$$

and the correction for the level will be

$$\tfrac{1}{2}(l' + l) = \tfrac{1}{2}(L' + L)D = \left(\frac{(n' + n) - (s' + s)}{4}\right)D \qquad (235)$$

Thus the correction for the level is found with its proper sign by subtracting the sum of the south end readings from the sum of the north end readings, and multiplying one-fourth the remainder by the value of a division.

228. *The correction for the micrometer.*—If we denote the actual micrometer readings for the south and north stars by M and M', expressed in revolutions of the screw, and put $R =$ the value of a revolution in seconds, we have

$$\tfrac{1}{2}(m' - m) = \tfrac{1}{2}(M' - M)R \qquad (236)$$

We have supposed the readings to increase as the zenith distances decrease, or, which is the same thing, that the readings increase from the upper part of the field towards the lower part. This is desirable only on account of the symmetry it gives to the reductions, the proper sign of the correction being determined, as in the case of the level, by always subtracting south readings from north readings. But it is well to reverse the instrument occasionally, using the telescope sometimes on the right and sometimes on the left of the vertical axis, in order to eliminate any unknown peculiar error of the instrument, and in conformity with the general principle of varying the circumstances under which different determinations of the same quantity are made. This reversal, of course, reverses the sign of the readings, and therefore when the readings are the reverse of those above supposed it will be sufficient to mark them all with the negative sign, and then to proceed by the same formulæ as before.

229. *Reduction to the meridian.*—When from any cause the server fails to obtain the meridian observation, a single extra-eridian observation is usually substituted. This observation ay be taken in either of two ways.

First. The instrument is left clamped in the meridian, and the ar is observed at a certain distance from the middle vertical hread, the time being noted. The reduction to the meridian is then the same as for the meridian circle (Art. 199), namely, τ being the hour angle of the star in seconds of time,

$$\tfrac{1}{4}(15\tau)^2 \sin 1'' \sin 2\delta$$

This is to be added to the observed zenith distance of a southern star, or subtracted from that of a northern star, and in either case one-half of it is to be added to the latitude. The correction to the latitude is, therefore,

$$x = \tfrac{1}{8}(15\tau)^2 \sin 1'' \sin 2\delta = [6.1347]\,\tau^2 \sin 2\delta \qquad (237)$$

when *one* of the stars of a pair is observed out of the meridian. If both are so observed, two such corrections, separately computed for each, must be added. If the star is south of the equator, the essential sign of the correction is negative.

Secondly. We may follow the star off the meridian by revolving the instrument in azimuth, keeping the star near the middle vertical thread. The reduction is then the same as that of circummeridian altitudes (Vol. I. Art. 170), namely,

$$\frac{(15\tau)^2 \sin 1''}{2} \cdot \frac{\cos\varphi \cos\delta}{\sin z}$$

which is always subtractive from the observed zenith distance, and therefore the correction to the latitude in this case will be

$$x = \pm \frac{(15\tau)^2 \sin 1''}{4} \cdot \frac{\cos\varphi \cos\delta}{\sin z} \qquad (238)$$

the upper sign for a northern and the lower for a southern star.

230. *Selection of stars.*—The fundamental stars whose declinations are determined with the highest degree of precision are too ew to afford suitable pairs for this method, and hence we must ave recourse to the smaller stars. Those of the 6th or 7th agnitude are the smallest that can be easily observed with a

portable instrument. But, as the declinations of these stars are not very precisely determined, we are obliged to employ a large number of pairs in order to eliminate their errors as far as possible by taking the mean of all the results. The British Association Catalogue will generally furnish from fifteen to thirty pairs for any given latitude on almost any night in the year, but, as the declinations of the stars selected will often be found to rest upon a single observation, or upon a single authority, these ought to be rejected unless they can be found also in more recent catalogues. In order to secure every available pair, the catalogue should be consulted from the earliest right ascension which the daylight at the time of the beginning of the series of observations permits, to the latest hour at which it is desirable to observe.

It is found expedient to prepare a table in which all the stars which culminate within 25° of the zenith, both north and south, are arranged in the order of their right ascensions. From this table suitable pairs are selected to satisfy as nearly as possible the following conditions: 1st, The difference of the zenith distances in a pair should not be more than 10′; in order not to have to observe either star near the edge of the field, and also in order to lessen the effect of an error in the determination of the value of the micrometer screw. 2d, The difference of the right ascensions of a pair should not be less than one minute, so as to give time to read the micrometer, and to revolve the instrument to be prepared for the second star; and not greater than about twenty minutes, to avoid changes in the state of the instrument. 3d, The interval between pairs should afford time for reading the micrometer and level, and for setting the instrument for the next pair. 4th, The greater zenith distance should be as often that of the northern as that of the southern star, as an error in the value of the micrometer screw will thereby be rendered less sensible. The effect of such an error would evidently be wholly insensible in the case of a pair whose zenith distances were exactly equal; and, in general, for any number of pairs the effec of such an error upon the final result will be the more nearly insensible the more nearly we approach to the condition

$$\Sigma z - \Sigma z' = 0 \qquad (239$$

231. EXAMPLE.—To illustrate the preceding method, I extrac from the records of the U. S. Coast Survey, by the kind permission of the Superintendent, a portion of the observations taken

the *Roslyn Station*, Virginia, in July, 1852, and shall give them ry nearly in the form in which they are recorded and reduced pon the survey. After selecting the most suitable pairs of stars y the process above described, a list is made out for the use of e observer in preparing for each observation, as follows:

Programme for Zenith Telescope.

U. S. C. S. Roslyn Station, Va. Approx. lat. = 37° 14′

Star.	Mag.	AR.	Dec.	Zen. Dist.		Setting.
B. A. C. 4843	6	$14^h\ 33^m\ 21^s$	+ 45° 3′	7° 49′	N.	7° 55′
" 4902	6	14 43 37	29 14	8 0	S.	
" 4902	6	14 43 37	29 14	8 0	S.	8 0
" 4965	5½	14 57 55	45 14	8 0	N.	
" 4991	6	15 2 2	26 52	10 22	S.	10 21
" 5092	7	15 20 21	47 35	10 21	N.	
" 5092	7	15 20 21	47 35	10 21	N.	10 24
" 5192	5	15 36 33	26 46	10 28	S.	
&c.		&c.				

The following are some of the observations taken by Mr. Dean:

Date, 1852.	Star.		Micrometer.		Level.			Merid. dist.
	No. B.A.C.	N. S.	Reading.	Diff. Z. Dist.	N.	S.	N — S.	
			Rev.	Rev.				
July 9	4843	N.	29.590		32.4	35.0		
	4902	S.	12.340	+ 17.250	34.0	35.3	— 8.9	
" 9	4902	S.	12.340		34.0	35.3		
	4965	N.	13.990	+ 1.650	33.8	37.0	— 4.5	
" 9	4991	S.	23.810		31.2	39.5		
	5092	N.	25.525	+ 1.715	39.2	33.0	— 2.1	
" 9	5092	N.	25.525		39.2	33.0		
	5192	S.	14.800	+ 10.725	32.8	41.0	— 2.0	
" 19	5911	N.	14.805		48.5	43.6		$10^s.9$
	5922	S.	26.675	— 11.870	43.0	49.0	— 1.1	
" 20	6453	S.	8.225		44.4	49.4		20 .5
	6530	N.	5.360	— 2.865	50.2	43.5	+ 1.7	

The stars 5911 and 6453 were observed out of the meridi the hour angles $10^s.9$ and $20^s.5$, respectively, the instru remaining in the meridian.

The next step is to deduce the apparent declinations fo dates of the observations from the catalogues, using for this pose not only the B. A. C., but also any later catalogues in w the stars can be found.

The value of a revolution of the micrometer was $R = 41'$ and that of one division of the level was $D = 1''.65$. The c putation of the latitude is then as follows:

Star.	δ and δ′	½(δ + δ′)	Corrections.				Latitud
			Microm.	Level.	Refr.	Merid.	
4843 4902	+45° 2′ 56″.56 +29 14 1 .85	37° 8′ 29″.21	+5′ 57″.08	−1″.61	+0″.10		37° 14′ 2
4902 4965	29 14 1 .85 45 13 43 .64	37 13 52 .75	+0 34 .15	−1 .86	+0 .01		2
4991 5092	26 52 24 .73 47 35 16 .37	37 13 50 .55	+0 35 .50	−0 .87	+0 .01		2
5092 5192	47 35 16 .37 26 46 13 .52	37 10 44 .95	+3 42 .01	−0 .83	+0 .06		2
5911 5922	48 23 22 .47 26 13 41 .36	37 18 31 .92	−4 5 .71	−0 .45	−0 .07	+0.02	2
6453 6530	22 27 47 .31 52 3 0 .31	37 15 23 .81	−0 59 .31	+0 .70	−0 .02	+0.04	2

Mean = 37 14 2

232. *Discussion of the results.*—In combining the results tained by this method, we should have regard to their respe weights. The weight of any result from a pair is a functio the probable error of the declinations of the stars and of probable error of observation.

The probable error of an observation of a single pair, w may be denoted by e, is found by comparing all the observat on the same pair with their mean, where a sufficient numbe observations have been taken. Assuming that the prob error of observation is the same for every pair of stars, we find its mean value from all the pairs, as follows. If v_1 der the residuals obtained by comparing the mean of the result the first pair with n_1 individual results from that pair, v_2 residuals obtained in like manner from a second pair on w there are n_2 observations, and so on, to m pairs, we have, acc ing to the theory of least squares,

$$(n_1 - 1)\, ee = q^2\, [v_1 v_1]$$
$$(n_2 - 1)\, ee = q^2\, [v_2 v_2]$$
$$\dots\dots\dots\dots$$
$$(n_m - 1)\, ee = q^2\, [v_m v_m]$$

ere $[v_1 v_1]$ &c. denote the sums of the squares of the values v_1, &c., and q is the factor for reducing mean errors to pro- le errors. (See Appendix, Art. 15.) The sum of these equa- ns gives

$$(n - m)\, ee = q^2\, [vv]$$

ere n denotes the whole number of individual results, or n $n_1 + n_2 + \dots\dots + n_m$, and $[vv]$ the sum of the squares of all residuals, or $[vv] = [v_1 v_1] + [v_2 v_2] + \dots\dots + [v_m v_m]$. Hence have

$$e = q\sqrt{\frac{[vv]}{n - m}} \qquad q = 0.6745 \qquad (240)$$

EXAMPLE.—The individual results of the whole series of ob- vations at Roslyn in July, 1852, from which the above are tracted, were as stated in the following table, in which only e seconds of latitude are given.

To find the error of observation.

No. of pair.	Lat.	Means	v	vv
1	24″.78			
2	25 .05			
3	25 .19	24″.83	.36	.1296
	24 .47		.36	.1296
4	26 .19	26 .20	.01	.0001
	25 .94		.26	.0676
	26 .47		.27	.0729
5	25 .52	25 .91	.39	.1521
	26 .08		.17	.0289
	26 .14		.23	.0529
6	22 .95	22 .73	.22	.0484
	22 .50		.23	.0529

To find the error of observation.—Continued.

No. of pair.	Lat.	Means.	*v*	*vv*
7	26″.26	25″.93	.33	.1089
	25 .42		.51	.2601
	25 .96		.03	.0009
	26 .01		.08	.0064
	25 .98		.05	.0025
	25 .96		.03	.0009
8	25 .47	25 .18	.29	.0841
	24 .97		.21	.0441
	24 .95		.23	.0529
	25 .30		.12	.0144
	24 .99		.19	.0361
	25 .38		.20	.0400
9	25 .17	25 .89	.72	.5184
	25 .64		.25	.0625
	26 .00		.11	.0121
	26 .45		.56	.3136
	26 .17		.28	.0784
10	25 .92	25 .79	.13	.0169
	25 .46		.33	.1089
	25 .70		.09	.0081
	26 .09		.30	.0900
11	25 .15	24 .53	.62	.3844
	24 .24		.29	.0841
	24 .43		.10	.0100
	24 .29		.24	.0576
12	26 .18	25 .15	1.03	1.0609
	24 .17		.98	.9604
	25 .10		.05	.0025
13	25 .73	25 .22	.51	.2601
	25 .78		.56	.3136
	24 .12		1.10	1.2100
	25 .23		.01	.0001
14	24 .86	24 .84	.02	.0004
	24 .55		.29	.0841
	25 .16		.32	.1024
	24 .80		.04	.0016
15	25 .91	25 .36	.55	.3025
	25 .00		.36	.1296
	25 .18		.18	.0324
	25 .35		.01	.0001

To find the error of observation.—Concluded.

No. of pair.	Lat.	Means.	*v*	*vv*
16	25″.94	26″.02	.08	.0064
	26 .74		.72	.5184
	26 .23		.21	.0441
	25 .18		.84	.7056
17	25 .82	25 .42	.40	.1600
	26 .01		.59	.3481
	24 .99		.48	.1849
	24 .86		.56	.3136
18	26 .37	26 .08	.29	.0841
	25 .94		.14	.0196
	25 .84		.24	.0576
	26 .16		.08	.0064
19	25 .97	25 .72	.25	.0625
	25 .92		.20	.0400
	25 .60		.12	.0144
	25 .37		.35	.1225
20	26 .02	25 .70	.32	.1024
	25 .67		.03	.0009
	25 .89		.19	.0361
	25 .20		.50	.2500
21	26 .32	25 .93	.39	.1521
	25 .49		.44	.1936
	25 .97		.04	.0016

$n = 73$

$m = 19$

$n - m = 54$

$[vv] = 11.0169$

ce, $e = 0.6745\sqrt{\frac{11.0169}{54}} = 0''.30$

his small probable error is a proof both of the great supe-
ity of this method over all previously known methods of
ing the latitude, and of the skill of the observer. Possibly
unusually favorable state of the atmosphere may have con-
ed to give this series an unusual degree of precision, as the
age experience of the observers of the Coast Survey gives
value of *e* somewhat greater. Not to assume too high a
ree of precision for the observations, the adopted value upon
Survey is

$$e = 0''.50$$

and even this value justifies us in asserting that the results by this method compare favorably with those obtained by first class fixed instruments of the observatory, where the measures depend upon graduated circles.

But the precision of the results is impaired by the defective state of the catalogues of the smaller stars, and the necessity for using such stars in order to find suitable pairs is the only "weak point of the method." The facility of multiplying the number of pairs, on account of the extreme simplicity of the observations, in a great degree compensates for this defect.

If now we denote the probable error of an observed zenith distance by e_z, we have the probable error of the observed difference $z - z' = \sqrt{2e_z^2}$, and the above value of e is the probable error of $\frac{1}{2}(z - z')$. Hence we have the relation

$$\sqrt{2e_z^2} = 2e$$

and, taking $e = 0''.50$,

$$e_z = e\sqrt{2} = 0''.71$$

which represents the combined effect of the error in bisecting the star, the culmination error, or error peculiar to a culmination arising from an anomalous variation in the refraction and affecting differently the two stars of a pair, the errors in the values of the micrometer and level divisions, and errors arising from changes in the instrument (resulting chiefly from changes of temperature) between the two observations of a pair. Of these, the most important is the error in bisecting the star which is strictly the error of observation.

233. Having found the probable error of observation, we can determine that of the declinations employed. For if ε is the probable error of observation of the mean value of φ deduced from all the observations of a pair, E_δ the probable error of the mean of two declinations, E_ϕ the probable error of the latitude composed of the errors of observation and declination, we have

$$E_\phi^2 = E_\delta^2 + \varepsilon^2$$

whence

$$E_\delta^2 = E_\phi^2 - \varepsilon^2 \qquad (241)$$

The mean value of E_δ for the stars employed (or for a given catalogue when all the declinations are taken from the same catalogue) will be obtained from this equation by employing in the second member mean values of E_ϕ^2 and ε^2. A mean value of E_ϕ will be obtained from the several means obtained from the several pairs (without here attempting to assign different weights to the observations) by the usual method from the residuals. The value of ε may be obtained for each pair from the single observations, when they are sufficiently numerous; but, as we wish in the present investigation to use all the observations even where a pair has been observed but once, it will be expedient to compute ε by the formula

$$\varepsilon^2 = \frac{e^2}{n}$$

in which e is the probable error of a single observation of a pair already found, and n is the number of observations of that pair. Then the mean of all these values of ε^2 is to be used in (241), and this mean is, for m pairs,

$$\varepsilon^2 = \frac{e^2}{m-1}\left[\frac{1}{n}\right] \qquad (242)$$

From the observations at Roslyn above given, we form the following table:

To find the probable error of declination.

No. of pair.	Lat.	v	v^2	No. of obs. $= n$	$\frac{1}{n}$
1	24″.78	.57	.3249	1	1.
2	25 .05	.30	.0900	1	1.
3	24 .83	.52	.2704	2	0.500
4	26 .20	.85	.7225	3	0.333
5	25 .91	.56	.3136	3	0.333
6	22 .73	2.62	6.8644	2	0.500
7	25 .93	.58	.3364	6	0.167
8	25 .18	.17	.0289	6	0.167
9	25 .89	.54	.2916	5	0.200
10	25 .79	.44	.1936	4	0.250
11	24 .53	.82	.6724	4	0.250
12	25 .15	.20	.0400	3	0.333
13	25 .22	.13	.0169	4	0.250
14	24 .84	.51	.2601	4	0.250
15	25 .36	.01	.0001	4	0.250
16	26 .02	.67	.4489	4	0.250
17	25 .42	.07	.0049	4	0.250
18	26 .08	.73	.5329	4	0.250
19	25 .72	.37	.1369	4	0.250
20	25 .70	.35	.1225	4	0.250
21	25 .93	.58	.3364	3	0.333

Mean $= 25\ .35$ $[vv] = 12.0083$ $\left[\frac{1}{n}\right] = 7.366$

$$E_\phi^2 = (\text{[redacted]} \times \frac{12.0083}{20} = 0.273 \qquad \epsilon^2 = \frac{(0.30)^2 \times 7.366}{20} = 0.033$$

$$E_\delta^2 = 0.240 \qquad E_\delta = 0''.49$$

The re [redacted] ie probable error of the quantity $\frac{1}{2}(\delta + \delta')$. That of a si lination is, therefore, $0''.49 \times \sqrt{2} = 0''.69$.

If al. :linations had been taken from the same authority, the prob :ror thus found would have determined the weight of that authority, and could afterwards be used in assigning weights to different observations. For this purpose, the probable errors of the different authorities have been determined from the numero [redacted] oast Survey by discussions essentially course, confining each discussion to s n source), with the following results: ϵ_δ de ting r of a single declination,

y.	ϵ_δ	ϵ_δ^2
Groombridge [redacted]	1''.5	2.25
B.A.C. on auth [redacted] of Br [redacted] iazzi, and Taylor..	1 .0	1.00
The same with additional modern authority.....	0 .85	0.72
Twelve Year (Gr.) Catalogue, with less than six observations..	0 .6	0.36
Nautical Almanac, or Twelve Year Catalogue, with six or more observations........................	0 .5	0.25

234. *Combination of the observations by weights.*—Let ϵ_δ and $\epsilon_{\delta'}$ denote the probable errors of the declinations of the stars of a pair on which there are n observations; then the probable error of $\frac{1}{2}(\delta + \delta')$ is

$$E_\delta = \tfrac{1}{2}\sqrt{(\epsilon_\delta^2 + \epsilon_{\delta'}^2)}$$

and that of the latitude is

$$E_\phi = \sqrt{E_\delta^2 + \frac{e^2}{n}}$$

The weight p of an observation is reciprocally proportional to E_ϕ^2; or, since the *scale* of weights is arbitrary, we may take

$$p = \frac{1}{4E_\phi^2}$$

$$= \frac{1}{\epsilon_\delta^2 + \epsilon_{\delta'}^2 + \frac{4e^2}{n}} \qquad (243)$$

Adopting the Coast Survey value $e = 0''.50$, we have, therefore,

$$p = \frac{1}{\varepsilon_\delta^2 + \varepsilon_{\delta'}^2 + \frac{1}{n}} \tag{244}$$

By this formula, the weight *unity* would be assigned to a value of the latitude found by a *single observation* of a pair of stars when the declinations were perfectly exact, or to a value found by *two observations* on a pair of Nautical Almanac stars.

The stars observed at Roslyn were really taken from various authorities, although, for the sake of illustration, we have discussed the probable error of their declinations as we should have done if but a single authority had been used. Let us now find the final value of the latitude from all the observations, having regard to their weights as determined by this formula. In the following table the values of ε_δ^2 are given according to the authorities from which their declinations are taken, as stated in the table at the end of the preceding article.

No. of pair.	ε_δ^2	$\varepsilon_{\delta'}^2$	n	p	ϕ	$p\phi$	$v = \phi \backsim \phi_0$	pvv
1	1.00	0.25	1	0.44	24''.78	10''.90	0''.76	0.25
2	0.25	0.25	1	0.67	25 .05	16 .78	0 .49	0.16
3	0.36	0.36	2	0.82	24 .83	20 .36	0 .71	0.41
4	0.36	1.00	3	0.59	26 .20	15 .46	0 .66	0.26
5	1.00	1.00	3	0.43	25 .91	11 .14	0 .37	0.06
[6]*	1.00	1.00	2		[22 .73]			
7	1.00	0.25	6	0.70	25 .93	18 .15	0 .39	0.11
8	0.36	1.00	6	0.65	25 .18	16 .37	0 .36	0.08
9	0.36	0.36	5	1.09	25 .89	28 .22	0 .35	0.13
10	0.25	0.25	4	1.33	25 .79	34 .30	0 .25	0.08
11	1.00	2.25	4	0.29	24 .53	7 .11	1 .01	0.30
12	0.36	1.00	3	0.59	25 .15	14 .84	0 .39	0.09
13	1.00	0.25	4	0.67	25 .22	16 .90	0 .32	0.07
14	1.00	0.25	4	0.67	24 .84	16 .64	0 .70	0.33
15	1.00	0.25	4	0.67	25 .36	16 .99	0 .18	0.02
16	1.00	0.36	4	0.62	26 .02	16 .13	0 .48	0.14
17	1.00	1.00	4	0.44	25 .42	11 .18	0 .12	0.01
18	1.00	1.00	4	0.44	26 .08	11 .48	0 .54	0.13
19	1.00	0.25	4	0.67	25 .72	17 .23	0 .18	0.02
20	0.25	0.25	4	1.33	25 .70	34 .18	0 .16	0.03
21	0.25	0.25	3	1.20	25 .93	31 .12	0 .39	0.18
$m = 20$				$[p] = 14.31$		$[p\phi] = 365\ .47$		$[pvv] = 2.86$

$$\varphi_0 = \frac{[p\varphi]}{[p]} = 25''.54$$

$$E_{\phi_0} = 0.6745 \sqrt{\frac{[pvv]}{(m-1)[p]}} = 0''.07$$

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* The result by the 6th pair of stars is rejected by *Peirce's Criterion* (see Appendix).

Hence, the final result from these observations is

$$\text{Lat. of Roslyn} = 37^\circ\ 14'\ 25''.54 \pm 0''.07$$

235. *To determine the value of a division of the level.*—It will generally be most convenient to find the value of the divisions of the level by the aid of the micrometer. It would seem, therefore, most natural to begin by determining the value of the micrometer screw; but it will be seen in the next article that in the investigation of the screw we must know the value of a division of the level *in parts of a revolution of the screw.* This value, then, we are here to find, and afterwards, when the micrometer value has been determined, we can convert it into arc.

Let the telescope be directed towards a well-defined terrestrial mark, or, which is better, to the cross-thread of a collimating telescope. Let the level be set to an extreme reading L. Bisect the mark by the micrometer, and let the reading be M. Now move the telescope and level together [by the tangent screw F, Plate XIII.] until the bubble gives a reading L' near the other extreme. Bisect the mark again by the micrometer, and let the reading be M'. Then the value d of a division of the level in terms of the micrometer will be

$$d = \frac{M - M'}{L' - L} \tag{245}$$

and if R is the value (in seconds of arc) of a revolution of the micrometer, we shall afterwards find the value D of a division of the level in seconds of arc, by the formula

$$D = Rd \tag{246}$$

Instead of a terrestrial mark we may use a circumpolar star at its culmination; for we can apply to each observation the reduction to the meridian (237), so that each will be referred to the fixed point in which the star culminates. In this method, however, we are exposed to errors arising from transient irregularities in the refraction, and also to any error arising from inclination of the micrometer thread. The latter error, however, may be avoided by revolving the instrument in azimuth, so as to observe the star always in the middle of the field, and then we should use the reduction to the meridian for circummeridian altitudes (238).

Example.—The following are some of the observations for etermining the value of a division of the level of a zenith teleope, taken by Mr. G. W. **Dean**, of the U. S. Coast Survey, at e Roslyn Station, Virginia, June 30, 1852, the telescope being rected upon a fixed terrestrial point.

Temp.	No. of obs.	Readings of			Difference.		d	v	v^2
		Micr.	Level.		Micr.	Level.			
			N.	S.					
		div.	div.	div.					
90°	1	1941	54.0	11.4					
		2106	11.2	53.9	165	42.65	3.869	0.176	.0310
	2	2111	56.1	8.2					
		2296	10.5	54.0	185	45.70	4.048	.003	.0000
	3	2305	55.5	8.8					
		2506	5.2	59.0	201	50.25	4.000	.045	.0020
	4	2517	55.0	9.1					
		2704	8.8	55.2	187	46.15	4.052	.007	.0000
	5	2709	59.0	4.8					
		2915	9.0	54.7	206	49.95	4.124	.079	.0062
	6	2919	56.0	7.8					
		3115	9.2	54.4	196	46.70	4.197	.152	.0231
	7	1176	58.2	5.8					
		1390	5.5	58.5	214	52.70	4.061	.016	.0003
	8	1396	59.6	5.0					
		1617	4.5	60.1	221	55.10	4.011	.034	.0012

Mean $d = 4.045$ Sum $= .0638$

The column of v gives the difference between each observed alue of d and the mean. From the sum of the squares of v we nd the probable error of the mean to be

$$= 0.6745\sqrt{\frac{0.0638}{8 \times 7}} = 0.023$$

The value of d is here expressed in divisions of the micrometer hread which represent hundredths of a revolution. Hence we ave, in parts of a revolution R of the micrometer, the value of division of the level,

$$D = 0.04045\,R \pm 0.00023\,R$$

From twenty-one observations of the same kind, the value found was

$$D = 0.03985\,R \pm 0.00013\,R$$

236. *To find the value of a revolution of the micrometer.*—The most convenient method with this instrument, as it avoids displacing the micrometer, is by transits of a circumpolar star near its eastern or western elongation (Art. 45). We first find the hour angle and zenith distance of the star at the elongation by the formulæ

$$\cos t_0 = \cot \delta \tan \varphi \qquad \cos z_0 = \operatorname{cosec} \delta \sin \varphi$$

and then, α being the star's right ascension, ΔT the correction of the chronometer, we find the chronometer time of the elongation by

$$T_0 = \alpha \pm t_0 - \Delta T \quad \begin{bmatrix} + \text{ western elong.} \\ - \text{ eastern } \quad `` \end{bmatrix}$$

Set the telescope for the zenith distance z_0, direct it upon the star some 20^m or 30^m before the time of elongation, bringing the star near the middle vertical thread, and clamp the instrument. Set the micrometer thread at any reading a little in advance of the star, and note the transit by the chronometer. Then advance the thread to a new reading, and again observe the transit, and so on until the star has been observed through the whole field or through the whole range of the micrometer screw. The repeated manipulation of the screw may slightly disturb the direction of the telescope, but the only change which can affect the determination of R will be shown by the level, which, therefore, must also be frequently observed during the transits. Of course, the relation of the level to the telescope must not be changed during the observations. Now, z_0 denoting as above the zenith distance of the star at the time T_0, and M_0 the corresponding reading of the micrometer when the level reading is zero, z the zenith distance at the time T of an observed transit when the micrometer reading is M and the level reading is L, we have (neglecting for the present the refraction)

$$z = z_0 + (M_0 - M)R - LD$$

or, since we as yet know the value of a level division only in parts of R,

$$z = z_0 + (M_0 - M)\,R - L\,Rd$$

In like manner, for another observation,

$$z' = z_0 + (M_0 - M')\,R - L'Rd$$

whence

$$R = \frac{(z - z_0) - (z' - z_0)}{M' - M + (L' - L)d} \tag{247}$$

The quantity $z - z_0$ may be computed (as we have shown in Art. 45) by the formula

$$\sin(z - z_0) = \pm \sin(T - T_0)\cos\delta$$

where the lower sign is to be used for the eastern elongation; or

$$z - z_0 = \pm \sin(T - T_0)\,\frac{\cos\delta}{\sin 1''} \tag{248}$$

The value of R thus found is corrected for refraction by subtracting from it the quantity $R\Delta r$, in which Δr = the change of refraction at the zenith distance z_0 for 1′ of zenith distance, and R is expressed in minutes.*

Example.—Observations of *Polaris* at its eastern elongation were taken June 30, 1852, at the Roslyn Station (Va.) of the U.S. Coast Survey, to determine the value of the micrometer of the same zenith telescope as was used in the example of the preceding articles.

To prepare for the observation, we have

$\varphi = 37° 14' 25''$

$\delta = 88° 30' 56''$

Hence, $z_0 = 52° 44' 42''$

	h	m	s
$\alpha =$	1	5	36.8
$t =$	5	55	29.1
Sid. time of elongation =	19	10	7.7
Chronometer fast,		24	46.8
$T_0 =$	19	34	54.5

The micrometer thread was set at every half revolution, and

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* The values of both R and D might be found at the same time from these observations. For by varying the level reading at the different observations (by means of the tangent screw F), we shall have from the observations, taken suitably in pairs, equations of condition of the form

$$z - z' = (M' - M)\,R + (L' - L)\,D$$

from which both R and D may be found. In this method $z - z'$ must be the apparent difference of zenith distance affected by the differential refraction.

59 transits were observed. I extract only those taken on the even whole revolutions, to illustrate the method.

Temp.	No. of obs.	Micr. M	Level.		L	T	$T - T_0$	$z - z_0$
			N.	S.				
		R.	div.	div.	div.			
77°	1	6	42.2	44.8	− 1.30	$19^h 11^m 39^s.0$	$- 23^m 15^s.5$	+ 541″.33
	2	8	"	"	"	15 14.2	19 40.3	458 .10
	3	10	"	"	"	18 46.8	16 7.7	375 .73
	4	12	"	"	"	22 23.4	12 31.1	291 .71
	5	14	42.5	44.2	− 0.85	25 58.8	8 55.7	208 .12
	6	16	"	"	"	29 29.4	5 25.1	126 .30
	7	18	"	"	"	33 4.4	− 1 50.1	+ 42 .77
	8	20	42.6	44.2	− 0.80	36 36.4	+ 1 41.9	− 39 .61
	9	22	"	"	"	40 11.6	5 17.1	123 .20
	10	24	42.7	44.2	− 0.75	43 43.3	8 48.8	205 .43
	11	26	"	"	"	47 15.0	12 20.5	287 .62
	12	28	41.9	45.1	− 1.60	50 46.7	15 52.2	369 .72
	13	30	"	"	"	54 19.3	19 24.8	452 .08
76	14	32	"	"	"	57 52.8	22 58.3	534 .70

We compare the 1st observation with the 8th, the 2d with the 9th, &c., and in each case we have $M' - M = 14$ Rev., or, taking $d = 0.04$, as found on p. 359, we have for the 1st and 8th observation $(L' - L)d = + 0.020$ revolutions of the micrometer; and hence, denoting the divisor in (247) by a, we obtain

$$a = M' - M + (L' - L)d = 14.020$$

Proceeding thus for each pair of transits, we have—

Obs.	a	$z - z'$	R	v	v^2
1 and 8	14.020	580″.94	41″.436	+ 0″.042	0.0018
2 " 9	14.020	581 .30	.462	+ 0 .068	.0046
3 " 10	14.022	581 .16	.446	+ 0 .052	.0027
4 " 11	14.022	579 .33	.316	− 0 .078	.0060
5 " 12	13.970	577 .84	.363	− 0 .031	.0010
6 " 13	13.970	578 .38	.402	+ 0 .008	.0001
7 " 14	13.970	577 .47	.336	− 0 .058	.0034
		Mean =	41 .394	Sum =	.0196

$$\text{Prob. error} = \frac{2}{3}\sqrt{\frac{.0196}{6 \times 7}} = 0''.014$$

The change of refraction for 1′ of zenith distance is, for $z_0 =$ 2° 45′, $\Delta r = 0''.046$, and hence the correction of the above mean $- 0''.046 \times \frac{41.4}{60} = - 0''.032$. These observations, therefore, ive us the result

$$R = 41''.362 \pm 0''.014$$

The final value, as found from all the observations on several ights, was

$$R = 41''.400 \pm 0''.011$$

d from this we find the value of a division of the level of this strument to be

$$D = 1''.65 \pm 0''.005$$

hich are the values employed above in reducing the observations r latitude at Roslyn.

237. A more thorough method of treating the preceding observations is the following. We have for each observed transit

$$z - z_0 = (M_0 - M) R - L R d$$

here M_0 is the unknown reading corresponding to z_0. Let us sume an approximate value for M_0, denoting it by M_1, and put $M_0 = M_1 + x$. Also let R_1 be an assumed approximate value of R, and put $R = R_1 + y$. Then

$$z - z_0 = (M_1 - M + x)(R_1 + y) - L R_1 d$$

here, on account of the small values of L, we can use R_1 instead f R in the last term. Then, neglecting the product xy as insenble when M_1 and R_1 are properly assumed, and putting

$$n = z - z_0 - (M_1 - M) R_1 + L R_1 d \qquad (249)$$

e have from each observation the equation of condition

$$R_1 x + (M_1 - M) y = n \qquad (250)$$

d from all these equations x and y can be found by the method f least squares.

Thus, in the above example, if we assume $M_1 = 19.0$, $R_1 = 41''.4$, which are easily seen from the observations to be near approximations, we have the following equations:

$$\begin{array}{ll}
41.4x + 13y = +0''.98 & 41.4x - y = +0''.47 \\
41.4x + 11y = +0\ .55 & 41.4x - 3y = -0\ .32 \\
41.4x + 9y = +0\ .98 & 41.4x - 5y = +0\ .33 \\
41.4x + 7y = -0\ .24 & 41.4x - 7y = +0\ .94 \\
41.4x + 5y = -0\ .29 & 41.4x - 9y = +0\ .23 \\
41.4x + 3y = +0\ .69 & 41.4x - 11y = +0\ .67 \\
41.4x + y = -0\ .04 & 41.4x - 13y = +0\ .85
\end{array}$$

from which we form the normal equations

$$23995.44x = +240.12$$

$$910y = -\ \ 1.72$$

whence

$$x = +0.01 \qquad y = -0.002$$

$$M_0 = 19.01 \qquad R = 41.398$$

If we substitute the values of x and y in the equations of condition, we shall find the sum of the squares of the residual to be $= 2.956$, and hence the mean error of a single observation i

$$= \sqrt{\frac{2.956}{14-2}} = 0''.496$$

and consequently the probable error of y, the weight of whic is its coefficient $(= 910)$ in the final equation, will be

$$= \frac{2}{3} \cdot \frac{0''.496}{\sqrt{910}} = 0''.011$$

Applying to the above value of R the correction for refractio as before, we have the final result by this method,

$$R = 41''.366 \pm 0.''011$$

The smaller probable error here found shows that the observations are better satisfied by the value of R found by the metho of least squares.

EXTRA-MERIDIAN OBSERVATIONS FOR LATITUDE WITH THE ZENITH TELESCOPE.

238. It has been seen above that, although the probable erro of observation with the zenith telescope is very small, the greate

le error of the declinations employed, when the observa-
re restricted to the meridian, renders it necessary to greatly
ly the number of pairs of stars observed. But if we are
g to observe one of the stars at some distance from the
ian, we can generally find a pair of fundamental stars, or
rom the most reliable catalogues, which can be observed
same zenith distance within a sufficiently brief interval of
o exclude the probability of sensible changes in the state
instrument; and by moderate attention to the determina-
f the time the probable error of observation will be very
increased, while the number of observations necessary to
to the desired degree of precision will be greatly reduced.
y not be superfluous, therefore, to deduce here the necessary
læ for this purpose.

δ and δ' be the declinations of two stars, the first of which
erved out of the meridian at the zenith distance z and hour
t, and the second on the meridian at the zenith distance z',
is very nearly equal to z. We have

$$\cos z = \cos (\varphi - \delta) - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$
$$z' = \varphi - \delta'$$

econd equation gives

$$z = \varphi - \delta' + z - z'$$

substituted in the first equation, gives

$$- \tfrac{1}{2} (\delta + \delta') - \tfrac{1}{2} (z' - z)] \sin \tfrac{1}{2} (\delta - \delta' + z - z') = \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

g then

$$\sin \gamma = \frac{\cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t}{\sin \tfrac{1}{2} (\delta - \delta' + z - z')} \qquad (251)$$

all have

$$\varphi = \tfrac{1}{2} (\delta + \delta') + \tfrac{1}{2} (z' - z) + \gamma \qquad (252)$$

quantity $z' - z$ will be given by the micrometer and level,
ely as in the case of meridian observations. The value of
l always be known with sufficient accuracy for the compu-
of γ.

e effect of an error in t upon γ, and consequently upon φ,
be computed by the formula

$$\Delta \gamma = (15 \Delta t) \frac{\tan \gamma}{\sin \tfrac{1}{2} t}$$

To prepare for the observation, put $\zeta = \varphi_1 - \delta'$, (or $\delta' - \varphi$ φ_1 being an assumed approximate value of φ, and set the instr ment at the zenith distance ζ for the observation of both star The hour angle at which the star out of the meridian is to b observed will be found by the formula

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\sin \frac{1}{2}(\zeta + \varphi_1 - \delta) \sin \frac{1}{2}(\zeta - \varphi_1 + \delta)}{\cos \varphi \cos \delta}\right)}$$

or rather,

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\sin [\frac{1}{2}(\delta' + \delta) - \varphi_1] \sin \frac{1}{2}(\delta' - \delta)}{\cos \varphi \cos \delta}\right)}$$

Then the sidereal time of the observation of this star may b either $\alpha + t$ or $\alpha - t$, α being the right ascension; and it ma often be convenient to observe the star at each of these times.

It will probably be most expedient to observe one of the star in the meridian; but, if both are observed out of the meridian we can find the latitude by the method of Vol. I. Art. 186.

239. The zenith telescope may be used with advantage i measuring any small difference of zenith distance. Its applicatio in finding the longitude from equal zenith distances of the moon' limb and a neighboring star is given in Vol. I. Art. 245. Th correction of the method there given for a small difference o the zenith distances of the moon and star, as found by th micrometer, is obvious.

240. We may determine both time and latitude with the zenit telescope, by observing a number of stars at the same altitud and combining them by the method of least squares. See Vo I. Art. 189.

ADAPTATION OF THE PORTABLE TRANSIT INSTRUMENT AS A ZENIT TELESCOPE.

241. Prof. C. S. LYMAN, of Yale College, has shown* that th transit instrument may be successfully used as a substitute fo the zenith telescope in the application of TALCOTT's method o finding the latitude by meridian observations. Indeed, it i evident that, if the level usually attached to the finding circle i made of the same delicacy as that applied to zenith telescopes, an a micrometer is added to the telescope, that method may be carrie out precisely in the same manner as with the zenith telescop

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* Am. Journal of Science and Arts, Vol. XXX. p. 52.

The different method of reversing the instrument by lifting it from its Vs instead of revolving directly about a vertical axis, does not in any way affect the principle, the essential condition of TALCOTT's method being always observed, namely, that the relation of the level and the telescope is to be absolutely the same at the observations of both stars of the pair.

CHAPTER IX.

THE EQUATORIAL TELESCOPE.

242. THE equatorial telescope is mounted with two axes of motion at right angles to each other, one of which is parallel to the axis of the earth. Of the various modes which have been employed for mounting the instrument according to these conditions, that which is now universally adopted is the one contrived by FRAUNHOFER and known by his name.

Plate XIV.* is a representation of the great FRAUNHOFER equatorial of the Pulkowa Observatory, constructed by MERZ and MAHLER. The lens has a clear aperture of 15 inches, with a focal length of 22.55 feet. The pier P is of stone (in smaller instruments a wooden stand is frequently used, resting on three feet). The upper face of the pier makes an angle with the horizon equal to the latitude of the place; secured to this face is a metallic bed, which supports at two points the *polar* or *hour axis* H of the instrument. This axis, being in the plane of the meridian, and making an angle with the horizon equal to the latitude of the place, is parallel to the earth's axis, and, consesequently, is directed towards the poles of the heavens. Permanently attached to the hour axis, and at right angles to it, is a metallic tube, DD, in which the *declination axis* revolves. The telescope is firmly attached to one extremity of this declination axis, and at right angles to it, the point of the tube at which it is attached being somewhat nearer to the eye end than to the object end.

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* Reduced from the drawing in the *Description de l'observatoire central* of STRUVE.

It is evident that as the instrument revolves upon the hour axis the declination axis remains in the plane of the celestial equator, and, consequently, the telescope, as it revolves upon the declination axis, always describes secondaries to the celestial equator, or declination circles. The declination of the point of the heavens towards which the telescope is at any time directed may, therefore, be indicated by the graduated *declination circle* $\delta\delta$, which is read by two opposite verniers. The hour angle of this point is at the same time shown by the graduated *hour circle* t, which is also read by two opposite verniers.

The great advantage of this mode of mounting the telescope is that we can follow a star in its diurnal motion by revolving the instrument upon the hour axis alone, the declination circle being clamped at the reading corresponding to the star's declination. Further, the star's motion being uniform, we can cause the instrument to follow it automatically by means of a clock f, which, by a train, turns an endless screw acting upon the circumference of the hour circle. The observer is thus left free either to make a careful examination of the physical appearance of the objects in the field, or to measure their relative positions with the micrometer m of the telescope.

It is important that all the parts of the instrument be so counterpoised that the telescope will be in equilibrium in all positions, and possess the greatest freedom of movement upon either axis. This is effected in the FRAUNHOFER arrangement in the most perfect manner. The equilibrium of the telescope with respect to the hour axis is produced by the counterpoises W, W, X, and Y, of which W, W are fixed cylindrical masses, but Y is adjustable, so that the equilibrium may be finally regulated with the utmost nicety. The weights X (of which there are two, one on each side of the declination axis) are attached to the extremities of levers whose fulcrums are at $\varkappa$. The opposite extremities of the levers seize upon a circular collar at K, in which there are four friction rollers. The weights X thus not only contribute to the equilibrium, but also reduce the friction of the declination axis. The centre of gravity of the telescope tube is not in the prolongation of the declination axis, but nearer to the object glass; its equilibrium with respect to the declination axis is produced by counterpoises a (one on each side of the tube) at the end of levers abc. Each of these levers consists of two conical tubes attached to a cube at b, which moves upon two axes; and ir extremities *c* seize upon a collar around the tube. The tremity *a*, at which the weight is placed, is free, and the weight n be adjusted by sliding upon the lever. In consequence of e double axis of each lever at *b*, the counterpoises act in all sitions of the telescope, and not only balance the tube, but pre-nt in a degree the flexure of the object end which would result m its weight, increased as it is by the great weight of the ject glass itself. The centre of gravity of the telescope and l its counterpoises is now in the hour axis at a point a little ove its upper journal; the result is a downward pressure upon is journal, and an upward pressure upon the lower journal. he weight *ω* at one extremity of a bent lever reduces the fric-n upon the upper journal by producing an opposite pressure *e* at right angles to the axis, two friction rollers upon the tremity *e* being thus pressed against the axis. The remaining all upward pressure of the inferior extremity of the axis is duced by a spring which presses two friction rollers against the is at *g*.

The weight of the Pulkowa telescope (including all the parts hich move, namely, the axes and tube with its counterpoises) very nearly 7000 pounds; and yet, with this admirable system counterpoises, it moves upon either axis with almost as much se as a small portable instrument. Without this perfect equi-rium and reduced friction, it would have been very difficult to oduce a regular automatic movement of the instrument by the iving clock. As this clock is required to produce a *continuous* gular movement, it is not regulated by an oscillating pendulum, t by the friction of centrifugal balls against the interior of a nical box *d*. The rate of movement is regulated by raising or pressing the pivot of this conical pendulum, which, in conse-ence of the conical form of the box, changes the degree of iction of the balls against its interior surface. The rate may us be adapted not only to the motion of a fixed star, but to that f the moon, or sun, or any planet, all of which have different tes of motion. In our own country, BOND'S *Spring Governor* as been successfully applied to produce the equable motion of quatorial telescopes.

A *finder F* is attached to the principal telescope (Art. 16).

The field of the telescope is illuminated by a lamp *q*, the light of which is reflected towards the reticule by a small mirror within the tube. The direct illumination of the threads, which is required when very faint objects are to be observed, is effect by two small lamps suspended at *n* and *n*. (See Transit Instr ment, p. 134).

The micrometer is provided with a position circle (Art. 49).

243. Any point of the heavens may be observed with th equatorial instrument in two different positions of its declin tion axis. For example, if the declination axis is at right angle to the plane of the meridian,—that is, horizontal,—the telescop will describe the plane of the meridian; and this, whether th *circle end* of the declination axis is east or west; and, in general the same declination circle of the heavens may be described by th telescope with this circle end of the axis on either side. Thes two positions are to be distinguished in the use of the instrument Let us suppose the declination axis to be produced through th *circle end* to the celestial sphere. The point in which it meet the sphere may be called the *pole* of the declination circle. I the hour angle of this pole is 90° greater than the hour angle o a star observed in the telescope, the circle is said to *precede* th telescope; if the hour angle of this pole is 90° less than that o the star, the circle is said to *follow* the telescope. Thus, for star on the meridian (at its upper culmination) the circle *precede* when it is *west* and *follows* when it is *east* of the meridian.

GENERAL THEORY OF THE EQUATORIAL INSTRUMENT.

244. Let us first consider the instrument in the most genera manner, that is, without supposing its hour axis to be eve approximately adjusted to the pole of the heavens. That poin of the celestial sphere towards which the hour axis is actuall directed will be called the *pole of the instrument*, or the pole of it hour axis, and that point in which the declination axis produced on the side of the declination circle meets the sphere will b called the pole of this axis or circle.

The instrument is designed to give, by means of its two circles the hour angle and declination of a star observed in the sigh line of the telescope. If the sight line were perpendicul to the declination axis, and if this axis were perpendicular the hour axis, the readings of the circles would give at once (merely correcting them for any index error) the hour angle a ıation *referred to the meridian and pole of the instrument.* The ;ions from perpendicularity being always very small in a constructed instrument, approximate formulæ will fully ; to reduce given readings to the proper values referred to ›le of the instrument. But an equatorial instrument may ;imes be used in a place for which it was not intended, and, g no adjustment by which the angle which its hour axis s with the horizon can be greatly changed, the pole of the ment may be so far from the celestial pole that the reduc- ›f the hour angles and declinations from their *instrumental* ›ir *true* values (referred to the celestial pole) will require the f rigorous formulæ. In order to provide for such a case, I first consider the method of deducing the instrumental ities by approximate but sufficiently exact formulæ; then the rigorous formulæ for reducing these to the celestial and finally give the approximate formulæ, most frequently red, for the case in which the deviation of the hour axis the celestial pole is very small. As some *flexure* of the ıation axis and of the telescope is always to be expected in strument of this kind, I shall include its effect in the ılæ.

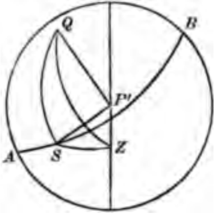


Fig. 52.

›. *To find the instrumental declination and hour angle of an ed point.*—Let the figure be a projection e celestial sphere upon the plane of the ›r *of the instrument;* P' its pole; Z the ı of the observer: then $P'Z$ may be l the *meridian of the instrument.*

; Q be the pole of the declination axis of nstrument. While the instrument re- s upon the hour axis, the point Q will ibe a circle of which P' is the pole, and which would be a circle if the axes were at right angles to each other, in ı case we should have $P'Q = 90°$. But we shall assume there is a deviation from this condition, and suppose the ›'Q to be $= 90° - i$: so that i will express the declination ıe point Q referred to the equator of the instrument.

›t us next suppose the declination axis to remain fixed while telescope revolves upon this axis and its sight line is brought n a star S. As the telescope revolves, the sight line (which may here suppose to be determined by a simple cross thread), describes a circle in the heavens of which Q is the pole which will be a great circle if this sight line is perpendicul the declination axis, and a small circle, ASB, in any other Let us suppose the polar distance of this small circle, or Q be $90° - c$: so that c will denote the collimation constant given thread.

The revolution of the instrument upon the hour axis is meas by the hour circle. When Q is 90° west of the meridian telescope should be in the meridian, and the reading of the circle, consequently, zero; but let us suppose the reading is $- x$. When Q is in the meridian and above the pole, the rea will be $- x - 90°$. If, then, for the actual position wher star is observed at S the reading is t, we have the angle Z $= t + x + 90°$.

Let the instrumental hour angle $ZP'S = t'$. Then we the angle $SP'Q = ZP'Q - ZP'S = t + x - t' + 90°$; and s from the construction of the instrument, this angle differs little from 90°, the quantity $t + x - t'$ will be very small.

As the telescope revolves upon the declination axis an sight line describes the circle ASB, the reading of the dec tion circle will vary directly with the angle $P'QS$, since Q i pole of this circle. If we denote the reading of the declin circle when the arc QS coincides with QP' by $90° - \Delta d$, an actual reading for the star at S by d, we shall have the a $P'QS = 90° - \Delta d - d$, provided the readings increase wit star's declination, as we here suppose.

Finally, let the instrumental declination be d'; that is $P'S = 90° - d'$.

We have then in the triangle $QP'S$ the given parts

$$P'Q = 90° - i \qquad QS = 90° - c$$
$$P'QS = 90° - (d + \Delta d)$$

and in order to determine t' and d' we are to find

$$SP'Q = 90° - (t' - t - x)$$
$$P'S = 90° - d'$$

From this triangle we obtain the general equations

$$\sin d' = \sin i \sin c + \cos i \cos c \sin (d + \Delta d$$
$$\cos d' \sin (t' - t - x) = \cos i \sin c - \sin i \cos c \sin (d + \Delta d$$
$$\cos d' \cos (t' - t - x) = \cos c \cos (d + \Delta d$$

But, as i and c are supposed to be so small that their squares and products are insensible, these equations give

$$\sin d' = \sin (d + \Delta d)$$
$$\cos d' = \cos (d + \Delta d)$$
$$(t' - t - x) \cos d' = c - i \sin (d + \Delta d),$$

whence

$$\left.\begin{aligned} d' &= d + \Delta d \\ t' &= t + x + c \sec d' - i \tan d' \end{aligned}\right\} \quad (253)$$

246. *Flexure.*—The flexure of the hour axis may be supposed to be altogether insensible, since the centre of gravity of the whole instrument falls very near to the upper journal of this axis, and the pressure at this point is relieved by a counterpoise.

The flexure of the declination axis, being assumed to result solely from the weight, changes the zenith distance of the point Q. Denoting the zenith distance of Q by ζ and the increased zenith distance by $\zeta + d\zeta$, we shall assume the flexure to be proportional to $\sin \zeta$ (Art. 204), and, therefore, put

$$d\zeta = \epsilon \sin \zeta$$

in which ϵ is the maximum of flexure of the declination axis corresponding to $\zeta = 90°$.

The flexure of the telescope changes the zenith distance ZS, so that, putting $ZS = \zeta'$, we can express this flexure by

$$d\zeta' = e \sin \zeta'$$

in which e is the maximum of flexure of the tube corresponding to $\zeta' = 90°$.

The flexure of the declination axis changes the arc $P'Q = 90° - i$, and the angle $ZP'Q = t + x + 90°$; but these changes (the flexure being supposed extremely small) evidently produce no sensible effect upon the declination d'. The flexure of the telescope, however, changes the arc $P'S = 90° - d'$, and thus also d'. Treating the changes as differentials, we have

$$d \,.\, P'S = d\,(90° - d') = d\zeta' . \cos P'SZ$$

If we denote the zenith distance of P' by $90° - \varphi_1$ (or let φ_1 be the observer's latitude referred to the equator of the instrument), the triangle $P'SZ$ gives

$$\cos P'SZ = \frac{\sin \varphi_1 \cos d' - \cos \varphi_1 \sin d' \cos t'}{\sin \zeta'}$$

and hence

$$dd' = -e\,(\sin\varphi_1 \cos d' - \cos\varphi_1 \sin d' \cos t') \qquad (m)$$

Again, we have

$$d\,.\,P'Q = d\,(90^\circ - i) = d\zeta \cos P'QZ$$
$$d\,.\,ZP'Q = dt = d\zeta \cdot \frac{\sin P'QZ}{\sin P'Q}$$

in which we may put $\sin P'Q = \cos i = 1$. Substituting also the values

$$\cos P'QZ = \frac{\sin\varphi_1 - \sin i \cos\zeta}{\cos i \sin\zeta}$$
$$\sin P'QZ = \frac{\cos(t+x)\cos\varphi_1}{\sin\zeta}$$

and neglecting the product of $d\zeta$ and i as insensible, we find

$$\left.\begin{aligned} di &= -\,\varepsilon \sin\varphi_1 \\ dt &= \,\varepsilon \cos\varphi_1 \cos(t+x) \end{aligned}\right\} \qquad (n)$$

Finally, the flexure of the telescope changes the arc $QS = 90^\circ - c$, and we have

$$d\,.\,QS = d\,(90^\circ - c) = d\zeta'\,.\cos ZSQ$$

in which

$$\cos ZSQ = \frac{\cos\zeta - \sin c \cos\zeta'}{\cos c \sin\zeta'}$$

Neglecting terms of the second order, therefore,

$$dc = -\,e \cos\zeta$$

in which we have

$$\cos\zeta = \sin i \sin\varphi_1 - \cos i \cos\varphi_1 \sin(t+x)$$

and in this we may put t' for $t + x$. Hence, again neglecting terms of the second order,

$$dc = e \cos\varphi_1 \sin t' \qquad (p)$$

By the formulæ for t' (253), we have

$$dt' = dt + dc \sec d' - di \tan d'$$

and hence, by (n) and (p),

$$dt' = \varepsilon\,(\sin\varphi_1 \tan d' + \cos\varphi_1 \cos t') + e \cos\varphi_1 \sec d' \sin t' \qquad (q)$$

Hence, applying the corrections (m) and (q) to d' and t' (253), the complete formulæ, including the effect of flexure, are*

$$\left.\begin{aligned} d' &= d + \Delta d - e\,(\sin\varphi_1 \cos d' - \cos\varphi_1 \sin d' \cos t') \\ t' &= t + x + c \sec d' - i \tan d' \\ &\quad + \varepsilon\,(\sin\varphi_1 \tan d' + \cos\varphi_1 \cos t') + e \cos\varphi_1 \sec d' \sin t' \end{aligned}\right\} \quad (254)$$

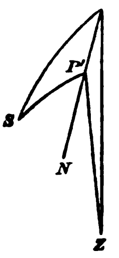
247. *To reduce the instrumental declination and hour angle* (d', t') *to the celestial declination and hour angle* (δ, τ).—Let PZ be the true meridian, P the celestial pole, P' the pole of the instrument, S the observed star. Let γ and ϑ denote the polar distance and hour angle of P'; that is, let

$$\gamma = PP' \qquad \vartheta = ZPP'$$

and, producing PP', let

$$\vartheta' = ZP'N = 180^\circ - ZP'P$$

Fig. 53.



The instrument gives, by the aid of (254), the values of $d' = 90^\circ - P'S$, $t' = ZP'S$, and we are to find $\delta = 90^\circ - PS$ and $\tau = ZPS$. The triangle $PP'S$, in which $PP'S = 180^\circ - (t' - \vartheta')$ and $P'PS = \tau - \vartheta$, gives

$$\left.\begin{aligned} \sin\delta &= \cos\gamma \sin d' - \sin\gamma \cos d' \cos(t' - \vartheta') \\ \cos\delta \cos(\tau - \vartheta) &= \sin\gamma \sin d' + \cos\gamma \cos d' \cos(t' - \vartheta') \\ \cos\delta \sin(\tau - \vartheta) &= \cos d' \sin(t' - \vartheta') \end{aligned}\right\} \quad (255)$$

which will determine δ and τ from d' and t' when the instrumental constants γ, ϑ, and ϑ' are known.

Putting $90^\circ - \varphi = PZ$, the relation between φ_1, ϑ', φ, ϑ, and γ is found from the triangle $PP'Z$, which gives

$$\left.\begin{aligned} \sin\varphi_1 &= \cos\gamma \sin\varphi + \sin\gamma \cos\varphi \cos\vartheta \\ \cos\varphi_1 \cos\vartheta' &= -\sin\gamma \sin\varphi + \cos\gamma \cos\varphi \cos\vartheta \\ \cos\varphi_1 \sin\vartheta' &= \cos\varphi \sin\vartheta \end{aligned}\right\} \quad (256)$$

248. In the preceding discussion I have not distinguished between the case in which the declination circle *precedes* and that in which it *follows* the telescope (Art. 243). The formulæ, nevertheless, will apply to either case, provided we reckon declinations over 90° when they require it. By Fig. 52, in which for a star at S the declination circle *precedes*, we see that when

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* These formulæ are essentially the same as BESSEL'S. See his *Astron. Untersuchungen*, Vol. I. p. 7.

the telescope is revolved from S towards B and passes beyond the pole, we shall have declinations exceeding 90° if we wish to employ the same formulæ as have been found for this position; but for these points beyond the pole the declination circle *follows* the telescope. The declination in that case, reckoned in the usual manner, will be $180° - d'$, and the hour angle will be $180° + t'$. We may, therefore, employ these formulæ in their present form in all cases, but when d' falls between 90° and 270° we must finally take $180° - d'$ and $180° + t'$ as the proper instrumental declination and hour angle. (See also Transit Instrument, Art. 128.)

If, however, we wish to distinguish the cases in the formulæ themselves, we shall have, *when the circle precedes*, the readings of the circle being d_1 and t_1,

$$\begin{aligned} d' &= d_1 + \Delta d - e(\sin\varphi_1 \cos d' - \cos\varphi_1 \sin d' \cos t') \\ t' &= t_1 + x + c\sec d' - i\tan d' \\ &\quad + \varepsilon(\sin\varphi_1 \tan d' + \cos\varphi_1 \cos t') + e\cos\varphi_1 \sec d' \sin t' \end{aligned}$$

and *when the circle follows*, the readings being d_2 and t_2, (257)

$$\begin{aligned} 180° - d' &= d_2 + \Delta d + e(\sin\varphi_1 \cos d' - \cos\varphi_1 \sin d' \cos t') \\ 180° + t' &= t_2 + x - c\sec d' + i\tan d' \\ &\quad - \varepsilon(\sin\varphi_1 \tan d' + \cos\varphi_1 \cos t') + e\cos\varphi_1 \sec d' \sin t' \end{aligned}$$

249. The rigorous formulæ (255) and (256) will be required only in the rare case in which the pole of the instrument is at a considerable distance from the celestial pole; but I will briefly indicate the methods of determining the instrumental constants for this case. It will always be possible to bring the hour axis of the instrument very nearly into the meridian of the place of observation, whatever may be the elevation of its pole above the horizon, so that the meridian of the instrument and the true meridian will nearly coincide.

If we observe a *fixed point* in both positions of the instrument, circle preceding and circle following, we shall have by (257), taking the sums of the respective equations,

$$\begin{aligned} 180° &= d_1 + d_2 + 2\Delta d \\ 180° + 2t' &= t_1 + t_2 + 2x + 2e\cos\varphi_1 \sec d' \sin t' \end{aligned}$$

the first of which determines the index correction (Δd) of the declination circle, and the second determines the value of $t' - x$, f we have independently found the flexure e, or if the fixed point is in the meridian of the instrument and consequently $t' = 0$.

Taking the differences of the same equations, the observation of the fixed point also gives

$$180° - 2d' = d_2 - d_1 + 2e(\sin\varphi_1 \cos d' - \cos\varphi_1 \sin d' \cos t')$$
$$180° = t_2 - t_1 - 2c \sec d' + 2i \tan d' - 2\varepsilon(\sin\varphi_1 \tan d' + \cos\varphi_1 \cos t')$$

The first of these determines d' when e is otherwise known, and, the value of d' thus found being substituted in the second, we have an equation of condition for determining c, i, and ε. The observation of at least three different points will be necessary in order to determine these quantities, or of at least two points if we neglect ε.

Upon the supposition that the pole of the instrument is very near the meridian, but at a considerable distance from the celestial pole, γ is a large arc, but ϑ is small, and we have from the first of the equations (256), by putting $\cos\vartheta = \pm 1$,

$$\varphi_1 = \varphi \pm \gamma$$

and the value of γ may be found from the observation of a star in the meridian and as far from the pole of the instrument as possible, since in this case we shall have very nearly

$$\pm \gamma = \delta - d'$$

in which d' will be known from two observations of the star in the two positions of the instrument.

When γ has been thus approximately found, let a star be observed on the six hour circle both west and east of the meridian. We deduce from (255)

$$\sin d' = \sin\delta \cos\gamma + \cos\delta \sin\gamma \cos(\tau - \vartheta)$$

Denoting the instrumental declination for the two observations by d_1' and d_2', and putting $\tau = 90°$ for the first observation, and $\tau = 270°$ for the second, we have

$$\sin d_1' = \sin\delta \cos\gamma + \cos\delta \sin\gamma \sin\vartheta$$
$$\sin d_2' = \sin\delta \cos\gamma - \cos\delta \sin\gamma \sin\vartheta$$

whence

$$\sin\vartheta = \frac{\sin d_1' - \sin d_2'}{2\cos\delta \sin\gamma}$$

This will give a sufficient approximation to ϑ, provided the star is not very near the pole.

A theoretically rigorous determination of both γ and ϑ would be found by observing two points whose declinations (δ_1, δ_2) and hour angles (τ_1, τ_2) are known, and then solving the equations

$$\begin{aligned}\sin d_1' &= \sin \delta_1 \cos \gamma + \cos \delta_1 \sin \gamma \cos (\tau_1 - \vartheta)\\ \sin d_2' &= \sin \delta_2 \cos \gamma + \cos \delta_2 \sin \gamma \cos (\tau_2 - \vartheta)\end{aligned}$$

When γ and ϑ have been found, we have, from the observation of one known point,

$$\begin{aligned}\cos d' \cos (t' - \vartheta') &= \sin \delta \sin \gamma - \cos \delta \cos \gamma \cos (\tau - \vartheta)\\ \cos d' \sin (t' - \vartheta') &= \cos \delta \sin (\tau - \vartheta)\end{aligned}$$

which determine $t' - \vartheta'$; and, since ϑ' will be known from (256), t' will also be known. Finally, the instrument gives the value of $t' - x$, as we have shown above, and thus x becomes known.

250. *When the pole of the instrument is very near the celestial pole*, γ is very small, but ϑ may have any value from 0° to 360°. Putting $\cos \gamma = 1$ in (256), and neglecting terms of the same order as γ^2, we find

$$\begin{aligned}\varphi_1 &= \varphi + \gamma \cos \vartheta\\ \vartheta - \vartheta' &= - \gamma \sin \vartheta \tan \varphi\end{aligned}$$

and (255) gives

$$\begin{aligned}\delta &= d' - \gamma \cos (t' - \vartheta')\\ \tau &= t' + \vartheta - \vartheta' - \gamma \sin (t' - \vartheta') \sin d' \sec \delta\end{aligned}$$

or, within terms of the second order,

$$\begin{aligned}\delta &= d' - \gamma \cos (\tau - \vartheta)\\ \tau &= t' - \gamma \sin \vartheta \tan \varphi - \gamma \sin (\tau - \vartheta) \tan \delta\end{aligned}$$

Substituting the values of d' and t' from (254), and putting $\Delta t = x - \gamma \sin \vartheta \tan \varphi$, which is constant, we have

$$\left.\begin{aligned}\delta &= d + \Delta d - \gamma \cos (\tau - \vartheta) - e (\sin \varphi \cos \delta - \cos \varphi \sin \delta \cos \tau)\\ \tau &= t + \Delta t - \gamma \sin (\tau - \vartheta) \tan \delta + c \sec \delta - i \tan \delta\\ &\quad + \varepsilon (\sin \varphi \tan \delta + \cos \varphi \cos \tau) + e \cos \varphi \sec \delta \sin \tau\end{aligned}\right\} \quad (258)$$

which are the formulæ usually required in practice. Here δ is to be reckoned beyond 90° when necessary, being then the supplement of the star's declination (Art. 248), and then τ is the star's hour angle increased by 180°.

The declination and hour angle are here *apparent*, that is—affected by refraction, &c. If we wish δ and τ to represent th ic position of the observed point, we may apply the ns for refraction, &c. to d and t.

prefer to distinguish the cases in the formulæ themselves, have—

$$\left.\begin{array}{l}
\textit{preceding:} \\
\Delta d - \gamma \cos(\tau - \vartheta) - e(\sin\varphi\cos\delta - \cos\varphi\sin\delta\cos\tau) \\
\Delta t - \gamma \sin(\tau - \vartheta)\tan\delta + c\sec\delta - i\tan\delta \\
\varepsilon(\sin\varphi\tan\delta + \cos\varphi\cos\tau) + e\cos\varphi\sec\delta\sin\tau \\
\\
\textit{following:} \\
= d + \Delta d + \gamma\cos(\tau - \vartheta) + e(\sin\varphi\cos\delta - \cos\varphi\sin\delta\cos\tau) \\
= t + \Delta t - \gamma\sin(\tau - \vartheta)\tan\delta - c\sec\delta + i\tan\delta \\
\qquad - \varepsilon(\sin\varphi\tan\delta + \cos\varphi\cos\tau) + e\cos\varphi\sec\delta\sin\tau
\end{array}\right\} \quad (259)$$

δ and τ will always denote the declination and hour the star reckoned in the usual manner.

ADJUSTMENT OF THE EQUATORIAL INSTRUMENT.

The adjustment of the instrument with respect to the pole eavens consists of two operations: 1st, bringing the hour o the plane of the meridian, and, 2d, giving this axis an n, with respect to the horizon, equal to the latitude of the

rough preliminary adjustment, place the declination axis izontal position, and move the stand until the telescope o a star at the computed time of its meridian passage. ur axis is then nearly in the plane of the meridian. ing the declination axis into the plane of the meridian (by g the instrument upon the hour axis through 90° by the cle), and direct the telescope upon a circumpolar star on our circle. The elevation of the axis should be changed make the star appear near the optical axis at the come when the star's hour angle is equal to 6^h.

he final adjustment, the outstanding deviations of the ent must be found by properly combined observations of ken in the two reverse positions of the declination axis, nethods given hereafter.

position of the pole of the instrument with respect to the the heavens may be expressed by the two quantities

$$\xi = \gamma\cos\vartheta \qquad \eta = \gamma\sin\vartheta \qquad (260)$$

are the distances of the pole of the instrument from the six hour circle and from the meridian, respectively. Accor to our definitions of γ and ϑ, a *positive* value of ξ will indicate the instrumental pole is above the true pole, and a positive v of η will indicate that the pole of the instrument is west o meridian. I proceed to consider the methods of finding t quantities, as well as the other instrumental constants.

252. *To find* ξ.—The most simple method is to observe declinations of known stars at their culmination in both posit of the declination axis, and to compare the instrumental va corrected for refraction, with the true declinations found f the best catalogues or ephemerides. By the instrumental va we shall hereafter understand the values inferred directly f the readings (d) of the circle.

As the two observations in reverse positions of the declina axis cannot both be absolutely in the meridian (unless obs tions on different days are combined), one of them is tak few seconds before the meridian passage, and the other a seconds after it. In consequence of the great facility with w even the largest equatorial instrument can be reversed, interval between the two observations will be so small that mean of the two values of cos $(\tau - \vartheta)$ will be sensibly the s as cos ϑ, τ being a very small quantity with opposite signs the two observations. Hence, we shall have for each pai observations on a star, by putting $\tau = 0$ in (259),

$$\delta = d_1 + \Delta d - \xi - e \sin(\varphi - \delta)$$
$$180^\circ - \delta = d_2 + \Delta d + \xi + e \sin(\varphi - \delta)$$

where d_1 and d_2 are the circle readings in the two positions. half sum of these equations gives the index correction of declination circle,

$$\Delta d = 90^\circ - \tfrac{1}{2}(d_1 + d_2)$$

Their half difference gives

$$\xi + e \sin(\varphi - \delta) = 90^\circ - \tfrac{1}{2}(d_2 - d_1) - \delta$$

If we put

$$D = 90^\circ - \tfrac{1}{2}(d_2 - d_1)$$

D will be the mean of the instrumental values of the declinat as inferred from the two readings, whatever may be the mod which the circle is graduated. A number of stars being observed, we shall have the equations of condition

$$\begin{aligned}
\xi + e\sin(\varphi - \delta\,) &= D - \delta \\
\xi + e\sin(\varphi - \delta') &= D' - \delta' \\
\xi + e\sin(\varphi - \delta'') &= D'' - \delta'' \\
\&c. \qquad & \qquad \&c.
\end{aligned}$$

which, treated by the method of least squares, will give both ξ and e.

EXAMPLE.—The declinations of ten stars were observed by OTTO STRUVE with the equatorial telescope of the Pulkowa Observatory, 1840, June 22, according to the preceding method, and the values of D, corrected for refraction, were as in the following table. The values of δ for the stars 1, 4, 5 and 8 were taken from the *Nautical Almanac*, for 2, 3, and 7 from ARGELANDER'S *Catalogue*, and for 6 and 9 from AIRY'S *Catalogue* for the year 1840. The latitude employed in computing the coefficient of e is $\varphi = 59° 46'.3$. The degrees and minutes of δ, omitted to save room, are the same as those of D. In order to apply the same formula to the stars observed below the pole, we have only to employ the supplements of their declinations instead of the declinations, that is, to reckon them *over the pole*. (Art. 128.)

Stars.	Instr. dec. $= D$.	δ	Equations.	v
1. *μ Sagittarii*	— 21° 5′ 55″.5	40″.6	$-14''.9 = \xi + 0.99e$	$-5''.4$
2. *η Serpentis*	— 2 56 23 .8	3 .4	$-20\ .4 = \xi + 0.89e$	$-7\ .7$
3. *θ Serpentis*	+ 3 59 47 .1	59 .5	$-12\ .4 = \xi + 0.83e$	$+2\ .2$
4. *ζ Aquilæ*	13 37 34 .6	48 .3	$-13\ .7 = \xi + 0.72e$	$+4\ .4$
5. *α Lyræ*	38 37 47 .1	70 .4	$-23\ .3 = \xi + 0.36e$	$+6\ .2$
6. *κ Cygni*	53 3 55 .5	83 .6	$-28\ .1 = \xi + 0.12e$	$+9\ .0$
7. *δ Draconis*	67 21 51 .6	99 .7	$-48\ .1 = \xi - 0.13e$	$-3.\ 1$
8. *δ Ursæ Min.*	86 34 22 .6	81 .2	$-58\ .6 = \xi - 0.45e$	$-3\ .4$
9. *2 Lyncis, s.p.*	120 55 12 .0	79 .9	$-67\ .9 = \xi - 0.88e$	$+0\ .9$
10. *ξ Aurigæ, s.p.*	124 19 4 .5	76 .9	$-72\ .4 = \xi - 0.90e$	$-3\ .0$

The solution of these 10 equations gives

$$\begin{aligned}
\xi &= -\,40''.9 \text{ with the probable error} = 1''.2 \\
e &= +\,31''.7 \quad \text{“} \quad \text{“} \qquad \text{“} \qquad \text{“} \quad = 1''.8
\end{aligned}$$

The last column gives the residuals v after the substitution of these values in the 10 equations. From these residuals we find the probable error of a single equation to be $3''.9$, which is composed of the error of observation and the error in the star's declination. This degree of accuracy in the determination of absolute declinations, with an equatorial instrument of such dimensions, is surprising, and is a striking proof of the perfection of its workmanship. At the same time we perceive that very crude determinations will be obtained if we neglect the flexure.

253. *To find* η.—This will be found by comparing the instrumental hour angles of different stars, near the meridian, with the observed clock times of their transits over a given thread. We shall, at the same time, find the instrumental constants i and c, and the index correction of the hour circle.

We shall suppose the thread on which the stars are to be observed to be placed in the direction of a circle of declination,—that is, as a transit thread,—and to be in the optical axis of the telescope. This optical axis may be defined to be the line drawn through the optical centre of the objective, and the centre of the position circle of the micrometer: consequently, when the thread is revolved 180° by this circle, it should still pass through the optical axis. As the thread may not be precisely adjusted in this respect, the error is to be eliminated by combining two observations taken in these two positions of the thread. Two such pairs of observations are to be taken on each star, one pair with circle preceding, and one with circle following. A second star, in a widely different declination, being observed in the same manner, we shall have all that is required for the determination of our constants. If we observe a greater number of stars, we can treat the observations by the method of least squares.

Supposing two stars to be observed, one near the pole and the other near the equator, the observations should be symmetrically arranged according to the following schedule, in which the position I denotes circle preceding, and II circle following, and the letters a and b refer to the two positions of the transit thread for the two readings of the position circle differing by 180°. We should endeavor to make the mean of the times of the four observations on a star coincide very nearly with the instant of its meridian passage.

Star.	Position.	Clock.	Means.	Hour circle.	Means.
	I. *a.*	$(T_1)_a$	$= T_1$	$(t_1)_a$	$= t_1$
1st Star	*b.*	$(T_1)_b$		$(t_1)_b$	
R. A. $= \alpha$	II. *b.*	$(T_2)_b$	$= T_2$	$(t_2)_b$	$= t_2$
Decl. $= \delta$	*a.*	$(T_2)_a$		$(t_2)_a$	
			Mean $= T_0$		Mean $= t_0$
	II. *a.*	$(T_2')_a$	$= T_2'$	$(t_2')_a$	$= t_2'$
2d Star	*b.*	$(T_2')_b$		$(t_2')_b$	
R. A. $= \alpha'$	I. *b.*	$(T_1')_b$	$= T_1'$	$(t_1')_b$	$= t_1'$
Decl. $= \delta'$	*a.*	$(T_1')_a$		$(t_1')_a$	
			Mean $= T_0'$		Mean $= t_0'$

The observations being very near the meridian, the flexure of the telescope (e) has no sensible effect. That term of the flexure (ε) of the declination axis which is multiplied by $\tan \delta$ may become sensible for stars near the pole, but, as it will always be combined with i, it will be convenient to put

$$i_1 = i - \varepsilon \sin \varphi \tag{261}$$

The term $\varepsilon \cos \varphi \cos t$, which is always less than ε, will be practically unimportant, and will here be neglected. A method of determining ε will, however, be given hereafter.

With this notation we find, by putting $\tau = 0$ in the second member of (259), for the observation at the clock time T_1,

$$\tau_1 = t_1 + \Delta t + \eta \tan \delta + c \sec \delta - i_1 \tan \delta$$

and if ΔT is the clock correction, we have also

$$\tau_1 = T_1 + \Delta T - \alpha$$

Hence, by putting

$$\lambda = \Delta t - \Delta T$$

we deduce

$$\eta \tan \delta + c \sec \delta - i_1 \tan \delta = T_1 - t_1 - \alpha - \lambda$$

In the same manner the observation at the clock time T_2 gives

$$\eta \tan \delta - c \sec \delta + i_1 \tan \delta = T_2 - t_2 - \alpha - \lambda$$

and from these two equations, with the notation of the above schedule,

$$\eta \tan \delta = T_0 - t_0 - \alpha - \lambda$$
$$c \sec \delta - i_1 \tan \delta = \tfrac{1}{2}[(T_1 - t_1) - (T_2 - t_2)]$$

The second star gives, in the same manner,

$$\eta \tan \delta' = T_0' - t_0' - \alpha' - \lambda$$
$$c \sec \delta' - i_1 \tan \delta' = \tfrac{1}{2}[(T_1' - t_1') - (T_2' - t_2')]$$

By combining the two equations in η, we have, therefore, the following three equations:

$$\left.\begin{array}{l} \eta\,(\tan \delta - \tan \delta') = (T_0 - T_0') - (t_0 - t_0') - (\alpha - \alpha') \\ c \sec \delta \;\; - i_1 \tan \delta \;\, = \tfrac{1}{2}\,[(t_2 - t_1) - (T_2 - T_1)] \\ c \sec \delta' - i_1 \tan \delta' = \tfrac{1}{2}\,[(t_2' - t_1') - (T_2' - T_1')] \end{array}\right\} \quad (262)$$

which determine η, i_1, and c from the observed clock times and the readings of the hour circle.

We can then find the value of λ by the formula

$$\lambda = T_0 - t_0 - \alpha - \eta \tan \delta = T_0' - t_0' - \alpha' - \eta \tan \delta' \quad (263)$$

and finally, if the clock correction is otherwise known, the index correction of the hour circle, by the formula

$$\Delta t = \Delta T + \lambda \quad (264)$$

EXAMPLE.—The following observations were taken, according to the above method, with the equatorial of the Pulkowa Observatory, on June 3, 1840.

	Clock times.		Hour circle.	
δ Ursæ Min. I. *a.*	$18^h\,21^m\,56^s.5$	$T_1 = 18^h\,22^m\,41^s.2$	$23^h\,58^m\,21^s.1$	$t_1 = 23^h\,58^m\,59^s.5$
b.	23 25 .8		59 37 .9	
II *b.*	27 6 .0	$T_2 = 18\;28\;22.4$	0 2 55 .0	$t_2 = 0\;4\;6.4$
a.	29 38 .8		5 17 .7	
		$T_0 = 18\;25\;31.8$		$t_0 = 0\;1\;33.0$
α Lyræ II. *a.*	34 10 .0	$T_2' = 18^h\,35^m\,2^s.7$	0 2 56 .7	$t_2' = 0^h\,3^m\,49^s.6$
b.	35 55 .4		4 42 .5	
I. *b.*	39 33 .1	$T_1' = 18\;40\;29.0$	8 23 .4	$t_1' = 0\;9\;19.4$
a.	41 24 .9		10 15 .4	
		$T_0' = 18\;37\;45.9$		$t_0' = 0\;6\;34.5$

The places of the stars, according to the Nautical Almanac, were—

δ Ursæ Min.	$\alpha = 18^h\,24^m\;5^s.8$	$\delta = 86°\;35'.2$
α Lyræ	$\alpha' = 18\;31\;34.0$	$\delta' = 38\;38.1$

Hence our equations (262) become

$$\begin{aligned} 15.97\,\eta &= +\,15^s.6 \\ 16.80\,c - 16.77\,i_1 &= -\,17\,.2 \\ 1.28\,c - 0.80\,i_1 &= -\,1\,.75 \end{aligned}$$

whence

$$\begin{aligned} \eta &= +\,0^s.98 = +\,14''.7 \\ i_1 &= -\,0^s.92 \qquad c = -\,1^s.94 \end{aligned}$$

The values of i_1 and c are here not separately so well determined as they would be if the second star were nearer to the equator. Their difference, however, $i_1 - c = +\,1^s.02$, is accurately determined by the first star. We next find, by (263),

$$\lambda = -\,23^s.4$$

and if the clock correction is $\Delta T = +\,20^s.0$, the index correction of the hour circle is, by (264),

$$\Delta t = -\,3^s.4$$

To give the reader some idea of the stability of a large equatorial properly mounted, I will here give the values of ξ and η, together with the coefficient of flexure of the tube (e), determined by the above methods, for the Pulkowa instrument during a year. They are taken from STRUVE's *Description de l' Observatoire Central*, p. 204, only changing the signs of ξ and η to agree with the preceding notation:

	ξ	e
840, May 15	$-41''.2$	$+32''.6$
June 3	$-46\ .4$	$+21\ .7$
" 22	$-40\ .9$	$+31\ .7$
July 3	$-54\ .3$	$+19\ .0$
" 24	$-48\ .3$	$+34\ .2$
Aug. 9	$-43\ .0$	$+36\ .2$
Sept. 24	$-43\ .2$	$+21\ .7$
" 26	$-53\ .0$	$+37\ .2$
Nov. 10	$-38\ .5$	$+35\ .4$
Dec. 26	$-44\ .1$	$+29\ .3$
1841, Mar. 15	$-43\ .5$	$+25\ .5$
Means	$-45\ .1$	$+29\ .5$

	η
1840, April 17	$+18''.9$
" 28	$+14\ .8$
June 3	$+14\ .7$
July 24	$+10\ .2$
Sept. 24	$+10\ .8$
Nov. 3	$+\ 4\ .4$
Dec. 26	$+11\ .4$
1841, Mar. 15	$+15\ .2$
Mean	$+12\ .5$

The temperature during this period varied from — 22° t Fahr. The constancy of the coefficient of flexure extremes of temperature is as remarkable as the stabilit axis.

254. By the preceding method of finding η we also f constants i_1 and c; but we can find η independently c constants by observing the declinations of stars on the s circle. When $\tau = \pm 6^h$, we have, by (259),

$$\delta = D \mp \eta - e \sin \varphi \cos \delta$$

where D is the mean instrumental declination from the o readings in the two positions of the instrument (the tw vations being taken in quick succession very near the s circle, and one on each side of it). If we put $p = D$ - shall have the equation of condition

$$\pm \eta + e \sin \varphi \cos \delta = p$$

and from a number of equations of this kind the values o e will be found.

If the same star is observed both at $\tau = + 6^h$ and τ = we shall have, for the two observations,

$$\eta + e \sin \varphi \cos \delta = p_1$$
$$- \eta + e \sin \varphi \cos \delta = p_2$$
$$\eta = \tfrac{1}{2}(p_1 - p_2)$$

in which $p_1 - p_2$ will be the difference of the observed mental declinations, corrected for any difference of ref that may result from changes in the meteorological instr in the interval between the observations.

But it is not always possible to observe stars on the si circle in both positions of the instrument, the pier or stan fering with one of the positions for stars within a certain d of the pole. We must then find D from a single obse by applying the index correction, previously found from m observations by Art. 252. The equations formed from s observation should have a weight of only one-half in con the equations according to the method of least squares.

255. Both ξ and η can be found in a general manne observations upon different stars, without limiting the

vations to the meridian or the six hour circle. If each observation of a star is *complete*,—that is, consists of the mean of two observations in the two positions of the declination axis,—we shall have for this mean

$$\begin{aligned}\delta &= D - \gamma\cos(\tau - \vartheta) - Be\\ \tau &= t + \Delta t - \gamma\sin(\tau - \vartheta)\tan\delta + B'e\end{aligned}$$

in which B and B' are the coefficients of e in (259). Developing $\sin(\tau - \vartheta)$ and $\cos(\tau - \vartheta)$, we find

$$\left.\begin{aligned}\xi\cos\tau \qquad + \eta\sin\tau \qquad + Be &= D - \delta\\ \Delta t - \xi\sin\tau\tan\delta + \eta\cos\tau\tan\delta + B'e &= \tau - t\end{aligned}\right\} \quad (267)$$

and, from a sufficient number of such equations, Δt, ξ, η, and e will be determined.

256. Again, ξ and η may be found from *single* observations,—that is, observations in but one of the positions of the declination axis,—by observing each star twice at very different hour angles. We shall have for two observations of the same star at the hour angles τ_1 and τ_2, circle preceding in both observations or following in both,

$$\begin{aligned}\tau_1 &= t_1 + \Delta t - \xi\sin\tau_1\tan\delta + \eta\cos\tau_1\tan\delta \pm c\sec\delta \mp i\tan\delta \pm A_1\varepsilon + B_1 e\\ \tau_2 &= t_2 + \Delta t - \xi\sin\tau_2\tan\delta + \eta\cos\tau_2\tan\delta \pm c\sec\delta \mp i\tan\delta \pm A_2\varepsilon + B_2 e\end{aligned}$$

where the signification of A and B is apparent from (259). The difference of these equations gives

$$\begin{aligned}-\xi(\sin\tau_2 - \sin\tau_1)\tan\delta + \eta(\cos\tau_2 - \cos\tau)\tan\delta \pm (A_2 - A_1)\varepsilon + (B_2 - B_1)e &=\\ \tau_2 - \tau_1 - (t_2 - t_1) &= 2q\end{aligned}$$

Now, suppose one series of observations in which each star is observed at equal or very nearly equal distances from the meridian, east and west: this equation will then be reduced to the form

$$-\xi\sin\tau\tan\delta + e\cos\varphi\sec\delta\sin\tau = q \qquad (268)$$

and from the whole series, embracing stars of very different declinations, ξ and e will be determined.

Suppose another series in which each star is observed at or very near to its upper and lower culminations: the equation will take the form

$$-\eta\tan\delta \mp \varepsilon\cos\varphi = q \qquad (269)$$

This series will, therefore, determine η and ε. The upper sign will here be used for a series in which the circle is west of the meridian at the upper culminations and east of the meridian at the lower culminations. This appears to be the most simple and satisfactory method of finding the flexure ε of the declination axis. Another method will be given in the next article.

257. All the preceding methods of determining the instrumental constants depend upon the accuracy of the graduations of the two circles of the instrument. Let us inquire how far it is possible to determine these constants independently of the circles, or without involving their errors.*

First.—The inclination $90° - c$ of the telescope to the hour axis can be separately determined, independently of the other constants, as follows. Bring the telescope into a horizontal position in the plane of the meridian, the declination axis being then also horizontal. Place two collimating telescopes in the prolongation of the optical axis, one north and one south, and, directing them towards each other, bring the cross threads in their foci into optical coincidence (the equatorial telescope being for this purpose temporarily moved out of the line joining the collimators by revolving it about the hour axis). Then, bringing the telescope upon one of the collimators, and clamping the hour circle, measure with the micrometer the distance between the fixed thread that marks the optical axis and the cross thread of the collimator. Revolve the telescope upon the declination axis, and measure the distance between its optical axis and the cross thread of the other collimator. The difference of the two micrometer measures is the value of $2c$. To eliminate any eccentricity of the fixed thread with respect to the optical axis, let each observation on a collimator be the mean of two taken in reverse positions of the thread corresponding to readings of the position circle differing 180°. This method is identical in principle with the process given for the transi instrument, and more fully explained in Art. 145. Instead o one of the collimators, a distant terrestrial point may be used.

We may, at the same time, determine the flexure e of th telescope, with the aid of the declination circle, but witho involving its errors of division (Art. 204).

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* See Bessel's *Astronom. Untersuch.*, Vol. I. p. 14.

Second.—An equation for determining the inclination, $90° - i$, of the declination and hour axes, can be obtained from the observation of the transits of two different stars in the same fixed position of the declination axis, that is, with the hour circle clamped at any assumed reading. If τ and τ' are the apparent hour angles of the stars, and T, T' the sidereal clock times of the transits (corrected for clock *rate*), the difference $2q$ of these hour angles will be known by the formula

$$2q = \tau' - \tau = T' - T - (\alpha' - \alpha) - (r' - r)$$

where r and r' are the corrections of τ and τ' for refraction; and, as the difference is very small, we may use τ for τ' in the second member of (259): hence, if the circle precedes, we shall find for this difference the expression

$$\begin{aligned} 2q = &- [\gamma \sin(\tau - \vartheta) + i - \varepsilon \sin\varphi] (\tan\delta' - \tan\delta) \\ &+ (c + e \cos\varphi \sin\tau) (\sec\delta' - \sec\delta) \end{aligned}$$

Now reverse the declination axis, setting the hour circle at a reading differing 12^h from the former reading, and repeat the observation on the same stars on the following day. We shall then have, in the same manner,

$$\begin{aligned} 2q' = &- [\gamma \sin(\tau - \vartheta) - i + \varepsilon \sin\varphi] (\tan\delta' - \tan\delta) \\ &- (c - e \cos\varphi \sin\tau) (\sec\delta' - \sec\delta) \end{aligned}$$

The half difference of these equations is

$$q' - q = (i - \varepsilon \sin\varphi) (\tan\delta' - \tan\delta) - c (\sec\delta' - \sec\delta) \quad (270)$$

from which, c being previously known, we find the value of $i - \varepsilon \sin\varphi$. The hour circle is here used only to set the instrument approximately in the reverse position, and so that the values of τ in the second members of all the equations may be regarded as equal to each other in the computation of the small terms. We thus find the combination $i - \varepsilon \sin\varphi$ independently of the circle reading; but we cannot separate i without such reading.

Third.—The quantities ξ and η may be found independently of the reading of the circles by observing the same star at its upper and lower culminations, and also at its east and west transits over the six hour circle, without revolving the telescope upon the declination axis, and measuring the distance of the star in declination from the sight line with the micrometer. Thus, for $\tau = 0$ and $\tau = 180°$, the reading of the declination circle being constant, and f_1 and f_2 the micrometer distances of the star from the sight line in the two observations, r_1 and r_2 the refractions, and δ the *true* declination, we have

$$\begin{aligned}\delta - r_1 &= d + \Delta d + f_1 - \xi - e(\sin\varphi\cos\delta - \cos\varphi\sin\delta)\\ \delta + r_2 &= d + \Delta d + f_2 + \xi - e(\sin\varphi\cos\delta + \cos\varphi\sin\delta)\end{aligned}$$

and the difference of these equations gives

$$\xi = \tfrac{1}{2}(f_1 - f_2) + \tfrac{1}{2}(r_1 + r_2) + e\cos\varphi\sin\delta \tag{271}$$

For $\tau = 90°$ and $\tau = 270°$, we have

$$\begin{aligned}\delta + r_1 &= d + \Delta d + f_1 - \eta - e\sin\varphi\cos\delta\\ \delta + r_2 &= d + \Delta d + f_2 + \eta - e\sin\varphi\cos\delta\end{aligned}$$

in which r_1 and r_2 will be equal if no change in the meteorological instruments has occurred. The difference of these equations gives

$$\eta = \tfrac{1}{2}(f_1 - f_2) - \tfrac{1}{2}(r_1 - r_2) \tag{272}$$

258. A precise determination of the constants would be required if the instrument were to be used for determining absolute hour angles and declinations. But so large an instrument is liable to be so much affected by its own weight and by changes of temperature that we could not rely upon the constancy of its condition for the intervals of time that must necessarily elapse between the determinations of its errors and its application to the observation of absolute positions of stars. Hence its chief application is to the measurement of small *differences* of right ascension and declination, or of distance and position angle of two stars with its micrometer. The advantages of the equatorial system of mounting for this application are obvious.

The methods of conducting these micrometer observations are discussed in the next chapter.

CHAPTER X.

MICROMETRIC OBSERVATIONS.

I SHALL confine myself to those micrometers which have been most generally approved by astronomers, either for their convenience or their accuracy, and which are more or less in common use at the present day.

THE FILAR MICROMETER.

259. This has already been fully described in Chapter II., where also the methods of finding the angular value of a revolution of the screw have been given. Those applications in which this micrometer is but an *auxiliary* of some principal instrument—as in the transit instrument, meridian circle, &c.—have already been treated of under their appropriate heads. We are here to consider it as the principal instrument, and the telescope as the auxiliary: consequently, we are to suppose the telescope to be mounted with special reference to the convenience of micrometric observations, or, in short, to be an equatorial telescope. We also suppose it to be furnished with a position circle, constituting it a *position micrometer* (Art. 49).

TO FIND THE DISTANCE AND POSITION ANGLE OF TWO STARS* WITH THE FILAR MICROMETER.

260. With the equatorial mounting, the telescope can be readily directed to the stars at any time by setting the circles to the known hour angle and declination of the middle point between the stars. Moreover, the automatic movement of this instrument (by the driving clock), by means of which the stars

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* I say "stars," in general, for brevity; but the methods given are obviously applicable to the measurement of the distance and position angle of any two near points, as the cusps in a solar eclipse, or to the measurement of apparent semi-diameters, &c.

are kept in a constant position in the field, is indispensable f the exact measurement of their distance and position angle.

The micrometer is to be revolved until its transverse threa which is parallel to the screw, passes through the two star The zero of the position circle (*i.e.* the reading when the tran verse thread is in the direction of a circle of declination) bein known $= P_0$, and P being the reading upon the stars, we hav at once the required position angle p, by the formula

$$p = P - P_0 \tag{273}$$

The distance of the stars is measured at the same time, by placing the fixed micrometer thread (which is perpendicular to the transverse thread) upon one of the stars, and the movabl thread upon the other. The reading of the micrometer now being M (revolutions), and its zero for coincidence of the threads being M_0, the required distance in revolutions of the micrometer is

$$m = M - M_0 \tag{274}$$

If R is the value of a revolution in seconds of arc (Arts. 42, 43, &c.), and $s =$ the observed distance in arc, we then have

$$\tan \tfrac{1}{2} s = m \tan R, \qquad \text{or, nearly,} \qquad s = mR \tag{275}$$

The distance m may also be found by placing the same thread successively upon the stars and taking the difference of the micrometer readings, thus dispensing with the fixed thread and with the determination of M_0. It will be still better to use two movable threads whose constant distance is known, as will be illustrated in Art. 265.

In this process, we should bring the images of the stars on opposite sides of the middle of the field, and at very nearly equal distances from it. The position angle measured is then the angle between the arc joining the stars and the circle of declination drawn to the middle point between the stars. Both the distance and position angle thus observed are *apparent;* the effect of refraction will be considered hereafter.

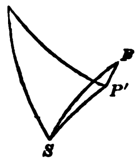
261. *Correction of the observed position angle for the errors of th equatorial instrument.*—The preceding process would be comple if the zero of the position circle always corresponded to th position of the transverse thread in which it coincided with

rcle of declination. The adjustment described in Art. 49—mely, placing the micrometer thread so that an equatorial sta the meridian runs along the thread—assumes, 1st, that the icrometer thread is perpendicular to the transverse thread, and, d, that the equatorial instrument is in perfect adjustment in all espects, so that the transverse thread, once adjusted to the meridian, will remain in the direction of a circle of declination in all other positions of the telescope.

The first source of error is avoided by adjusting the transverse thread independently of the micrometer threads. This will be most readily done by directing the telescope upon a distant terrestrial point, and revolving the micrometer until a motion of the telescope upon the declination axis alone causes the point to move exactly along the thread. The thread then represents a declination circle of the instrument, or rather a circle whose pole is that of the declination axis; and we take the reading P_0 in this position as the zero of the position circle.

The second source of error is next to be removed by *computation*, based upon the actual state of the instrument. The distance of the stars is correctly obtained independently of the errors of he equatorial adjustment, and we therefore have only to investigate the effect of these errors upon the position angle. The djustment of the thread by the method just described causes he thread to be at right angles to the arc QS, Fig. 54, which joins the pole of the declination axis and the tar. If P is the celestial pole and λ is the required correction of the observed position angle, we have the angle $QSP = 90° - \lambda$. Let P' be the pole of the instrument, and put

Fig. 54.



$$QSP' = 90° - Q, \qquad PSP' = q$$

we shall then have

$$\lambda = q + Q$$

The triangle QSP' gives, with the notation of Art. 245,

$$\sin Q = \frac{\sin i - \sin c \sin d'}{\cos c \cos d'}.$$

or, with sufficient precision,

$$Q = i \sec \delta - c \tan \delta$$

To take the flexure of the declination axis and telescope int account, we see, by Art. 246, that we must increase i by the correction $di = -\varepsilon \sin \varphi$, and c by the correction $dc = e \cos \varphi \sin \tau$ Hence, putting, as in Art. 253,

$$i_1 = i - \varepsilon \sin \varphi$$

we have

$$Q = i_1 \sec \delta - c \tan \delta - e \cos \varphi \tan \delta \sin \tau$$

The triangle PSP', with the notation of Arts. 245 and 247, give

$$\sin q = \frac{\sin \gamma \sin (\tau - \vartheta)}{\cos d'}$$

or, with sufficient precision,

$$q = \gamma \sin (\tau - \vartheta) \sec \delta$$

and it is evident that the flexure produces no sensible effect upo this angle. We have, therefore,

$$\lambda = \gamma \sin (\tau - \vartheta) \sec \delta + i_1 \sec \delta - c \tan \delta - e \cos \varphi \tan \delta \sin \tau \quad (276)$$

This formula can be used for either position of the declination axis by observing the precepts of Art. 248; but if we wish to let δ always represent the actual declination, and regard (276) as applicable to the case in which the declination circle precedes, we shall have, for the case in which it *follows*,

$$\lambda = \gamma \sin (\tau - \vartheta) \sec \delta - i_1 \sec \delta + c \tan \delta - e \cos \varphi \tan \delta \sin \tau \quad (276^*)$$

The value of δ must be that which belongs to the middle of the field, or the mean of the apparent declinations of the two stars.

The position angle resulting from the observation will now be

$$p = P - P_0 + \lambda \qquad (277)$$

262. The constant c expresses the angle between the optical axis and the axis of collimation; and it may be well to repeat here the definitions of these terms as we have used them. The optical axis is the straight line drawn through the optical centre of the objective and the centre of the position circle; and th axis of collimation, the straight line drawn through the optic centre of the objective perpendicular to the declination axi Now, the transverse thread may not pass through the optic

axis, but may have a certain *eccentricity:* hence, to obtain the position angle according to the above formula with the utmost rigor, we must take the mean of two observations in reversed positions of the thread, corresponding to readings of the position circle differing 180°.

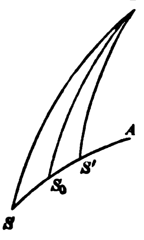
The correction λ, if the equatorial adjustment is good, will seldom amount to one minute of arc, and may usually be disregarded. The importance of a correct determination of the position angle increases with the distance of the stars, since an error in this angle will produce errors in the deduced relative right ascension and declination of the stars which are directly proportional to this distance: at the same time, the greater distance is favorable to accuracy in the *observation* of the position angle. The field of the filar micrometer, however, is small, diminishing as we increase the magnifying power for the sake of increased accuracy; and, since for this observation both stars must be seen in the field at once, we are obliged to use low powers for the greater distances (from 10′ to 20′), and thus lose, in a degree, the advantage which the increased distance would otherwise afford. This difficulty does not exist in the use of the *heliometer*, for which, therefore, a greater degree of refinement in the deduction of the position angle is requisite, and the above correction becomes of greater importance.

263. *Reduction of the observed position angle to the mean of the position angles at the two stars.*—Let S and S', Fig. 55, be the stars, P the celestial pole, S_0 the middle point between the stars, and let the arc SS' be produced through the star S' towards A. Let

$$p' = PSA, \qquad p'' = PS'A, \qquad p = PS_0A.$$

It is usual to assume p to be the mean of p' and p'', but for large distances, and when the stars are near the pole, a correction becomes necessary. If we put

Fig. 55.



$$\delta,\ \delta',\ \delta_0 = \text{the declinations of } S,\ S',\ S_0,$$
$$s = \text{the distance } SS',$$

the triangle PS_0S gives

$$\cos\delta\cos p' = \cos\tfrac{1}{2}s\cos\delta_0\cos p + \sin\tfrac{1}{2}s\sin\delta_0$$
$$\cos\delta\sin p' = \cos\delta_0\sin p$$

whence

$$\cos\delta\sin(p'-p) = -\sin\tfrac{1}{2}s\sin\delta_0\sin p + \sin^2\tfrac{1}{4}s\cos\delta_0\sin 2p$$
$$\cos\delta\cos(p'-p) = \cos\delta_0 + \sin\tfrac{1}{2}s\sin\delta_0\cos p - 2\sin^2\tfrac{1}{4}s\cos\delta_0\cos^2 p$$

and, developing $\sin\frac{1}{2}s$ and $\sin\frac{1}{4}s$ in series,

$$\cos\delta\sin(p'-p) = -\tfrac{1}{2}s\sin\delta_0\sin p + \tfrac{1}{16}s^2\cos\delta_0\sin 2p + \&c.$$
$$\cos\delta\cos(p'-p) = \cos\delta_0 + \tfrac{1}{2}s\sin\delta_0\cos p - \&c.$$

Dividing the first by the second, and putting for $\tan(p'-p)$ its value in series, we find

$$p'-p = -\tfrac{1}{2}s\tan\delta_0\sin p + \tfrac{1}{16}s^2\sin 2p(1+2\tan^2\delta_0) - As^3 + \&c$$

In like manner, the triangle PS_0S' gives

$$\cos\delta'\cos p'' = \cos\tfrac{1}{2}s\cos\delta_0\cos p - \sin\tfrac{1}{2}s\sin\delta_0$$
$$\cos\delta'\sin p'' = \cos\delta_0\sin p$$

from which we see that the development of $p''-p$ will be obtained from that of $p'-p$ by merely changing the sign of s: hence

$$p''-p = +\tfrac{1}{2}s\tan\delta_0\sin p + \tfrac{1}{16}s^2\sin 2p(1+2\tan^2\delta_0) + As^3 + \&c.$$

Neglecting only the 4th and higher powers of s, we have, therefore,

$$\tfrac{1}{2}(p'+p'') - p = \tfrac{1}{16}s^2\sin 2p(1+2\tan^2\delta_0) \qquad (278)$$

which is the required correction to be added to the observed position angle p to reduce it to the mean $\frac{1}{2}(p'+p'')$. When s is expressed in seconds of arc, the second member must be multiplied by sin 1″.

We also find, within terms of the 3d order,

$$\tfrac{1}{2}(p''-p') = \tfrac{1}{2}s\tan\delta_0\sin p \qquad (279)$$

The purpose of the observation is usually to determine the place of one star from that of another which is given. It will be convenient hereafter to consider the observed position angle as expressing the position of the unknown star referred to the known: thus, in the above formulæ the three position angles p', p'', p are all reckoned in the direction from the known to the unknown star, p' being the angle at the former, p'' the angle at the latter, and p the angle at the middle point between the two stars.

FIND THE APPARENT DIFFERENCE OF RIGHT ASCENSION AND DECLINATION OF TWO STARS WITH THE FILAR MICROMETER.

264. First Method.—Observe the distance s, and the position gle p, of the unknown star from the known star, by the pre-ding method. For a rigorous method of computation we ust first reduce the observed angle to the mean of the angles the stars, by (278). Thus, if we denote this mean by p_0, we rst find

$$p_0 = p + \tfrac{1}{16} s^2 \sin 1'' \sin 2p\,(1 + 2 \tan^2 \delta_0) \qquad (280)$$

n which we may take $\delta_0 =$ the mean of the declinations of the tars, which may be found with sufficient precision by a rough preliminary computation. If we also put $\Delta p = \tfrac{1}{2}(p'' - p')$, we find in the next place, by (279),

$$\Delta p = \tfrac{1}{2} s \tan \delta_0 \sin p \qquad (281)$$

Now, α, δ denoting the right ascension and declination of the known star, α', δ' those of the unknown star, the triangle formed by the two stars and the pole gives, by the Gaussian equations of Spherical Trigonometry,

$$\begin{aligned}
\sin \tfrac{1}{2}(\delta' - \delta) \cos \tfrac{1}{2}(\alpha' - \alpha) &= \sin \tfrac{1}{2} s \cos p_0 \\
\cos \tfrac{1}{2}(\delta' - \delta) \cos \tfrac{1}{2}(\alpha' - \alpha) &= \cos \tfrac{1}{2} s \cos \Delta p \\
\sin \tfrac{1}{2}(\delta' + \delta) \sin \tfrac{1}{2}(\alpha' - \alpha) &= \cos \tfrac{1}{2} s \sin \Delta p \\
\cos \tfrac{1}{2}(\delta' + \delta) \sin \tfrac{1}{2}(\alpha' - \alpha) &= \sin \tfrac{1}{2} s \sin p_0
\end{aligned}$$

The 1st and 2d give

$$\tan \tfrac{1}{2}(\delta' - \delta) = \tan \tfrac{1}{2} s \cdot \frac{\cos p_0}{\cos \Delta p} \qquad (282)$$

Having thus found $\tfrac{1}{2}(\delta' - \delta)$, we also have $\tfrac{1}{2}(\delta' + \delta) = \delta + \tfrac{1}{2}(\delta' - \delta)$; and then the 4th equation gives

$$\sin \tfrac{1}{2}(\alpha' - \alpha) = \frac{\sin \tfrac{1}{2} s \sin p_0}{\cos \tfrac{1}{2}(\delta' + \delta)} \qquad (283)$$

For an approximate method of computation, sufficient in most cases, we can neglect the difference between p and p_0, and, consequently, also neglect terms in s^3 in (282) and (283), so that these equations will become

$$\left.\begin{aligned}
\delta' - \delta &= s \cos p \\
\alpha' - \alpha &= s \sin p \sec \tfrac{1}{2}(\delta' + \delta)
\end{aligned}\right\} \qquad (284)$$

EXAMPLE.—In 1846, November 29, at the Washington Observatory, Mr. SEARS C. WALKER observed the position angle and distance of the planet Neptune from a star as follows:

$$\text{Sid. time} = 0^h\,17^m\,52^s \qquad P = 82°\,35'.7 \qquad m = 20.576\,\text{rev.}$$

For the zero of the position circle he found $P_0 = 272°\,38'$, and the value of a revolution of the micrometer was $R = 15''.406$. The star's apparent place was

$$\alpha = 21^h\,51^m\,50^s.69 \qquad \delta = -\,13°\,25'\,52''.76$$

Hence we have, by (284),

$$\begin{array}{llll}
P - P_0 = p = 169°\,57'.7 & \log\cos p \;\; n9.99330 & & \log\sin p \;\; 9.24132 \\
 & \log mR = \log s \;\; \underline{2.50105} & \ldots\ldots & \log s \quad 2.50105 \\
\delta' - \delta = -\,5'\,12''.14 & \log(\delta' - \delta) \;\; n2.49435 & & \log\sec\tfrac{1}{2}(\delta' + \delta) \;\; \underline{0.01212} \\
\tfrac{1}{2}(\delta' + \delta) = -13°\,28'\,29''. & \alpha' - \alpha = +56''.82 = +3^s.79 & & \log(\alpha' - \alpha) \;\; 1.75449
\end{array}$$

The computation by the rigorous formulæ (282) and (283) gives the same results. Neglecting the differential refraction, which will be treated of hereafter, these differences applied to the given place of the star give for the place of Neptune at the sidereal time $0^h\,17^m\,52^s$,

$$\alpha' = 21^h\,51^m\,54^s.48 \qquad \delta' = -\,13°\,31'\,4''.90$$

In the case of a planet the place thus found has also to be corrected for its parallax. (Arts. 102, 103, of Vol. I.)

265. *When one of the stars has a proper motion*, the mean of several observed distances and position angles will not correspond precisely to the mean of the times. To proceed rigorously in that case, we must compute the differences of right ascension and declination from each observation; and, as these differences may be regarded as proportional to the time, their mean will correspond to the mean of the times. But a briefer method of reduction consists in employing the mean of the observed distances and position angles *corrected for second differences*. Let s_1, s_2, s_3, &c. be the observed distances, and s_0 their arithmetical mean; p_1, p_2, p_3, &c. the observed position angles, and p_0 their arithmetical mean; T_1, T_2, T_3, &c. the corresponding observed times, and T their arithmetical mean. Let s and p denote the values of the distance and position angle corresponding to the

time T. We have only to find s and p, with which a single computation of the differences of right ascension and declination will give the quantities required for the time T.

Let $\Delta\alpha$, $\Delta\delta$ be the changes of right ascension and declination in one sidereal second. If α', δ' are the values which correspond to the time T, we have

$$\begin{aligned} s \sin p &= (\alpha' - \alpha) \cos \tfrac{1}{2} (\delta' + \delta) \\ s \cos p &= \delta' - \delta \end{aligned}$$

and, consequently,

$$\begin{aligned} s_1 \sin p_1 &= (\alpha' - \alpha) \cos \tfrac{1}{2} (\delta' + \delta) + \Delta\alpha\, (T_1 - T) \cos \tfrac{1}{2} (\delta' + \delta) \\ s_1 \cos p_1 &= \delta' - \delta \qquad\qquad\qquad\qquad + \Delta\delta\, (T_1 - T) \end{aligned}$$

Put

$$T_1 - T = \tau_1, \qquad T_2 - T = \tau_2, \qquad T_3 - T = \tau_3, \text{ \&c.}$$

and, also,

$$\left.\begin{aligned} f \sin \vartheta &= \Delta\alpha \cos \tfrac{1}{2} (\delta' + \delta) \\ f \cos \vartheta &= \Delta\delta \end{aligned}\right\} \quad (285)$$

then

$$\begin{aligned} s_1 \sin p_1 &= s \sin p + f \sin \vartheta \,.\, \tau_1 \\ s_1 \cos p_1 &= s \cos p + f \cos \vartheta \,.\, \tau_1 \end{aligned}$$

whence

$$\left.\begin{aligned} s_1 \sin (p - p_1) &= f \sin (p - \vartheta) \,.\, \tau_1 \\ s_1 \cos (p - p_1) &= s + f \cos (p - \vartheta) \,.\, \tau_1 \end{aligned}\right\} \quad (A)$$

These equations give, first,

$$\tan (p - p_1) = \frac{\frac{f}{s} \sin (p - \vartheta) \,.\, \tau_1}{1 + \frac{f}{s} \cos (p - \vartheta) \,.\, \tau_1}$$

which developed in series [Pl. Trig. Art. 257] gives

$$p = p_1 + \frac{f}{s} \cdot \frac{\sin (p - \vartheta)}{\sin 1''} \cdot \tau_1 - \frac{f^2}{s^2} \cdot \frac{\sin 2 (p - \vartheta)}{\sin 1''} \cdot \frac{\tau_1^2}{2} + \text{\&c.}$$

Each observation gives an equation of this form; and the mean of n such equations, observing that $\Sigma\tau = 0$, is

$$p = p_0 - \frac{f^2}{s^2} \cdot \frac{\sin 2 (p - \vartheta)}{\sin 1''} \cdot \frac{\Sigma\tau^2}{2n}$$

where we neglect terms of the third and higher orders. He is expressed in seconds of time, and we have, very nearly,

$$\frac{\tau^2}{2} = \frac{2 \sin^2 \frac{1}{2}\tau}{(15 \sin 1'')^2}$$

If we employ the quantity m given by Table V.,—*i.e.*

$$m = \frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$$

our formula will become

$$p = p_0 - \left(\frac{f}{15\, s \sin 1''}\right)^2 \sin 2(p - \vartheta) \frac{\Sigma m}{n} \qquad ($$

Again, the sum of the squares of the equations (A) gives

$$s_1^2 = s^2 + 2fs \cos(p - \vartheta) . \tau_1 + (f\tau_1)^2$$

whence

$$\frac{s_1}{s} = \left[1 + \frac{2f \cos(p - \vartheta)}{s} . \tau_1 + \left(\frac{f\tau_1}{s}\right)^2\right]^{\frac{1}{2}}$$

$$= 1 + \frac{f}{s} \cos(p - \vartheta) . \tau_1 + \frac{1}{2}\left(\frac{f\tau_1}{s}\right)^2 \sin^2(p - \vartheta)$$

where the terms of the third order are neglected. The m of n equations of this kind is

$$\frac{s_0}{s} = 1 + \frac{f^2 \sin^2(p - \vartheta)}{s^2} . \frac{\Sigma\tau^2}{2n}$$

and, if M is the modulus of common logarithms, we have, v nearly,

$$\log s = \log s_0 - M\left(\frac{f}{15 s}\right)^2 \frac{\sin^2(p - \vartheta)}{\sin 1''} . \frac{\Sigma m}{n} \qquad ($$

It will be convenient to find the correction of p_0 in mim of arc, and the correction of $\log s_0$ in units of the fifth deci place; for which purpose we have to divide the last term (286) by 60, and multiply the last term of (287) by 10^5. It also be convenient to let $\Delta\alpha$ and $\Delta\delta$ be the changes of right cension and declination in *one minute of mean time*, as they usually be given in this form; and then we must divide f 60.164 (= no. of sid. seconds in 1^m of mean time). With th modifications our formulæ will become

$$\left.\begin{aligned} p &= p_0 - [2.93984] \frac{f^2}{s^2} \sin 2(p - \vartheta) \frac{\Sigma m}{n} \\ \log s &= \log s_0 - [4.04135] \frac{f^2}{s^2} \sin^2(p - \vartheta) \frac{\Sigma m}{n} \end{aligned}\right\} \qquad ($$

where the logarithm of the constant factor is given. The quantities $\Delta\alpha$, $\Delta\delta$, f, and s are supposed to be expressed in seconds of arc.

266. **Second Method.**—Set the declination circle of the equatorial instrument to the mean declination of the two stars; direct the telescope to a point a little in advance of the stars, and clamp the hour circle. The telescope being fixed, the diurnal motion will carry the stars across the field. Set the transit threads (*i.e.* the transverse thread and the threads parallel to it) in the direction of a circle of declination, and, as the stars pass across the field, observe the clock times of their transits over the threads. At the same time, set the micrometer thread upon the two stars successively as each passes the middle of the field, and read the micrometer interval between them; this will give at once the difference of declination. The difference of right ascension will be the difference between the observed clock times of transit of the two stars over the same threads, this difference being, of course, reduced to a sidereal interval when necessary, and also corrected for clock rate.

For the reduction of defective transits, it is necessary to know the intervals of the threads, which will be found as in the transit instrument (Art. 131).

If one of the bodies has a proper motion, the differences obtained are those which belong to the instant when this body was observed.

It is usual, in observations of this kind, to avoid all consideration of the errors of the equatorial instrument, by adjusting the movable micrometer thread at the time of the observation so that the star runs along the thread.* If the transit threads are exactly perpendicular to the micrometer thread, they will be (very nearly) parallel to a circle of declination drawn through the centre of the field; but, to eliminate any error arising fro a defect of perpendicularity, the threads should be revolved 180 by the position circle, and the observation repeated; and in series of consecutive observations there should be a like numbe of observations in these two positions.

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* This method is, however, not strictly correct; for the apparent path of a star is not precisely perpendicular to the circle of declination, on account of the difference of the refraction at different points of this path. The error is, indeed, extremely small, except when the zenith distance is very great; but, if we wish to proceed with the utmost precision, we can set the threads by means of the position circle. If the zero P_0 of the position circle has been determined as in Art. 261, and the circle is set to this reading, the threads will make the angle λ with a true circle of declination; consequently, δ and δ' being the declinations of the stars, we must add the correction $\frac{1}{15}(\delta' - \delta) \sin\lambda \sec\delta'$ to the observed time of transit of the star whose declination is δ'. The angle λ will be found by (276).

The slide moved by the screw is often provided with thre micrometer threads the constant distance of which from eac other is known, and each of the two bodies is observed on th thread which is nearest to it. By this arrangement we ar enabled to measure a large difference of declination with but a small motion of the screw, which often facilitates the observa- tion, especially when the stars have nearly the same right ascen- sion, and, consequently, pass the middle of the field nearly at the same time.

The equatorial mounting enables us to repeat the observatio as often as we please, with the greatest facility. After each ob- servation we have only to revolve the instrument a small dis- tance upon the hour axis and clamp it again a little in advanc of the objects.

EXAMPLE.—In 1846, November 29, at the Washington Obser- vatory, Mr. WALKER observed the difference of right ascension and declination of the planet Neptune and a star as below. The micrometer was adjusted so that the star ran along a micrometer thread. There were three micrometer threads, numbered 1, 2, 3, of which 1 was nearest the micrometer head, and the constan distance between 2 and 3 was 29.983 revolutions. The reading of the micrometer increased with the declination. The valu of a revolution was $R = 15''.406$.

	Transit Thread.			Mean of threads.	Micrometer.	
	I	II	III		Thread.	M
						Rev.
Star	$2^s.7$	$15^s.2$	$27^s.4$	$23^h\ 30^m\ 15^s.10$	2	54.564
Neptune	48 .2	0 .5	12 .5	" 32 0 .40	3	55.453

$$\alpha' - \alpha = +\ 1\ \ 45\ .30$$

$$+\ 0.889$$

$$-\ 29.983$$

$$m = -\ 29.094$$

$$\delta' - \delta = mR = -\ 7'\ 28''.22$$

The star's place was

$$\alpha = 21^h\,50^m\,8^s.99 \qquad \delta = -13^\circ\,23'\,35''.11$$

and therefore, neglecting the differential refraction and the planet's parallax, we have

$$\alpha' = 21^h\,51^m\,54^s.29 \qquad \delta' = -13^\circ\,31'\,3''.33$$

which belong to the time when Neptune was observed. The clock correction was $-3^m\,31^s.7$, and therefore the place determined corresponds to the sid. time $23^h\,28^m\,28^s.7$.

Five observations of the same kind were taken successively, which gave at the sid. time $23^h\,30^m\,56^s$, $\alpha' - \alpha = +1^m\,45^s.23$, $\delta' - \delta = -7'\,29''.40$.

267. Third Method.—When the telescope follows the motion of the stars automatically with great accuracy, we may measure the difference of right ascension by placing the micrometer threads at right angles to the diurnal motion and setting the fixed thread upon one star and the movable thread upon the other. The middle point of the arc joining the stars should be as nearly as possible in the centre of the field. If, then, m is the distance of the threads, and its equivalent in arc is $s = mR$, we shall have, very nearly, $\sin(\alpha' - \alpha) = 2 \sin \frac{1}{2} s \sec \delta_0$, in which δ_0 is the mean declination. This method will not be used for stars far from the equator, and therefore in all practical cases we may take $\alpha' - \alpha = s \sec \delta_0$. The objection to this method is, that the difference of declination is not found at the same time.

THE HELIOMETER.

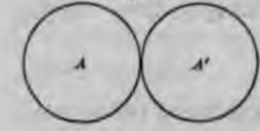
268. This instrument belongs to the class of *double image* micrometers. The object glass of an equatorially mounted telescope is bisected, the plane of the section passing through the optical axis of the lens, and the two semi-lenses, set in separate metallic frames, slide upon each other in a direction parallel to the line of section.* Either semi-lens can be moved, and the amount of its motion measured, by a micrometer screw. Each semi-lens forms a complete image of a distant object at the prin-

---

* The duplication of the image by means of two complete lenses was invented by Bouguer, in 1748. The improvement of substituting the two halves of a single lens was shortly after made by John Dollond.

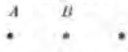
cipal focus. These images (in a perfect instrument) are superposed, and form but a single image at the focus, when the two semi-lenses are in their primitive position forming a single circular lens; but when the optical centres of the two semi-lenses are separated by the sliding motion, the two images at the focus are separated from each other by a distance equal to the distance of the centres of the semi-lenses. The instrument thus arranged becomes a micrometer adapted for the measurement of small angular distances in general, but, from its supposed peculiar adaptation to the measurement of the sun's diameter, has received the name of the *heliometer*. Thus, if A (Fig. 56) is the image of the sun formed at the focus when the centres of the semi-lenses are coincident, and one semi-lens is then moved until the image it forms is in the position A', so that its limb is in apparent contact with that formed by the other semi-lens, the motion of the semi-lens, as measured by the micrometer screw, gives the measure of the angular diameter of the sun as soon as the angular value of a revolution of the screw is known.

Fig. 56.



Again, if A and B (Fig. 57) are the images of two stars when the semi-lenses are coincident, and if (the direction of the line of section of the lens being made to coincide with that of the line joining the stars) one semi-lens is moved until the image of A is seen at B, while that of B is moved to B', the motion of the lens as given by the screw determines the angular distance of the stars. The position angle of the two stars will also be determined by the angle which the line of section makes with a declination circle; and for this purpose the whole lens is mounted so as to be revolved in a plane at right angles to its optical axis, and its position at any time is shown by a graduated position circle attached to the tube of the telescope.

Fig. 57.



Such is the general principle of the instrument; but in order to give precision to the observation, it is necessary that the observed point of coincidence of two images should be in the optical axis of the complete lens, and that these images should be separated by moving the semi-lenses in opposite directions and equal distances on each side of this axis; or, if these conditions are not exactly or approximately satisfied, that we should have the means of computing the correction which the observed measure requires. For this purpose, the ocular is also provided with a micrometer screw and a position circle, and the position of the point of contact of two images, with respect to the line joining the centres of the two position circles, can be determined. The mode of using the data thus obtained will be discussed in the general theory of the instrument hereafter given.

269. Plate XV. represents the heliometer of the Königsberg Observatory, with which BESSEL determined the parallax of 61 *Cygni*. The focal length of the telescope is 102 inches, the diameter of the lens is 6¼ inches. The equatorial mounting needs no special explanation, as it is essentially the same as that described in the preceding chapter, except that the stand is here of wood and adjustable by means of four foot screws. The sliding motion of the semi-lenses is produced by the micrometer screws *a*, *b*, which are moved by the observer by means of the rods *a′* and *b′*. The measure of the motion is obtained either from the graduated heads of the micrometer screw or from two graduated scales, which are read by the microscopes *e* and *f*. The latter method is, however, chiefly used as a check upon the former, and also to verify the regularity of the screw. The revolution of the lens about the axis of the tube is effected by a rack (*hh*) and pinion, which is out of view in the drawing, but is acted upon by the rod *c*. In order to read the micrometer and position circle after an observation is completed, the telescope has only to be revolved upon the declination axis until its object end is brought to a convenient position for reading.

It greatly facilitates the successive *repetitions* of the observation to employ the automatic movement by clock-work; for after an observation the telescope can be revolved upon the declination axis *without stopping the clock*, and after reading the micrometer and position circle it can be restored to its former position in declination, and the objects will be still in the field.

It is one of the chief advantages of the heliometer that the precision of the observation is not impaired by the diurnal motion; for even when we do not employ the driving clock, a good result is obtained whenever we have made a *contact* of the images of the observed points near the centre of the field. The automatic movement is, therefore, not essential to secure the accuracy of the observation (as it is in the case of the filar micrometer), but is chiefly important as facilitating the repetition of the observation.

It has been objected to the heliometer that the optical performance of a semi-lens is imperfect. In fact, it appears that, although the correction for spherical aberration of a complete lens may be perfect, it is not perfect for each half of the lens,—at least, it has not been found perfect in the instruments of this kind heretofore constructed. There is also some *inflexion* of the rays of light produced at the line of section. The combined effect of these causes is an elongation of the separated images in a direction at right angles to the line of section. Another objection is, that the brightness of each of the images is but one-half that of an image formed by the whole lens. It has also been found that when the two semi-lenses are in their primitive position, forming a single complete lens, the two superposed images do not always form a single constant image, but that in a disturbed state of the air the images are frequently seen to separate momentarily. This effect, of which no entirely satisfactory explanation has been suggested, has been observed in most if not all the heliometers.

But these optical defects are more than compensated by the superior accuracy in the measurement of distances, resulting from the great precision with which contacts and coincidences of images can be observed. The elongation of the images, being in a direction at right angles to the observed distance, has no sensible effect upon its measure, and its minute effect upon the position angle is eliminated by repeating the observation with opposite motions of the semi-lenses, that is, by interchanging the images. The tremulous motion of stars arising from a disturbed state of the air is in general common to the images of both objects, and, therefore, does not affect the observation of a contact; and the momentary separation of the images above referred to, which when the semi-lenses are separated produces a slight tremulous motion of each image, does not cause the images to appear so unsteady *relatively* to each other as the single image formed by a complete lens relatively to the thread of the filar micrometer. Finally, the experience of BESSEL and others in the actual use of the instrument has proved that the probable error of a single measure, whether of distance or position angle, is less than in the use of any other micrometer.*

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* See BESSEL's account of the Königsberg heliometer, *Astron. Nach.* Vol. VIII. pp. 411–426.

The heliometer possesses a great advantage over all other micrometers in the measurement of comparatively large distances. With a filar micrometer the distances observed must be the less the higher the magnifying power employed, since the whole distance must be in the field of view; but no such restriction exists with the heliometer, where only the point of contact or coincidence of two objects is required to be in the field. With the Königsberg instrument above described, a distance of 1° 52′ can be measured.

GENERAL THEORY OF THE HELIOMETER.

270. In the following discussion of the mathematical theory of the heliometer I shall chiefly follow BESSEL.[*]

I shall first investigate the general formulæ which determine the position of any point of the celestial sphere observed with one semi-lens, the data being—1st, the declination and hour angle of the point of the sphere which is in the *heliometer axis*, which point may be called the pole of the heliometer axis; 2d, the position of the semi-lens with respect to this axis, as given by the micrometer and position circle of the objective; 3d, the position of the point in the field where the image is observed, as given by the micrometer and position circle of the ocular.

By the heliometer axis is here meant the straight line which joins the centres of the position circles of the objective and ocular; and we shall here apply to this axis the notation which in the theory of the equatorial instrument (Art. 245) was applied to the sight line. Thus, 90° — c will now express the distance of the pole of the heliometer axis from the pole of the declination axis. If then we denote by δ_1 and τ_1 the declination and hour angle of the pole of the heliometer axis, we shall have, by (258),

$$\left.\begin{aligned}\delta_1 &= d + \Delta d - \gamma \cos(\tau_1 - \vartheta)\\ \tau_1 &= t + \Delta t - \gamma \sin(\tau_1 - \vartheta)\tan\delta_1 + c\sec\delta_1 - i_1\tan\delta_1\end{aligned}\right\}\quad(289)$$

where d and t are the readings of the declination and hour circles, and Δd, Δt, γ, ϑ, c, and i_1 are the constants of the equatorial instrument, supposed known. The terms depending on the flexure are here omitted, as not sensibly affecting micrometric observations, excepting only the term $\varepsilon \sin \varphi \tan \delta_1$, which, on account of the factor $\tan \delta_1$, may be supposed to become sensible for stars very near the pole; and this term is included in our formulæ by the substitution of $i_1 = i - \varepsilon \sin \varphi$.

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[*] *Astronomische Untersuchungen*, Vol. I., *Theorie eines mit einem Heliometer versehenen Equatoreal-Instruments.* See, however, also HANSEN'S *Ausführliche Methode mit dem Fraunhoferschen Heliometer Versuche anzustellen*, 4to. Gotha, 1827.

It is assumed that the images of infinitely distant points formed by each semi-lens are mathematical points, that they all lie in the same focal plane perpendicular to the heliometer axis, and that the straight lines joining these points and their images pass through the optical centre of the semi-lens. Let this optical centre be denoted by O. The point O is moved by the micrometer screw in a plane which is at right angles to the heliometer axis and in a line which should pass through that axis; but a perfect adjustment in this respect will not be assumed, and we shall suppose that the line in which the point O moves is at the distance b from the heliometer axis. The position of the point O in this line at any time will be determined by the micrometer reading m, together with the reading that corresponds to some assumed point of the line as an origin. Let this origin be the point of the line which is at the least distance ($= b$) from the heliometer axis, and let a be the reading when O is at this point; then the distance of O from this origin at any time will be expressed by $m - a$.

The direction of the line of motion of the point O at any time will be given by the position circle. The zero of the position circle will be the reading when this line coincides in direction with a celestial circle whose pole is the pole (Q) of the declination circle of the instrument, as in Art. 261. If we here denote this zero reading by n_0, and the reading at any time by n, the position angle of the line of motion will be

$$= n - n_0 + \lambda$$

in which we have, by (276),

$$\lambda = [\gamma \sin (\tau_1 - \vartheta) + i_1] \sec \delta_1 - (c + e \cos \varphi \sin \tau_1) \tan \delta_1 \quad (290)$$

271. Now, in order to express the position of the point O in a general manner, let us take two planes of reference at right angles to each other passing through the heliometer axis, and let one of these planes be the plane of the circle of declination passing through the pole of this axis. Let AY, Fig. 58, be the intersection of the plane of the circle of declination with the plane of motion of the semi-lens; AX the intersection of the

second plane of reference with the plane of motion; BO the line in which the optical centre O of the semi-lens moves; AO_1 the perpendicular from A upon BO. Then, according to the notation above adopted, we have $AO_1 = b$, $O_1O = m - a$, and $ABO = n - n_0 + \lambda = n - k$, where, for brevity, we put

$$k = n_0 - \lambda \tag{291}$$

Hence the distance of O from the two planes of reference, or its co-ordinates on the axes AX and AY, are evidently

$$x = (m - a) \sin (n - k) + b \cos (n - k)$$
$$y = (m - a) \cos (n - k) - b \sin (n - k)$$

The position of the point in the field of the ocular, at which the image of the celestial point is observed, which point we shall call the point o, will be determined by referring it to the same two planes: so that if μ, α, ν, κ, β have the same signification for the point o that m, a, n, k, b have for the point O, the co-ordinates of o, with reference to these planes are

$$\xi = (\mu - \alpha) \sin (\nu - \kappa) + \beta \cos (\nu - \kappa)$$
$$\eta = (\mu - \alpha) \cos (\nu - \kappa) - \beta \sin (\nu - \kappa)$$

Fig. 58.

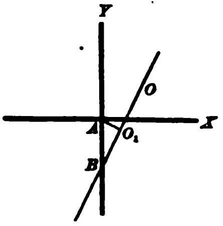
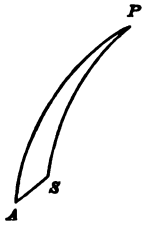


Fig. 59.



The direction of the sight line oO, or that of a star whose image is observed at o, can now be determined by means of these co-ordinates and the distance f' between the planes of motion of o and O. Conceive a straight line to be drawn through o, parallel to the heliometer axis. This line and the heliometer axis have the same vanishing point in the celestial sphere, namely, the pole of the heliometer axis. Let A, Fig. 59, be this point of the sphere, S the star in the sight line oO, P the pole of the heavens. The plane passed through the line oA and the line oO makes with the plane of the circle of declination PA the angle $PAS = \pi$; and the angle between the lines oA and oO is measured by the arc $AS = \Delta$. The distance of O from the line oA is $f' \tan \Delta$, and its distances from the plane of PA and the plane drawn through oA at right angles to the plane of PA are $f' \tan \varDelta \sin \pi$ and $f' \tan \varDelta \cos \pi$. These distances are also expressed by $x - \xi$ and $y - \eta$; and hence we have the equations

$$\begin{aligned} f' \tan \varDelta \sin \pi &= x - \xi \\ f' \tan \varDelta \cos \pi &= y - \eta \end{aligned}$$

If we take the linear distance of the threads of the micrometer screw of the objective as the common unit of measure of all the quantities m, a, b, μ, α, β, f', and if R is the angular value of one revolution of the screw, we have, since f' is the focal length of the lens,

$$\tan R = \frac{1}{f'}$$

Hence, the above expressions divided by f' give

$$\left.\begin{aligned} \tan \varDelta \sin \pi = \tan R\,[(m - a) \sin (n - k) + b \cos (n - k) \\ - (\mu - \alpha) \sin (\nu - \varkappa) - \beta \cos (\nu - \varkappa)] \\ \tan \varDelta \cos \pi = \tan R\,[(m - a) \cos (n - k) - b \sin (n - k) \\ - (\mu - \alpha) \cos (\nu - \varkappa) + \beta \sin (\nu - \varkappa)] \end{aligned}\right\} \quad (292)$$

These determine $\varDelta$ and π, with which the declination δ and hour angle τ of the star are determined by means of the formulæ, derived from the triangle PAS,

$$\left.\begin{aligned} \sin \delta &= \sin \delta_1 \cos \varDelta + \cos \delta_1 \sin \varDelta \cos \pi \\ \cos \delta \cos (\tau_1 - \tau) &= \cos \delta_1 \cos \varDelta - \sin \delta_1 \sin \varDelta \cos \pi \\ \cos \delta \sin (\tau_1 - \tau) &= \sin \varDelta \sin \pi \end{aligned}\right\} \quad (293)$$

272. We can now proceed to the determination of the relative position of two stars S and S' whose images have been brought into coincidence by giving the two semi-lenses different positions. This relative position is expressed (as in the use of the filar position micrometer) by the distance $s = SS'$, and the position angle at the middle point of $SS' = p$. Thus, in Fig. 55, p. 395, S_0 being the middle point of SS', we have $PS_0S' = p$. The declination δ_0 and hour angle τ_0 of S_0 will be regarded as known.

Let us distinguish the two semi-lenses by the numerals I. and II., and let the formulæ (292) and (293) refer to the semi-lens I. and to the image of the star S formed by it. Let the image of the star S' be formed by the semi-lens II., and let the several quantities referring to this star be distinguished by accents, excepting those which are common to both stars. These common quantities are—1st, the readings n and ν of the position circles; 2d, the ometer reading $\mu - \alpha$ and the constants β and $\varkappa$ of the ar, since these refer to a single point of the field. But we suppose the lines of motion of the two semi-lenses to be not ectly parallel, and shall therefore express the angle which line of motion of the semi-lens II. makes with a declina-circle by $n - k'$; so that, n_0' denoting the zero reading of the ion circle when this semi-lens is used, we have

$$k' = n_0' - \lambda \qquad (294)$$

$$\left.\begin{aligned} {}' \sin \pi' &= \tan R \,[(m' - a') \sin (n - k') + b' \cos (n - k') \\ &\quad - (\mu - \alpha) \sin (\nu - \varkappa) - \beta \cos (\nu - \varkappa)] \\ {}' \cos \pi' &= \tan R \,[(m' - a') \cos (n - k') - b' \sin (n - k') \\ &\quad - (\mu - \alpha) \cos (\nu - \varkappa) + \beta \sin (\nu - \varkappa)] \end{aligned}\right\} \qquad (295)$$

$$\left.\begin{aligned} \sin \delta' &= \sin \delta_1 \cos \Delta' + \cos \delta_1 \sin \Delta' \cos \pi' \\ \cos \delta' \cos (\tau_1 - \tau') &= \cos \delta_1 \cos \Delta' - \sin \delta_1 \sin \Delta' \cos \pi' \\ \cos \delta' \sin (\tau_1 - \tau') &= \sin \Delta' \sin \pi' \end{aligned}\right\} \qquad 96)$$

triangles PS_0S and PS_0S' (Fig. 55, p. 395) give

$$\left.\begin{aligned} \sin \tfrac{1}{2} s \sin p &= -\cos \delta \sin (\tau_0 - \tau) \\ \sin \tfrac{1}{2} s \cos p &= -\sin \delta \cos \delta_0 + \cos \delta \sin \delta_0 \cos (\tau_0 - \tau) \\ \cos \tfrac{1}{2} s &= \sin \delta \sin \delta_0 + \cos \delta \cos \delta_0 \cos (\tau_0 - \tau) \\ \sin \tfrac{1}{2} s \sin p &= \cos \delta' \sin (\tau_0 - \tau') \\ \sin \tfrac{1}{2} s \cos p &= \sin \delta' \cos \delta_0 - \cos \delta' \sin \delta_0 \cos (\tau_0 - \tau') \\ \cos \tfrac{1}{2} s &= \sin \delta' \sin \delta_0 + \cos \delta' \cos \delta_0 \cos (\tau_0 - \tau') \end{aligned}\right\} \qquad (297)$$

m these equations we must eliminate δ, τ, δ', and τ', since the ues of s and p, resulting from the observation, are to be rived only from the declination δ_0 and hour angle τ_0 of the ddle point between the stars, and from the data obtained from e instrument. For brevity, let us write u and v instead of n $\Delta \sin \pi$ and $\tan \Delta \cos \pi$, and u' and v' instead of $\tan \Delta' \sin \pi'$ nd $\tan \Delta' \cos \pi'$. Also, put r and r' for $\sqrt{(1 + uu + vv)}$ and $/(1 + u'u' + v'v')$. The equations (293) and (296) become

$$\begin{aligned} r \sin \delta &= \sin \delta_1 + v \cos \delta_1 \\ r \cos \delta \cos (\tau_1 - \tau) &= \cos \delta_1 - v \sin \delta_1 \\ r \cos \delta \sin (\tau_1 - \tau) &= u \end{aligned}$$

and

$$\begin{aligned} r' \sin \delta' &= \sin \delta_1 + v' \cos \delta_1 \\ r' \cos \delta' \cos (\tau_1 - \tau') &= \cos \delta_1 - v' \sin \delta_1 \\ r' \cos \delta' \sin (\tau_1 - \tau') &= u' \end{aligned}$$

These, combined with (297), give

$$
\begin{aligned}
r\sin\tfrac{1}{2}s\sin p &= -\cos\delta_1\sin(\tau_0-\tau_1)-u\cos(\tau_0-\tau_1)+v\sin\delta_1\sin(\tau_0-\tau_1)\\
r\sin\tfrac{1}{2}s\cos p &= -\sin\delta_1\cos\delta_0+\cos\delta_1\sin\delta_0\cos(\tau_0-\tau_1)-u\sin\delta_0\sin(\tau_0-\tau_1)\\
&\qquad -v\,[\cos\delta_1\cos\delta_0+\sin\delta_1\sin\delta_0\cos(\tau_0-\tau_1)]\\
r\cos\tfrac{1}{2}s &= \sin\delta_1\sin\delta_0+\cos\delta_1\cos\delta_0\cos(\tau_0-\tau_1)-u\cos\delta_0\sin(\tau_0-\tau_1)\\
&\qquad +v\,[\cos\delta_1\sin\delta_0-\sin\delta_1\cos\delta_0\cos(\tau_0-\tau_1)]
\end{aligned}
\tag{298}
$$

and

$$
\begin{aligned}
r'\sin\tfrac{1}{2}s\sin p &= \cos\delta_1\sin(\tau_0-\tau_1)+u'\cos(\tau_0-\tau_1)-v'\sin\delta_1\sin(\tau_0-\tau_1)\\
r'\sin\tfrac{1}{2}s\cos p &= \sin\delta_1\cos\delta_0-\cos\delta_1\sin\delta_0\cos(\tau_0-\tau_1)+u'\sin\delta_0\sin(\tau_0-\tau_1)\\
&\qquad +v'\,[\cos\delta_1\cos\delta_0+\sin\delta_1\sin\delta_0\cos(\tau_0-\tau_1)]\\
r'\cos\tfrac{1}{2}s &= \sin\delta_1\sin\delta_0+\cos\delta_1\cos\delta_0\cos(\tau_0-\tau_1)-u'\cos\delta_0\sin(\tau_0-\tau_1)\\
&\qquad +v'\,[\cos\delta_1\sin\delta_0-\sin\delta_1\cos\delta_0\cos(\tau_0-\tau_1)]
\end{aligned}
\tag{299}
$$

These equations not only determine s and p, but also give a relation between δ_0, τ_0 and δ_1, τ_1. To find this relation, multiply the first two equations of (298) by r', and the first two of (299) by r, and subtract the former products from the latter: we find

$$
\begin{aligned}
0 &= (r+r')\cos\delta_1\sin(\tau_0-\tau_1)+(r'u+ru')\cos(\tau_0-\tau_1)-(r'v+rv')\sin\delta_1\sin(\tau_0-\tau_1)\\
0 &= (r+r')\,[\sin\delta_1\cos\delta_0-\cos\delta_1\sin\delta_0\cos(\tau_0-\tau_1)]+(r'u+ru')\sin\delta_0\sin(\tau_0-\tau_1)\\
&\qquad +(r'v+rv')\,[\cos\delta_1\cos\delta_0+\sin\delta_1\sin\delta_0\cos(\tau_0-\tau_1)]
\end{aligned}
$$

which, if we put

$$
\left.
\begin{aligned}
\tan g\sin G &= \frac{r'u+ru'}{r+r'}\\
\tan g\cos G &= \frac{r'v+rv'}{r+r'}
\end{aligned}
\right\}
\tag{300}
$$

may be written in the following form:

$$
\begin{aligned}
0 &= [\cos\delta_1-\sin\delta_1\tan g\cos G]\sin(\tau_0-\tau_1)+\tan g\sin G\cos(\tau_0-\tau_1)\\
\frac{\sin\delta_1+\cos\delta_1\tan g\cos G}{\tan\delta_0} &= [\cos\delta_1-\sin\delta_1\tan g\cos G]\cos(\tau_0-\tau_1)-\tan g\sin G\sin(\tau_0-\tau_1)
\end{aligned}
$$

If we multiply each of these by $\cos g$, and then introduce the auxiliaries h and H, determined by the conditions

$$
\left.
\begin{aligned}
\sin h &= \sin g\sin G\\
\cos h\sin H &= \sin g\cos G\\
\cos h\cos H &= \cos g
\end{aligned}
\right\}
\tag{301}
$$

we shall have

$$
\begin{aligned}
0 &= \cos h\cos(\delta_1+H)\sin(\tau_0-\tau_1)+\sin h\cos(\tau_0-\tau_1)\\
\frac{\cos h\sin(\delta_1+H)}{\tan\delta_0} &= \cos h\cos(\delta_1+H)\cos(\tau_0-\tau_1)-\sin h\sin(\tau_0-\tau_1)
\end{aligned}
$$

om which we deduce

$$\frac{\cos h \sin(\delta_1 + H)}{\tan \delta_0} \cos(\tau_0 - \tau_1) = \cos h \cos(\delta_1 + H)$$

$$\frac{\cos h \sin(\delta_1 + H)}{\tan \delta_0} \sin(\tau_0 - \tau_1) = -\sin h$$

d the sum of the squares of these gives, by a simple reduction,

$$\cos h \sin(\delta_1 + H) = \sin \delta_0$$

y the combination of the last three equations we have, therefore,

$$\left.\begin{aligned} \sin \delta_0 &= \cos h \sin(\delta_1 + H) \\ \cos \delta_0 \cos(\tau_0 - \tau_1) &= \cos h \cos(\delta_1 + H) \\ \cos \delta_0 \sin(\tau_0 - \tau_1) &= -\sin h \end{aligned}\right\} \quad (302)$$

If we regard δ_1 and τ_1 as given by the declination and hour circles of the instrument, with the aid of (289), we can employ these equations to obtain δ_0 and τ_0; or, if δ_0 and τ_0 be regarded as known, we can employ the same equations to obtain δ_1 and τ_1, and then the reading of the declination and hour circles is altogether dispensed with.

The values of s and p will be derived from the following equations, which are obtained by adding (298) and (299):

$$\left.\begin{aligned} (r+r')\sin\tfrac{1}{2}s\sin p &= (u'-u)\cos(\tau_0-\tau_1)-(v'-v)\sin\delta_1\sin(\tau_0-\tau_1) \\ (r+r')\sin\tfrac{1}{2}s\cos p &= (u'-u)\sin\delta_0\sin(\tau_0-\tau_1) \\ &\quad +(v'-v)\left[\cos\delta_1\cos\delta_0+\sin\delta_1\sin\delta_0\cos(\tau_0-\tau_1)\right] \\ (r+r')\cos\tfrac{1}{2}s &= 2\left[\sin\delta_1\sin\delta_0+\cos\delta_1\cos\delta_0\cos(\tau_0-\tau_1)\right] \\ &\quad -(u'+u)\cos\delta_0\sin(\tau_0-\tau_1) \\ &\quad +(v'+v)\left[\cos\delta_1\sin\delta_0-\sin\delta_1\cos\delta_0\cos(\tau_0-\tau_1)\right] \end{aligned}\right\} \quad (303)$$

In these rigorous formulæ, every thing in the second members is known. But it will never be necessary to employ them in this rigorous form, except when the two stars are so near to the pole that the quantities u, v, u', v' can no longer be regarded as small in relation to the polar distance. In almost all cases, therefore, an approximate development of the formulæ will suffice; and this I proceed to consider.

273. The approximate development of the equations (303), when the terms involving the third and higher powers of u, v, u', v' are neglected, is extremely simple, and would lead us to the formulæ usually given for the heliometer. But it is easy to see that such a development is not sufficiently exact, even for s near the equator, when their distance approaches to the maxin limit (of about 2°) which the instrument is capable of measur unless a special method of observation is exclusively emplc by which the terms of the higher orders are rendered practic insensible. The nature of such methods of observation wil seen hereafter; but, in order to obtain the most generally us formulæ, which can afterwards be simplified and adapte special cases, I shall follow out the very precise developm given by BESSEL, in which the terms of the third order retained.

In order to develop the equations (303) as far as terms of third order in u, v, u', v', it is necessary to develop the fac by which $u' - u$, $v' - v$, $u' + v$, $v' + v$ are multiplied, as fa terms of the second order only. If in (300) we substitute values of $r = \sqrt{(1 + uu + vv)}$ and $r' = \sqrt{(1 + u'u' + v'v')}$, develop the expressions, we shall find that when terms of third order are neglected they are reduced to

$$\begin{aligned}\tan g \sin G &= \tfrac{1}{2}(u' + u)\\ \tan g \cos G &= \tfrac{1}{2}(v' + v)\end{aligned}$$

and consequently we shall have, with the same degree of appr mation,

$$\begin{aligned}\sin g \sin G &= \tfrac{1}{2}(u' + u)\\ \sin g \cos G &= \tfrac{1}{2}(v' + v)\\ \cos g &= 1 - \tfrac{1}{8}(u' + u)^2 - \tfrac{1}{8}(v' + v)^2\end{aligned}$$

The equations (302), by the substitution of the values of h an according to (301), become

$$\begin{aligned}\sin \delta_0 &= \sin \delta_1 \cos g + \cos \delta_1 \sin g \cos G\\ \cos \delta_0 \cos(\tau_0 - \tau_1) &= \cos \delta_1 \cos g - \sin \delta_1 \sin g \cos G\\ \cos \delta_0 \sin(\tau_0 - \tau_1) &= - \sin g \sin G\end{aligned}$$

from which follow, also,

$$\begin{aligned}\cos g &= \sin \delta_1 \sin \delta_0 + \cos \delta_1 \cos \delta_0 \cos(\tau_0 - \tau_1)\\ \sin g \cos G &= \cos \delta_1 \sin \delta_0 - \sin \delta_1 \cos \delta_0 \cos(\tau_0 - \tau_1)\\ \sin g \sin G &= - \cos \delta_0 \sin(\tau_0 - \tau_1)\end{aligned}$$

With the aid of these equations the required developmen (303) is readily obtained. We find

$$
\begin{aligned}
(r+r')\sin\tfrac{1}{2}s\sin p &= (u'-u)\left[1-\tfrac{1}{8}(u'+u)^2-\tfrac{1}{8}(u'+u)^2\tan^2\delta_0\right] \\
&\quad +(v'-v)\left[\tfrac{1}{2}(u'+u)\tan\delta_0-\tfrac{1}{4}(u'+u)(v'+v)\right] \\
(r+r')\sin\tfrac{1}{2}s\cos p &= (v'-v)\left[1-\tfrac{1}{8}(v'+v)^2-\tfrac{1}{8}(u'+u)^2\tan^2\delta_0\right] \\
&\quad -\tfrac{1}{2}(u'-u)(u'+u)\tan\delta_0 \\
(r+r')\cos\tfrac{1}{2}s &= 2\left[1+\tfrac{1}{8}(u'+u)^2+\tfrac{1}{8}(v'+v)^2\right]
\end{aligned}
$$

or, dividing the first two of these by the third,

$$
\left.\begin{aligned}
2\tan\tfrac{1}{2}s\sin p &= (u'-u)\left[1-\tfrac{1}{4}(u'+u)^2-\tfrac{1}{8}(v'+v)^2-\tfrac{1}{8}(u'+u)^2\tan^2\delta_0\right] \\
&\quad +(v'-v)\left[\tfrac{1}{2}(u'+u)\tan\delta_0-\tfrac{1}{4}(u'+u)(v'+v)\right] \\
2\tan\tfrac{1}{2}s\cos p &= (v'-v)\left[1-\tfrac{1}{4}(v'+v)^2-\tfrac{1}{8}(u'+u)^2-\tfrac{1}{8}(u'+u)^2\tan^2\delta_0\right] \\
&\quad -\tfrac{1}{2}(u'-u)(u'+u)\tan\delta_0
\end{aligned}\right\} \quad (304)
$$

in which we are now to substitute convenient expressions for $u'-u$, $v'-v$, $u'+u$, $v'+v$.

It is expedient in practice to make all our observations depend upon but one of the micrometer screws of the two semi-lenses, since all the time that we may have to devote to the investigation of the errors of the screws may then be expended upon this one. Let us suppose the micrometer screw of the semi-lens II. to be thus adopted, and let w denote the angle between the lines of motion of the semi-lens II. and of the ocular, so that

$$w = (n-k') - (\nu - \varkappa)$$

and let f and F be determined by the conditions

$$
\begin{aligned}
f\sin F &= \tan R\left[(m-a)\sin(k'-k)+b\cos(k'-k)+(\mu-\alpha)\sin w-\beta\cos w\right] \\
f\cos F &= \tan R\left[(m-a)\cos(k'-k)-b\sin(k'-k)-(\mu-\alpha)\cos w-\beta\sin w\right]
\end{aligned}
\quad (305)
$$

Multiplying these respectively by $\cos(n-k')$ and $\sin(n-k')$, and also by $-\sin(n-k')$ and $\cos(n-k')$, the sums of the products are, by (292),

$$
\left.\begin{aligned}
u &= f\sin(n-k'+F) \\
v &= f\cos(n-k'+F)
\end{aligned}\right\} \quad (306)
$$

from which it follows that f and $n-k'+F$ are the same as $\tan\varDelta$ and π.

If we also assume S and E to be determined by the conditions

$$
\begin{aligned}
2\tan\tfrac{1}{2}S\sin E &= \tan R\left[-(m-a)\sin(k'-k)+b'-b\cos(k'-k)\right] \\
2\tan\tfrac{1}{2}S\cos E &= \tan R\left[(m'-a')-(m-a)\cos(k'-k)+b\sin(k'-k)\right]
\end{aligned}
\quad (307)
$$

we shall find, by means of the multiplication and addition above employed, and by comparison with (292) and (295),

$$\left.\begin{aligned} u' - u &= 2\tan\tfrac{1}{2}S\sin(n - k' + E) \\ v' - v &= 2\tan\tfrac{1}{2}S\cos(n - k' + E) \end{aligned}\right\} \quad (308)$$

and from (306) and (308),

$$\left.\begin{aligned} \tfrac{1}{2}(u' + u) &= \tan\tfrac{1}{2}S\sin(n - k' + E) + f\sin(n - k' + F) \\ \tfrac{1}{2}(v' + v) &= \tan\tfrac{1}{2}S\cos(n - k' + E) + f\cos(n - k' + F) \end{aligned}\right\} \quad (309)$$

To facilitate the substitution of these values in (304), let us put

$$q = n - k' + E \qquad u_1 = \tfrac{1}{2}(u' + u) \qquad v_1 = \tfrac{1}{2}(v' + v)$$

we shall then have

$$\frac{\tan\frac{1}{2}s}{\tan\frac{1}{2}S}\sin p = \sin q\,(1 - u_1^2 - \tfrac{1}{2}v_1^2 - \tfrac{1}{2}u_1^2\tan^2\delta_0) + \cos q\,(u_1\tan\delta_0 - u_1 v_1)$$

$$\frac{\tan\frac{1}{2}s}{\tan\frac{1}{2}S}\cos p = \cos q\,(1 - v_1^2 - \tfrac{1}{2}u_1^2 - \tfrac{1}{2}u_1^2\tan^2\delta_0) - \sin q\,.\,u_1\tan\delta_0$$

Multiplying these respectively by cos q and $-$ sin q, and again by sin q and cos q, the sums of the products are

$$\frac{\tan\frac{1}{2}s}{\tan\frac{1}{2}S}\sin(p - q) = u_1\tan\delta_0 - \tfrac{1}{2}\cos q\,[2v_1(u_1\cos q - v_1\sin q) + (u_1^2 + v_1^2)\sin q]$$

$$\frac{\tan\frac{1}{2}s}{\tan\frac{1}{2}S}\cos(p - q) = 1 - (u_1^2 + v_1^2) - \tfrac{1}{2}u_1^2\tan^2\delta_0 + \tfrac{1}{2}(u_1\cos q - v_1\sin q)^2$$

The square root of the sum of the squares of these equations, neglecting terms of the 4th degree in their second members, gives

$$\tan\tfrac{1}{2}s = \tan\tfrac{1}{2}S\,[1 - (u_1^2 + v_1^2) + \tfrac{1}{2}(u_1\cos q - v_1\sin q)^2]$$

and their quotient gives tan $(p - q)$, for which we may write $p - q$; whence

$$p - q = u_1\tan\delta_0 - \tfrac{1}{2}\cos q\,[2v_1(u_1\cos q - v_1\sin q) + (u_1^2 + v_1^2)\sin q]$$

But with the notation just adopted, the expressions (309) become

$$\begin{aligned} u_1 &= \tan\tfrac{1}{2}S\sin q + f\sin(q + F - E) \\ v_1 &= \tan\tfrac{1}{2}S\cos q + f\cos(q + F - E) \end{aligned}$$

whence, also,

$$u_1^2 + v_1^2 = \tan^2\tfrac{1}{2}S + 2f\tan\tfrac{1}{2}S\cos(F - E) + f^2$$

$$u_1\cos q - v_1\sin q = f\sin(F - E)$$

by the substitution of which we obtain

$$\begin{aligned}\tan\tfrac{1}{2}s &= \tan\tfrac{1}{2}S\left\{1-\tan^2\tfrac{1}{2}S-2f\tan\tfrac{1}{2}S\cos(F-E)-\tfrac{1}{2}f^2[1+\cos^2(F-E)]\right\}\\ p &= q+[\tan\tfrac{1}{2}S\sin q+f\sin(q+F-E)]\tan\delta_0\\ &\quad -\tfrac{1}{2}\cos q\,[\tan^2\tfrac{1}{2}S\sin q+2f\tan\tfrac{1}{2}S\sin(q+F-E)\\ &\quad +f^2\sin(q+2F-2E)]\end{aligned}\tag{310}$$

In the terms of the order of $\tan^2\tfrac{1}{2}S$, we may put p for q; but in those of the order of $\tan\tfrac{1}{2}s$, in the first line of the value of p, we shall employ the more accurate value

$$q = p-[\tan\tfrac{1}{2}S\sin p+f\sin(p+F-E)]\tan\delta_0$$

Dividing the first equation of (310) by $1-\tan^2\tfrac{1}{2}S$, the first member becomes $\tfrac{1}{2}\tan s$, within the degree of approximation here adopted, and in the small terms we may put $\tfrac{1}{2}s$ for $\tan\tfrac{1}{2}S$. The equations thus become

$$\begin{aligned}\tan s &= 2\tan\tfrac{1}{2}S\left\{1-fs\cos(F-E)-\tfrac{1}{2}f^2[1+\cos^2(F-E)]\right\}\\ p &= n-k'+E+[\tfrac{1}{2}s\sin p+f\sin(p+F-E)]\tan\delta_0\\ &\quad -[\tfrac{1}{8}s^2\sin p+\tfrac{1}{2}fs\sin(p+F-E)+\tfrac{1}{2}f^2\sin(p+2F-2E)]\cos p\\ &\quad -[\tfrac{1}{8}s^2\sin 2p+\tfrac{1}{2}fs\sin(2p+F-E)+\tfrac{1}{2}f^2\sin(2p+2F-2E)]\tan^2\delta_0\end{aligned}$$

These may, however, be still further simplified. The angle E is, in general, either very small or very nearly 180°, according as $m'-a'-(m-a)$ is a positive or negative quantity in (307). The case must be excepted in which the distance s is itself so small as to be regarded as of the same order as $k'-k$ and $b'-b$; but in this case the terms involving E are themselves so small that they can be wholly neglected. Putting, therefore, in the small terms, $E=0$ or $=180°$, and also substituting the value of $k'=n_0'-\lambda$, and of λ by (290), we have, finally,

$$\left.\begin{aligned}\tan s &= 2\tan\tfrac{1}{2}S[1\mp fs\cos F-\tfrac{1}{2}f^2(1+\cos^2F)]\\ p &= n-n_0'+E+[\gamma\sin(\tau_0-\vartheta)+i_1]\sec\delta_0\\ &\quad +[\tfrac{1}{2}s\sin p\pm f\sin(p+F)-c-e\cos\varphi\sin\tau_0]\tan\delta_0\\ &\quad -[\tfrac{1}{8}s^2\sin p\pm\tfrac{1}{2}fs\sin(p+F)+\tfrac{1}{2}f^2\sin(p+2F)]\cos p\\ &\quad -[\tfrac{1}{8}s^2\sin 2p\pm\tfrac{1}{2}fs\sin(2p+F)+\tfrac{1}{2}f^2\sin(2p+2F)]\tan^2\delta_0\end{aligned}\right\}\tag{311}$$

in which the upper or the lower sign is to be taken according as $m'-a'-(m-a)$ is positive or negative. In the value of λ (290), we have here substituted τ_0 and δ_0 for τ_1 and δ_1, which will produce no appreciable error.

The angle p here expresses the position angle *from* the star whose image is formed by the semi-lens I. *to* the star whose image is formed by the semi-lens II. It is also to be observed that we have employed the formulæ for the equatorial instrument as given for the case in which the declination circle *precedes* the telescope: so that, according to Arts. 248 and 250, when the declination circle *follows*, τ_0 will be the hour angle increased by 180°, and δ_0 will be the supplement of the declination; consequently, also, p will be the position angle increased by 180°.

274. The coincidence of the images of the two stars S and S' can be produced at the point O (Art. 271) in two different ways, namely, by opposite motions of the semi-lens II. relatively to the semi-lens I. By the combination of the observations made in these two ways, we shall be able to eliminate a, a', b, b', $k'-k$, and it will no longer be necessary to determine these quantities.

Let us suppose the semi-lens I. to remain in the same position as in the first observation, and that the semi-lens II. is now moved in a direction opposite to that of its former motion until the second coincidence of the images is produced. This will, in general, require a common revolution, to a small extent, of the two lenses about the heliometer axis, thus slightly changing the reading of the position circle, which reading we shall now denote by n_1. Let the reading of the micrometer in this observation be m_1', and let the corresponding values of S, E, and p be denoted by S_1, E_1, and p_1. The formulæ (307) and (311), with these changes, will then apply to this second observation, and (307) will become

$$2\tan\tfrac{1}{2}S_1\sin E_1 = \tan R\,[-(m-a)\sin(k'-k)+b'-b\cos(k'-k)]$$
$$2\tan\tfrac{1}{2}S_1\cos E_1 = \tan R\,[m_1'-a'-(m-a)\cos(k'-k)+b\sin(k'-k)]$$

Since $m_1'-a'$ and $m'-a'$ fall upon opposite sides of $m-a$, the quantities $2\tan\frac{1}{2}S_1\cos E_1$ and $2\tan\frac{1}{2}S\cos E$ have opposite signs, but $2\tan\frac{1}{2}S_1\sin E_1$ and $2\tan\frac{1}{2}S\sin E$ are equal; from which it follows (since S_1 and S can differ only by terms of the 3d order) that E_1 differs from $180°-E$ only by terms of the order of the product of $k'-k$ into s^2, and this difference may be regarded as altogether insensible. In the application of (311) to the second observation, therefore, the meaning of the double sign will be reversed. We can, however, avoid all the difficulty in distinguishing the cases in which E is to be taken greater or less than 90°, by calling that observation *the first*, for which $E < 90°$, and applying to it the notation m', n. Under this condition, the upper signs of (311) will be used for the first observation and the lower signs for the second; and the value of p_1 for the second observation will be $180° + p$.

The formulæ for the two observations may, therefore, be expressed as follows, where we introduce the value of $2\tan\frac{1}{2}S$ given by the second equation of (307) after neglecting the insensible terms (which terms, however, even if they were sensible, would be eliminated by the subsequent combination of the two observations):

1st Observation.

$$\begin{aligned}
\tan s &= \tan R\,\frac{(m'-a'-m+a)}{\cos E}\left[1-fs\cos F-\tfrac{1}{2}f^2(1+\cos^2 F)\right]\\
p &= n-n_0'+E+[\gamma\sin(\tau_0-\vartheta)+i_1]\sec\delta_0\\
&\quad+[\tfrac{1}{2}s\sin p+f\sin(p+F)-c-e\cos\varphi\sin\tau_0]\tan\delta_0\\
&\quad-[\tfrac{1}{8}s^2\sin p+\tfrac{1}{2}fs\sin(p+F)+\tfrac{1}{2}f^2\sin(p+2F)]\cos p\\
&\quad-[\tfrac{1}{8}s^2\sin 2p+\tfrac{1}{2}fs\sin(2p+F)+\tfrac{1}{2}f^2\sin(2p+2F)]\tan^2\delta_0
\end{aligned}$$

2d Observation.

$$\begin{aligned}
\tan s &= \tan R\,\frac{(m-a-m_1'+a')}{\cos E}\left[1+fs\cos F-\tfrac{1}{2}f^2(1+\cos^2 F)\right]\\
p &= n_1-n_0'-E+[\gamma\sin(\tau_0-\vartheta)+i_1]\sec\delta_0\\
&\quad+[-\tfrac{1}{2}s\sin p+f\sin(p+F)-c-e\cos\varphi\sin\tau_0]\tan\delta_0\\
&\quad-[\tfrac{1}{8}s^2\sin p-\tfrac{1}{2}fs\sin(p+F)+\tfrac{1}{2}f^2\sin(p+2F)]\cos p\\
&\quad-[\tfrac{1}{8}s^2\sin 2p-\tfrac{1}{2}fs\sin(2p+F)+\tfrac{1}{2}f^2\sin(2p+2F)]\tan^2\delta_0
\end{aligned}$$

From the mean of the two observations, we have

$$\left.\begin{aligned}
\tan s &= \tan R\,\frac{m'-m_1'}{2\cos E}\left[1-\tfrac{1}{2}f^2(1+\cos^2 F)\right]\\
p &= \frac{n+n_1}{2}-n_0'+[\gamma\sin(\tau_0-\vartheta)+i_1]\sec\delta_0\\
&\quad+[f\sin(p+F)-c-e\cos\varphi\sin\tau_0]\tan\delta_0\\
&\quad-\tfrac{1}{16}s^2\sin 2p\,(1+2\tan^2\delta_0)\\
&\quad-\tfrac{1}{2}f^2[\sin(p+2F)\cos p+\sin(2p+2F)\tan^2\delta_0]
\end{aligned}\right\}\quad(312)$$

The value of E, obtained from the difference of the two values of p, is

$$\begin{aligned}
E=\frac{n_1-n}{2}&-\tfrac{1}{2}s\sin p\tan\delta_0\\
&+\tfrac{1}{2}fs[\sin(p+F)\cos p+\sin(2p+F)\tan^2\delta_0]
\end{aligned}\quad(313)$$

But it will not usually be necessary to regard the divisor $\cos E$ in the formula for $\tan s$, for it can differ sensibly from unity only in those cases in which s is an extremely small quantity, and these cases we may take $E = \frac{1}{2}(n_1 - n)$.

The method of observation with the heliometer, in which tw corresponding observations in opposite positions of the sem lenses are combined, may be regarded as fundamental and essential. The same degree of accuracy which it affords cannot b attained by single observations, the reduction of which requir an accurate determination of the quantities a, a', b, b', $k' - k$ for, in addition to the uncertainty of such determinations fo any given position of the instrument, it is not certain that th values of these quantities are really constant for all positions of the telescope with respect to the horizon. It is true that our formulæ still involve f and F, which depend upon a, a', &c.; but a *precise* determination of these quantities is no longer necessary, since they enter only into the small terms of the formulæ. Moreover, by a proper method of observation, f and F may be dispensed with altogether, as I next proceed to show.

275. Assuming that a complete observation always consists of two corresponding observations, as in the preceding article, there are yet three different methods of making such an observation, each of which offers some advantage over the others. These I propose to consider separately.

First Method of Observation.—Let the semi-lens which is to remain fixed during the observation be set so that its sight line shall be parallel to the heliometer axis. This will be effected by making $m - a = \mu - \alpha$, and at the same time $n - k = \nu - \varkappa$, or, in the most simple manner, by making $m - a = \mu - \alpha = 0$. We shall then have $f = 0$, and the formulæ (312) become

$$\left.\begin{aligned} \tan s &= \tan R\,\frac{m' - m_1'}{2\cos E} \\ p &= \frac{n + n_1}{2} - n_0' + [\gamma \sin(\tau_0 - \vartheta) + i_1] \sec \delta_0 \\ &\quad - (c + e\cos\varphi \sin\tau_0)\tan\delta_0 - \tfrac{1}{16} s^2 \sin 2p\,(1 + 2\tan^2\delta_0) \end{aligned}\right\} \quad (314)$$

This method recommends itself by the symmetry which it give to the observations, as well as by the simplicity of their reduction

Second Method.—In this method, we make the lines of motio of the objective and ocular parallel, or $w = 0$, and also mak $m = a$; but the ocular is moved between the two observations being set for one observation so that $\mu - \alpha = \frac{1}{2}(m' - a')$, an for the other so that $\mu - a = \frac{1}{2}(m_1' - a')$. We then have $f = \frac{1}{2}s$ and $F = 180°$ for one observation, but $F = 0$ for the other. These changes must be made in the two sets of formulæ from which (312) were obtained; for in the combination expressed by (312) the ocular was supposed to have the same position in both observations. Here, however, we must put $F = 180°$ in the first and $F = 0$ in the second, at the same time substituting $\frac{1}{2}s$ for f, and then make the combination: we thus obtain

$$\left.\begin{aligned} \tan s &= \tan R \frac{m' - m_1'}{2 \cos E} \\ p &= \frac{n + n_1}{2} - n_0' + [\gamma \sin(\tau_0 - \vartheta) + i_1] \sec \delta_0 \\ &\qquad - (c + e \cos \varphi \sin \tau_0) \tan \delta_0 \end{aligned}\right\} \quad (315)$$

In this method, the rays from the two stars make the same angle ($= \frac{1}{2}s$) with the optical axis of each semi-lens; whereas in the first method the rays from one star make the angle s with this axis and those from the other star are parallel to the axis. The second method, therefore, offers the advantage of bringing both images at equal distances from the axis, thereby producing equal distinctness and accuracy of definition in them, and avoiding the defects of the lens, which appear more prominently as the rays fall more obliquely. The greater simplicity of the first method in the observation will, however, give it the preference so long as the distance to be measured is not so great as to carry one of the objects beyond the limits of distinct vision.

Third Method.—This combines the advantage of the second method with the simplicity of the first. We place the ocular permanently in the heliometer axis, and make each observation with the semi-lenses at equal distances from that axis and on opposite sides of it. The chief objection to this method is that, since both lenses are moved, it becomes necessary to know the value of a revolution of the screws of both; but, as has been already remarked in Art. 273, it is expedient to devote all our attention to the investigation of the errors of but one screw. It may also be objected to this method that, when the distance to be measured is rapidly changing, time will be lost in effecting the requisite symmetrical arrangement of the observations. This objection, however, may be made with even greater force against the second method; but the first method is free from it.

With any of these methods, if we wish to free the results from the effects of flexure of the declination axis and from the inclination of this axis to the hour axis, without supposing i_1 an c to be known, we take two complete observations (*i.e.* pairs o observations) in the two positions of the declination circle, *preceding* and *following;* for we see by (314) and (315) that i_1 and c will vanish from the mean of these two observations.

In Art. 263, we have seen that $\frac{1}{16}s^2 \sin 2p\,(1 + 2\tan^2\delta_0)$ is the correction to be added to the position angle at the middle point between the two stars to reduce it to the mean $(= p_0)$ of the position angles at the two stars: consequently, if we neglect this term in the first method of observation above given, the resulting position angle will be at once the mean position angle p_0, with which and the distance s we find the differences of declination and right ascension of the stars, by Art. 264. The results are yet to be freed from the effect of refraction, by the methods hereafter to be given.

276. I have thus far assumed that the contact of the images is always produced at a certain *known* point (o) of the plane of motion of the ocular. It will be well always to make the contacts at the middle point of the field, but the position of thi point will usually be *estimated* only, unless it is indicated by square formed of intersecting threads or some equivalent contrivance, which, however, involves the necessity of illuminating the field or the threads. Let us inquire, therefore, to wha extent an erroneous estimate of the position of the middle of the field will affect the observed measures.

The quantities f and F, determined by (305), express the actua position of the middle of the field (o); but if the point of contact is a different point (o'), the values given by the formula require a correction.

Let h denote the angular distance of o' from o, and H th angle which oo' makes with the observed arc SS', H and w being reckoned in the same direction. The quantities $\tan R.\,(\mu - \alpha) \sin w$ and $\tan R.\,(\mu - \alpha) \cos w$, which express the angular distances of the point o from SS', and from a perpendicular to SS' drawn through the heliometer axis, must be increased by $h \sin H$ and $h \cos H$ respectively. Consequently, $f \sin F$ and $f \cos F$ will require the corrections $h \sin H$ and $- h \cos H$: hence, if we suppos h to be so small that its square may be neglected, the effect upo $\tan s$ will be, by (311),

$$\pm hs^2 \cos H + hsf\,(2 \cos F \cos H - \sin F \sin H)$$

and the effect upon the position angle will be

$$\mp h \sin(p - H) \tan \delta_0 \pm \tfrac{1}{2} hs[\sin(p - H) \cos p + \sin(2p - H) \tan^2 \delta_0]$$
$$+ hf[\sin(p + F - H) \cos p + \sin(2p + F - H) \tan^2 \delta]$$

Since h will be but a few minutes in any case, it follows that the effect upon the distance will be usually inappreciable even for he greatest values of s and f. The first and principal term of e effect upon the position angle is proportional to the tangent f the declination; but it vanishes when $\sin(p - H) = 0$, that , when $H = p$, or $H = p + 180°$, or when the point at which e contact is made lies in the declination circle which passes rough the centre of the field. When the telescope follows the urnal motion accurately, and a contact has once been made in e centre of the field, the subsequent observations will all be ry near this point. The greater the declination, the more reful must we be to make the contacts near the declination rcle of the centre of the field; but it is evident from the preding discussion that we shall probably always be able to effect is with sufficient accuracy by estimating the position of this ntre, without resorting to the use of illuminated threads.

DETERMINATION OF THE CONSTANTS OF THE HELIOMETER.

277. *To find a, a', α.*—Direct the telescope to any fixed point, and, having brought the centre of the semi-lens I. nearly into the heliometer axis (by estimation), revolve the lens 180° about the axis. If the image of the point appears still in the same point of the field of view, the reading m of the micrometer is then evidently $= a$. If the image has moved, we have only to move the semi-lens by its micrometer screw until the image has been carried to the middle point between its first and second positions, and, if this middle point has been correctly estimated, the semi-revolution will no longer affect the apparent position of the image. By repeating this process, we shall very quickly find he exact position of the semi-lens when its centre is at the ninimum distance from the heliometer axis, for which $m = a$. n the same manner, a' will be found for the semi-lens II.; and, y a similar process, revolving the ocular 180°, α will be found.

278. *To find $k' - k$, $b' - b$.*—These quantities produce the eater influence upon the readings of the position circle, the

smaller the distance between two points whose images brought into coincidence. They will, therefore, be most ac rately determined by *complete* observations (Art. 275) of the d tance and position angle of the components of a double st Since s is in this case extremely small, we shall have $E = \frac{1}{2}(n_1 -$ and, neglecting the insensible terms in (307), the single observ tions will give

$$s \sin \tfrac{1}{2}(n_1 - n) = R\,[(m - a)(k - k') + b' - b]$$
$$s \cos \tfrac{1}{2}(n_1 - n) = R\,[m' - a' - m + a]$$

and (since in the second observation we put $180° - E$ for E)

$$s \sin \tfrac{1}{2}(n_1 - n) = R\,[(m - a)(k - k') + b' - b]$$
$$s \cos \tfrac{1}{2}(n_1 - n) = R\,[m - a - m_1' + a']$$

from the combination of which we derive

$$(m - a)(k - k') + b' - b = \tfrac{1}{2}(m' - m_1') \tan \tfrac{1}{2}(n_1 - n) \quad (31$$

in which the second member and also the coefficient of $k - k'$ a known from the observation. By setting the semi-lens I. at vario readings m, and making the contacts by moving the semi-len II., we shall thus for each complete observation have an equ tion of condition of the form (316); and since the coefficients o $k - k'$ in these equations may be made to have very differe values, the combination by the method of least squares will gi a very accurate determination of both $k - k'$ and $b' - b$.

We may here observe that it is not necessary, nor is it adva tageous, to bring the images of the stars into coincidence. will be better to bring the image of one of the componen formed by one semi-lens to the middle point between the t images of the two components formed by the other semi-le Thus, if a and b are the images of the two components form by the semi-lens I., a' and b' those formed by the semi-lens I in the first observation the images will stand thus:

$$\begin{matrix} a' & a & b' & b \\ * & * & * & * \end{matrix}$$

and in the second observation thus:

$$\begin{matrix} a & a' & b & b' \\ * & * & * & * \end{matrix}$$

As the components are supposed very close together, the bisection of their distance will be more accurately estimated than a coincidence of superposed images. This method of observation is always advisable when the distance to be measured is but a few seconds.

I should have remarked before that the quantity $k - k'$ is the difference of the index errors of the position circle for the two semi-lenses, since from the values of k and k' (291) and (294) we have

$$k - k' = n_0 - n_0'$$

279. *To find the index error* (n_0') *of the position circle.*—This is the index error for the semi-lens II., with which we suppose all our observations to be made. Let the semi-lenses be separated to any assumed distance (by setting $m - a$ and $m' - a'$ to different readings), direct the telescope upon a *fixed* point, and revolve the objective until a motion of the telescope upon the hour axis (the declination circle being clamped) causes the two images of the fixed point to come successively into the sight line, that is, into the centre of the field of the ocular. The position angle of the line joining the two images is then nearly ± 90°; but it will vary with the distance by which the semi-lenses are separated.

If the hour circle is clamped and the objective is revolved until a motion of the telescope, upon the declination axis only, causes the images to come successively into the centre of the field, the position angle of the images will be nearly 0° or 180°, but will also vary with the distance of the centres of the semi-lenses. The relation between the reading (n) of the position circle and the distance of the lenses will be investigated for each of these methods.

In either method, I shall suppose that the sight line of the semi-lens I. is made to coincide with the heliometer axis, which will be effected by setting the micrometers so that $m - a = 0$ and $\mu - \alpha = 0$.

1st. *When the telescope is revolved upon the hour axis.*—It is obviously unnecessary to consider the position of the instrument with respect to the pole of the heavens, and we may therefore express the position of the heliometer axis by formulæ which give the *instrumental* hour angle and declination of the axis. In order to show the effect of flexure, let us return to the general formulæ (258), which, by omitting the terms $\gamma \cos(\tau - \vartheta)$ and

$\gamma \sin(\tau - \vartheta) \tan \delta$, will express the declination δ_1 and hour ang τ_1 of the heliometer axis referred to the pole of the instrument. Putting D for $d + \Delta d$ and T for $t + \Delta t$, and $i_1 = i - \varepsilon \sin \varphi$, we shall put

$$\begin{aligned} \delta_1 &= D - e(\sin\varphi \cos D - \cos\varphi \sin D \cos T) && = D + \Delta D \\ \tau_1 &= T + c \sec D - i_1 \tan D + \varepsilon \cos\varphi \cos T + e \cos\varphi \sec D \sin T && = T + \Delta T \end{aligned}$$

in which φ will now denote the latitude of the instrument. The equations (293), under the form given to them in Art. 272, will now become

$$\begin{aligned} r \sin \delta &= \sin D + v \cos D + r \cos \delta \cos(T - \tau) \,.\, \Delta D \\ r \cos \delta \cos(T - \tau) &= \cos D - v \sin D - r \sin \delta \,.\, \Delta D + r \cos \delta \sin(T - \tau) \,.\, \Delta T \\ r \cos \delta \sin(T - \tau) &= u - r \cos \delta \cos(T - \tau) \,.\, \Delta T \end{aligned} \tag{317}$$

in which δ and τ are the declination and hour angle of the fixed point.

In the revolution about the hour axis, D remains constant. If the preceding equations are assumed for the case in which the image produced by the semi-lens I. is in the sight line, and we distinguish by accents those quantities which vary when the second image is brought into the sight line, we shall have, since δ is fixed,

$$\begin{aligned} \sin \delta &= \frac{1}{r} \sin D + \frac{v}{r} \cos D + \cos \delta \cos(T - \tau) \,.\, \Delta D \\ &= \frac{1}{r'} \sin D + \frac{v'}{r'} \cos D + \cos \delta \cos(T' - \tau) \,.\, \Delta D' \end{aligned}$$

as the expression of the condition that the two images of the same point are successively brought into the sight line. But, as we may neglect the products of the small quantities c, i_1, ε, e, by the squares and products of u, v, u', v', we can in the last terms put $\cos(T - \tau) = \cos(T' - \tau) = 1$, and then give the equation the form

$$\begin{aligned} \left(\frac{v'}{r'} - \frac{v}{r}\right) \cos D &= \left(\frac{1}{r} - \frac{1}{r'}\right) \sin D + \cos \delta \, (\Delta D - \Delta D') \\ &= \left(\frac{1}{r} - \frac{1}{r'}\right) \sin D + e \cos \varphi \sin D \cos \delta \, (\cos T - \cos T') \end{aligned}$$

From the second and third equations of (317) we have, with the degree of approximation here required,

$$\cos \delta \cos T = \cos D \cos \tau - v \sin D \cos \tau - u \sin \tau$$

and, therefore, also

$$\cos \delta \cos T' = \cos D \cos \tau - v' \sin D \cos \tau - u' \sin \tau$$

by means of which our equation becomes

$$\frac{v'}{r'} - \frac{v}{r} = \left(\frac{1}{r} - \frac{1}{r'}\right) \tan D + e \cos \varphi \tan D [(v' - v) \sin D \cos \tau + (u' - u) \sin \tau]$$

he mode of observation above proposed, by which we have $- a = 0$ and $\mu - \alpha = 0$, leads to a simplification of this equaion; for these conditions give also $f = 0$, and consequently, by 306), $u = v = 0$, and $r = \sqrt{(1 + uu + vv)} = 1$. We have also, y (308), under the same conditions,

$$u' = 2 \tan \tfrac{1}{2} S \sin (n - k' + E)$$
$$v' = 2 \tan \tfrac{1}{2} S \cos (n - k' + E)$$

and, consequently,

$$r' = 1 + \tfrac{1}{2}(u'u' + v'v') = 1 + 2 \tan^2 \tfrac{1}{2} S$$

Substituting these values, and neglecting terms of the order of $e \tan^2 \frac{1}{2} S$, we deduce

$$\cos (n - k' + E) = \tan \tfrac{1}{2} S \tan D + e \cos \varphi \tan D [\sin D \cos \tau \cos (n - k' + E)$$
$$+ \sin \tau \sin (n - k' + E)]$$

from which it follows that $\cos (n - k' + E)$ is of the same order as $\tan \frac{1}{2} S$, and $n - k' + E$ is nearly $= \pm 90°$. We may, therefore, in the last term, put $\cos (n - k' + E) = 0$ and $\sin (n - k' + E) = \pm 1$, and write the equation in the following form:

$$\sin [90° \mp (n - k' + E)] = \tan \tfrac{1}{2} S \tan D \pm e \cos \varphi \tan D \sin \tau$$

We shall here have to distinguish between the cases in which $n - k'$ is nearly $= 90°$ or nearly $= - 90°$. The angle E is nearly $= 0$ or nearly equal $180°$, according as $m' - a'$ is positive or negative in (307). When $n - k'$ is nearly $= + 90°$ and E is nearly $= 0$, we have $n - k' + E$ nearly $= + 90°$, and the upper ign in the second member must be used. Under the same onditions, the upper sign in the first member makes $90° - (n - k' + E)$ nearly $= 0$, and the angle may be put for its sine. hen $n - k'$ is nearly $= + 90°$ and E is nearly $= 180°$, the wer signs must be used. Hence, if we write $\sin E$ for E or r $180° - E$, we shall have, when $n - k'$ is nearly $= + 90°$,

$$(n - k' - 90°) - \sin E = \tan \tfrac{1}{2} S \tan D \pm e \cos \varphi \tan D \sin \tau \quad (318a)$$

and similarly, when $n - k'$ is nearly $= - 90°$,

$$\pm (n - k' + 90°) + \sin E = \tan \tfrac{1}{2} S \tan D \mp e \cos \varphi \tan D \sin \tau \quad (318$$

The value of k', according to (294) and (290), when we refer λ the pole of the instrument, is

$$k' = n_0' - i_1 \sec \delta_1 + c \tan \delta_1 + e \cos \varphi \tan \delta_1 \sin \tau_1$$

where the last term is equivalent to the last term of (318). I therefore, we neglect this term in (318), the value of k', whi the equations then determine, will be

$$= n_0' - i_1 \sec \delta_1 + c \tan \delta_1$$

If we suppose $k' - k$ and $b' - b$ to be known, we shall kno E from (307), and a single observation will determine k' by (318 But it will be preferable always to combine two correspondin observations in which $m' - a' - m + a$ and $m_1' - a' - m + a$ ar numerically equal but have opposite signs; then, n and n_1 bein the readings of the position circle in the two observations, w shall have from their mean

$$n_0' - i_1 \sec \delta_1 + c \tan \delta_1 = \tfrac{1}{2} (n_1 + n) \mp 90° \quad (319$$

If we set the micrometer at various readings in making thes pairs of observations, and assume that the weight of the resultin determinations is proportional to $\tfrac{1}{2} (m_1' - m')$, and if we denot the several values of $\tfrac{1}{2} (m_1' - m')$ by M, M', M'', &c., and o $\tfrac{1}{2} (n_1 + n) \mp 90°$ by N, N', N'', &c., we shall have the final mea by the formula (see Appendix, Method of Least Squares)

$$(N) = \frac{MN + M'N' + M''N'' + \&c.}{M + M' + M'' + \&c.}$$

and then

$$n_0' - i_1 \sec \delta_1 + c \tan \delta_1 = (N)$$

To eliminate the terms involving i_1 and c, we take observatio in the two opposite positions of the declination axis,—circle pr ceding and circle following,—and if (N) and (N') are the gener means found in the two positions, we shall have

$$n_0' = \tfrac{1}{2} [(N) + (N')] \quad (32$$

We see that the index error will be found independently of a other quantities, by taking the mean of the readings in four observations, two in each position of the declination axis.

2d. *When the telescope is revolved upon the declination axis.*—In this case T is constant and D varies. The condition that the two images are successively brought into the centre of the field will be expressed by equating the two values of $\cos\delta \sin(T - \tau)$ given by the last equation of (817). Putting $\cos(T - \tau) = 1$ in the last term of this equation, we find

$$\frac{u}{r} - \cos\delta \,.\, \Delta T = \frac{u'}{r'} - \cos\delta \,.\, \Delta T'$$

or, by the same method of observation as we employed above, making $f = 0$, and, consequently, also $u = v = 0$, and $r = 1$,

$$\begin{aligned} u' &= r' \cos\delta\,(\Delta T' - \Delta T) \\ &= r' \cos\delta\,[i_1(\tan D - \tan D') - (c + e\cos\varphi \sin T)(\sec D - \sec D')] \end{aligned}$$

which, with the same degree of approximation as was observed above, may be reduced to

$$u' = r'v'\,[i_1 \sec\delta - (c + e\cos\varphi \sin T)\tan\delta]$$

Substituting $\tan(n - k' + E)$ for $\frac{u'}{v'}$, and $r' = 1$ (which involves only errors of the order of $\tan^2 \frac{1}{2} S$ multiplied by i_1, c, e), we have

$$\tan(n - k' + E) = i_1 \sec\delta - (c + e\cos\varphi \sin T)\tan\delta$$

Hence $n - k' + E$ is very small or very nearly $= 180°$. When $n - k'$ is nearly $= 0$, we shall have, for the two cases of E,

$$n - k' \pm \sin E = i_1 \sec\delta - (c + e\cos\varphi \sin T)\tan\delta \qquad (821a)$$

and, when $n - k'$ is nearly $= 180°$,

$$n - k' \mp \sin E = i_1 \sec\delta - (c + e\cos\varphi \sin T)\tan\delta \qquad (821b)$$

If we omit all the terms in the second member, the value of k' which these equations determine will be that of n_0' itself. If, then, two observations are taken in which $m' - a' - m + a$ and $m_1' - a' - m + a$ are numerically equal but have opposite signs, and if n and n_1 are the two readings of the position circle, we shall have

$$n_0' = \tfrac{1}{2}(n_1 + n)$$

Regarding the weights of the several determinations thus ma as proportional to the values of $\frac{1}{2}(m_1' - m')$, a general mean (will be found as above, and then we shall have $n_0' = (N)$.

280. From the preceding article it appears that by revolvi the telescope upon the declination axis the index error of t position circle is found independently of all other quantitie and without reversing the declination axis. We should expec therefore, that when this method is followed in both positions o that axis—that is, both with circle preceding and with circl following—the same value of n_0' will be obtained. Besse found, however, that this was by no means the case with th Königsberg heliometer; for the difference of the resulting values was sometimes as great as 4′, which is too great a difference to be ascribed wholly to errors of observation. He explains the discrepancy by supposing the telescope to have a tendency to revolve (so far as the elasticity of its materials will permit) about the point at which it is secured to the declination axis; a revolution which has the same effect upon the position angles as a revolution of the tube about the heliometer axis, and which is clearly to be distinguished from a flexure of the declination axis. Supposing the amount of the revolution to be proportional to the force which tends to produce it, the law which it follows in all positions of the instrument is easily assigned; for this force is merely that part of the weight of the telescope which acts at right angles to a plane passing through the declination axis and the heliometer axis, and is, consequently, proportional to the cosine of the zenith distance of the point of the heavens towards which the perpendicular to this plane is directed. The hour angle of this point is the same as that of the heliomete axis $= \tau_1$, and its declination differs 90° from that of the heliometer axis $= 90° + \delta_1$. Denoting the zenith distance of th point by ζ, we shall have

$$\cos \zeta = \sin \varphi \cos \delta_1 - \cos \varphi \sin \delta_1 \cos \tau_1$$

and the amount of revolution will be expressed by $\psi \cos \zeta$, i which ψ is its maximum. The observed position angles must b corrected by adding this quantity, or

$$\psi (\sin \varphi \cos \delta_1 - \cos \varphi \sin \delta_1 \cos \tau_1) \qquad (322$$

hich term must, therefore, be annexed to the formulæ for p in 14) and (315).*

281. *To find the index error* (x) *of the position circle of the ocular.*—let the semi-lens II. at any assumed distance $= m' - a'$ from he heliometer axis, and the ocular at an equal distance $= \mu - \alpha$ rom that axis. Revolve the ocular about its axis until the image f a fixed point is seen in the centre of the field. Let n and ν be the readings of the position circles of the objective and ocular. Without moving the telescope or changing n, repeat the observation with the distance $-(m' - a') = -(\mu - \alpha)$, and let ν' be the new reading of the position circle of the ocular. Then, $n - n_0'$ being the true direction of the line of motion of the semi-lens II., we have $x = \frac{1}{2}(\nu + \nu') - (n - n_0')$. It will be well to adjust the index of this circle so that its readings will agree with those of the position circle of the objective.

For the fixed point in the preceding methods of determining the index error of the position circles, it will be expedient to employ the intersection of a cross thread in the focus of an auxiliary telescope, mounted in the observing room, with its objective turned towards the heliometer; the two threads of the cross making an angle of 45° with a declination circle.

282. *To find the distance* (β) *of the line of motion of the ocular from the heliometer axis.*—Set the ocular at an assumed distance $\mu - \alpha$ from the axis, and bring the image of a fixed point into the centre of the field. Keeping the telescope fixed, set the ocular at a reading μ' such that $\mu' - \alpha = -(\mu - \alpha)$, and revolve it until the image is again seen in the centre of the field. Let ν and ν' be the readings of its position circle in the two positions; then we evidently have

$$\pm\beta = \frac{\mu - \mu'}{2} \tan \tfrac{1}{2}(180° - \nu + \nu') \qquad (323)$$

It will be easy to adjust the ocular, by means of the proper adjusting screws, so that its line of motion passes through the heliometer axis, and thus make $\beta = 0$. A small error in this adjustment will have no sensible effect upon the observations, as our formulæ show.

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* See **Bessel's** *Astron. Untersuch.*, Vol. I. pp. 45, 72. In the latter place he finds r the Königsberg heliometer ψ (which he there denotes by μ) $= 1'.914$.

283. Finally, the value of a revolution of the micrometer screw ($= R$) is to be determined with the utmost precision. Of the methods given in Chapter II. for the filar micrometer, we may regard the following as the most suitable for the heliometer:

1st. By the measurement of the focal length of the lens and of the distance between two successive threads of the micrometer screw.

2d. By the Gaussian process, or the observation of a thread in the focus of the lens with a theodolite.

3d. By the measurement of a distance otherwise known, as, for example, the distance of two stars in the group *Pleiades* determined by meridian observations.

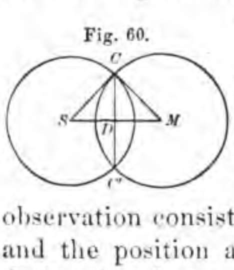
By the third method, however, we cannot expect to reach the degree of accuracy which is necessary to give the heliometer all the advantage which it should possess as a *micrometer*. This objection is obviated in a degree by measuring the successive distances between a number of stars which are nearly in the same great circle, and, having reduced these distances to the great circle joining the extreme stars, comparing the total reduced distance with the distance of the extreme stars as determined by meridian observations.

Bessel, after a careful trial of all these methods with the Königsberg heliometer, gave the preference to the first. I must refer the reader to his elaborate researches upon this instrument (already referred to) for his very precise method of determining the focal length of the lens. These researches include also some optical investigations of great elegance and importance.

OBSERVATIONS UPON THE CUSPS OF THE SUN IN A SOLAR ECLIPSE.

284. In the general discussion of eclipses in Vol. I., I omitted to speak of the use that may be made of these observations in determining the corrections of the elements of the eclipse. The omission may be appropriately supplied here in connection with the heliometer, with which the observations are most accurately made.

Fig. 60.



Let M and S (Fig. 60) be the *apparent* places of the centres of the moon and sun, CC' the common chord of the intersecting discs. The observation consists in measuring the distance of the cusps C, C', and the position angle of CC' with reference to the circle of declination drawn to its middle point. This distance, as well as the position angle, will be affected by refraction, the correction for which will be investigated hereafter. Let s and p here denote the distance and position angle deduced from the observation by the formulæ above given for the heliometer, and also corrected for refraction.

The local time of each measure must be accurately known. For this time, let the parallaxes of the two bodies in right ascension and declination be computed (by Vol. I. Art. 98), and let α and α' denote the resulting apparent right ascensions of the moon and sun respectively, δ and δ' their apparent declinations. Let σ denote the apparent distance of the centres $= SM$, and π the position angle of SM with reference to a circle of declination drawn through its middle point, reckoning this angle from the moon towards the sun. We have, with sufficient accuracy,

$$\left.\begin{aligned} \sigma \sin \pi &= (\alpha' - \alpha) \cos \tfrac{1}{2} (\delta' + \delta) \\ \sigma \cos \pi &= \delta' - \delta \end{aligned}\right\} \quad (324)$$

which determine σ and π.

For the same time, the apparent semidiameters of the moon and sun, which we shall denote by S and S' respectively, will be computed by Vol. I. Art. 131. We then have given the three sides of the triangle SCM, and, denoting the angles at M and S by μ and μ', we may find these angles by the usual formulæ of plane trigonometry, or by the following formulæ, which in the present case are somewhat more convenient:

$$\left.\begin{aligned} \tfrac{1}{2} (S \cos \mu + S' \cos \mu') &= \tfrac{1}{2} \sigma \\ \tfrac{1}{2} (S \cos \mu - S' \cos \mu') &= \frac{(S + S')(S - S')}{2\sigma} = A \end{aligned}\right\} \quad (325)$$

With either of these angles and the value of S or S', we can compute the value of CC'. Let this *computed* value of CC' be denoted by s'; we have

$$s' = 2S \sin \mu = 2S' \sin \mu' \qquad (326)$$

The difference between this computed value and the observed value s will determine the corrections which the elements of the eclipse require in order to satisfy the observation. Put $s = s' - ds'$. Differentiating (326), we find

$$ds' = 2S \cos \mu \,.\, d\mu + 2 \sin \mu \,.\, dS$$

and from the formula

$$2S\sigma \cos \mu = \sigma^2 + S^2 - S'^2$$

we find

$$-S\sigma \sin \mu \, d\mu = (\sigma - S \cos \mu)\, d\sigma + (S - \sigma \cos \mu)\, dS - S' dS'$$

whence, with the aid of the known relations between the parts of the plane triangle, we readily find

$$\frac{2S}{\sigma \tan \mu'}\, dS + \frac{2S}{\sigma \tan \mu}\, dS' - \frac{2S \cos \mu}{\sigma \tan \mu'}\, d\sigma = s - s'$$

But, since $d\sigma$ varies with π, we must replace it by corrections which will have the same value in all the equations of condition thus formed. By putting

$$\begin{aligned} \sigma \sin \pi &= (\alpha' - \alpha) \cos \tfrac{1}{2} (\delta' + \delta) = x \\ \sigma \cos \pi &= \delta' - \delta \qquad\qquad\qquad\quad = y \end{aligned}$$

we shall find

$$d\sigma = dx \,.\, \sin \pi + dy \,.\, \cos \pi$$

in which

$$\begin{aligned} dx &= \cos \tfrac{1}{2} (\delta' + \delta) \,.\, d\,(\alpha' - \alpha) \\ dy &= d\,(\delta' - \delta) \end{aligned}$$

and we may regard $d\,(\alpha' - \alpha)$ and $d\,(\delta' - \delta)$, and, consequently, also dx and dy, as constant for the duration of the eclipse. We then have

$$\frac{2S}{\sigma \tan \mu'}\, dS + \frac{2S'}{\sigma \tan \mu}\, dS' - \frac{2S \cos \mu}{\sigma \tan \mu'} \sin \pi \, dx - \frac{2S \cos \mu}{\sigma \tan \mu'} \cos \pi \, dy = s - s' \tag{327}$$

This will be the final form of our equations of condition if the distance s is fully corrected for the instrumental errors. If, however, the zero of the micrometer is uncertain, we should make observations on opposite sides of the zero, (with the heliometer, by placing the movable semi-lens alternately in opposite positions with respect to the stationary one,) and if c is the unknown error of the micrometer zero, we must write $s \pm c$ for s in the above equation, taking $s + c$ for one series of observations and $s - c$ for the other. The resolution of all the equations of condition by the method of least squares will then determine dS, dS', dx, dy, and c.

It will usually, however, be inexpedient to retain dS', as its coefficient will differ very little from that of dS. The value of the sun's semidiameter is now so well determined that in discussions of this kind it will be quite allowable to put $dS' = 0$.

We may also form equations of condition from the position angles. The angle π is formed by SM and a circle of declination drawn to the middle point of SM, while p is formed at the point D. Denoting the middle point of SM by E, we have $DE = \frac{1}{2}\sigma - S' \cos \mu' = \frac{1}{2}(S \cos \mu - S' \cos \mu') = A$; and we can now compute the position angle of CC' at the point D from the known parts of the triangle formed by the points D, E, and the pole. Let p' denote this computed value; we readily find

$$p' = \pi - 90^\circ + A \sin \pi \tan \tfrac{1}{2} (\delta' + \delta) \qquad (328)$$

Putting the observed value $p = p' + dp'$, we have, by neglecting the insensible variations of the last term of (328), $dp' = d\pi$, and, consequently,

$$\frac{\cos \pi \, dx}{\sigma \sin 1'} - \frac{\sin \pi \, dy}{\sigma \sin 1'} = p - p' \qquad (329)$$

where dx, dy, and σ are expressed in seconds and dp' in minutes. From all the equations thus formed, we can find dx and dy; or we can combine all the equations of the forms (327) and (329) in a single discussion. We see that the corrections of the semidiameters cannot be determined from the position angles alone.

When the observations are made with the heliometer, each must be a *single* observation, for the chord s changes so rapidly that we cannot combine two opposite observations, as has been supposed in Art. 275. We must, therefore, reduce each observation by the general formula (311), in which, however, we may make $f = 0$, by making all the contacts in the heliometer axis or middle of the field. The angle E in these formulæ must then be known; but if it has not been determined with certainty, we may introduce it into our equations of condition as an additional unknown quantity. For one series of observations, we must write $p + E$ in the place of p in (329), and for the other series, in opposite positions of the semi-lenses, we must write $p - E$ in the place of E. But, as E varies inversely with the distance s, it will be necessary to put

$$E = \frac{r}{s \, . \sin 1'}$$

in which γ is a constant which will be expressed in seconds, since s is in seconds and E in minutes. The equation (329) may then be put under the form*

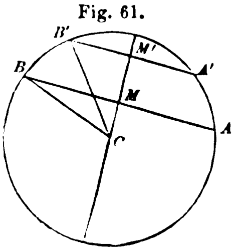
$$\frac{s}{\sigma}\cos\pi\,dx - \frac{s}{\sigma}\sin\pi\,dy \mp \gamma = s\sin(p - p') \qquad (329^*)$$

For some observations of the cusps of the solar eclipse of July 28, 1851, made with the heliometer of the Königsberg Observatory and reduced by the preceding method by the Director WICHMANN, see *Astron. Nach.*, Vol. XXXIII. p. 309.

THE RING MICROMETER.

285. This is simply a thin metallic ring, exactly circular, placed in the focus of the objective, with its plane at right angles to the optical axis. From the times of transit of two stars across its edge, the telescope remaining fixed throughout the observation, we can find both the difference of right ascension and the difference of declination of the stars. Although inferior in accuracy to the filar micrometer and the heliometer, it possesses the advantage over the former of not requiring illumination, and over both in not requiring an equatorial mounting of the telescope.

Fig. 61.



Let $ABB'A'$ represent the inner edge of the ring. Denote by t_1 and t_2 the observed sidereal times of ingress and egress of a star at the points A and B; by t_1' and t_2' the same for a star observed at A' and B'. Upon the supposition that the paths of the stars across the field are rectilinear, the straight line CMM', drawn from the centre C of the ring perpendicular to the chords AB and $A'B'$, will coincide with the declination circle of the point C. The time of the transit of the first star over this circle is the arithmetical mean of the times t_1 and $t_2 = \frac{1}{2}(t_1 + t_2)$; that of the transit of the second star over the same circle is $\frac{1}{2}(t_1' + t_2')$; and, hence, if α and α' are the right ascensions of the stars, we have

$$\alpha' - \alpha = \tfrac{1}{2}(t_1' + t_2') - \tfrac{1}{2}(t_1 + t_2) \qquad (330)$$

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* By (307), we perceive that γ is here the value of the quantity $(m - a)(k - k') + b' - b$ expressed in seconds; and by putting its value found from the discussion of the equations (329) in the second member of (316), and also the true value of $m - a$ found from the value of c by (327), we shall have an equation for determining $k - k'$ and $b' - b$.

t r denote the radius of the ring expressed in seconds of δ and δ' the declinations of the stars, and put

$$\begin{aligned} \tau &= t_2 - t_1 & \tau' &= t_2' - t_1' \\ \gamma &= BCM & \gamma' &= B'CM' \\ d &= MC & d' &= M'C \\ \mu &= BM & \mu' &= B'M' \end{aligned}$$

we have

$$\left.\begin{aligned} \mu &= \frac{15}{2}\,\tau\cos\delta & \mu' &= \frac{15}{2}\,\tau'\cos\delta' \\ \sin\gamma &= \frac{\mu}{r} & \sin\gamma' &= \frac{\mu'}{r} \\ d &= r\cos\gamma & d' &= r\cos\gamma' \end{aligned}\right\} \quad (331)$$

hence the difference of declination of the stars:

$$\delta' - \delta = d' - d \qquad (332)$$

signs of $\cos\gamma$ and $\cos\gamma'$ are not determined by the second .tions of (331); consequently, either sign may be used in puting d or d'. To remove the ambiguity, it is necessary the observer note the positions of the stars with respect to entre of the ring: then d or d' will be *positive* when the star s *north* and *negative* when *south* of the centre.

AMPLE.*—On the 11th of April, 1848, at the Observatory of , the planet Flora and a neighboring star were compared by g micrometer of a six feet refractor. The observed sidereal s were as follows:

Flora (N. of centre).	Star (N. of centre).
$t_1' = 11^h\ 16^m\ 35^s.0$	$t_1 = 11^h\ 17^m\ 53^s.0$
$t_2' = 11\ \ 17\ \ 25.5$	$t_2 = 11\ \ 19\ \ 46.5$
$\tau' = \quad 50.5$	$\tau = \quad 1\ \ 53.5$

approximate declination of Flora was $\delta' = +24°\ 5'.4$. The arent place of the star was

$$\begin{aligned} \alpha &= 6^h\ 4^m\ 51^s.93 \\ \delta &= +24°\ 1'\ \ 9''.01 \end{aligned}$$

e radius of the ring was $r = 1126''.25$; and hence

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* Brünnow's Sphärische Astronomie, p. 546.

$\log \tau'$	1.70329	$\log \tau$	2.05500
$\log \cos \delta'$	9.96043	$\log \cos \delta$	9.96067
$\log \mu'$	2.53878	$\log \mu$	2.89073
$\log \sin \gamma'$	9.48715	$\log \sin \gamma$	9.83910
$\log \cos \gamma'$	9.97850	$\log \cos \gamma$	9.85940
$\log d'$	3.03013	$\log d$	2.91103
$d' = +17' \, 51''.9$		$d = +13' \, 34''.8$	

The planet and star being both observed on the north side of the centre of the field, d' and d are both positive, and hence

$$\delta' - \delta = d' - d = + 4' \, 17''.1$$

For the times of transit over the declination circle of the middle of the field, we have

$$\begin{array}{lrrr} \text{Flora, } \tfrac{1}{2}(t_1' + t_2') = & 11^h & 17^m & 0^s.25 \\ \text{Star, } \tfrac{1}{2}(t_1 + t_2) = & 11 & 18 & 49.75 \\ \alpha' - \alpha = - & & 1 & 49.50 \end{array}$$

Hence we have for the planet

$$\begin{aligned} \alpha' &= 6^h \, 3^m \, 2^s.43 \\ \delta' &= + 24^\circ \, 5' \, 26''.1 \end{aligned}$$

which values express the planet's apparent place at the time of its passage over the declination circle of the middle of the field, that is, at the sidereal time $11^h \, 17^m \, 0^s.25$. But the effect of refraction has not yet been allowed for. See Art. 300.

286. *Correction for curvature.*—The correction which the preceding method requires, in consequence of the curvature of the paths of the stars, may be found as follows. In the spherical triangle of which the three angular points are the pole, the centre of the ring, and the point where the star enters or leaves the ring, we have

$$\sin \delta = \sin D \cos r + \cos D \sin r \cos \gamma$$

where D is the declination of the centre of the ring. For the second star, we have

$$\sin \delta' = \sin D \cos r + \cos D \sin r \cos \gamma'$$

and the difference of these equations gives

$$2 \sin \tfrac{1}{2}(\delta' - \delta) \cos \tfrac{1}{2}(\delta' + \delta) = (\sin r \cos \gamma' - \sin r \cos \gamma) \cos D$$

or, very nearly,

$$\begin{aligned}(\delta' - \delta)\cos\tfrac{1}{2}(\delta' + \delta) &= (r\cos\gamma' - r\cos\gamma)\cos D\\ &= (d' - d)\cos D\end{aligned}$$

in which $d' - d$ is the approximate difference found by the preceding article. But we have, very nearly,

$$D = \delta - d \qquad\qquad D = \delta' - d'$$

the mean of which is

$$D = \tfrac{1}{2}(\delta' + \delta) - \tfrac{1}{2}(d' + d)$$

and we may, therefore, put

$$\cos D = \cos\tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(d' + d)\sin 1''\sin\tfrac{1}{2}(\delta' + \delta)$$

so that we obtain

$$\delta' - \delta = d' - d + \tfrac{1}{2}(d' + d)(d' - d)\sin 1''\tan\tfrac{1}{2}(\delta' + \delta) \qquad (333)$$

Hence, the correction of the difference of declination found upon the supposition that the path of the star is rectilinear, is

$$+\tfrac{1}{2}(d' + d)(d' - d)\sin 1''\tan\tfrac{1}{2}(\delta' + \delta)$$

The correction disappears when d' and d are numerically equal, that is, when the stars are observed at equal distances from the centre of the ring.

In the example of the preceding article, this correction amounts to $+0''.52$, and the corrected difference of declination is

$$\delta' - \delta = +4'\,17''.62$$

287. If the outer edge of the ring is also an exact circle, it may be used in the same manner as the inner edge. Let the four transits of a star over the edges of both rings be observed at the times t_1, t_2, t_3, t_4; then, if r is the radius of the outer ring, r_1 that of the inner ring, we put

$$\mu = \tfrac{15}{2}(t_4 - t_1)\cos\delta \qquad\qquad \mu_1 = \tfrac{15}{2}(t_3 - t_2)\cos\delta$$

$$\sin\gamma = \frac{\mu}{r} \qquad\qquad \sin\gamma_1 = \frac{\mu_1}{r_1}$$

so that with the outer ring we find

$$d = r\cos\gamma$$

and with the inner ring,

$$d = r_1 \cos \gamma_1$$

and the mean of these values will be taken as the true value of d. In the same manner d' for the second star will be found, after which $\delta' - \delta = d' - d$.

But when the four observations have been obtained, the process of reduction may be slightly abridged, as follows:*

The sum and difference of the values of d^2 give

$$d^2 = \tfrac{1}{2}[r^2 + r_1^2 - (\mu^2 + \mu_1^2)]$$
$$r^2 - r_1^2 = \mu^2 - \mu_1^2$$

Putting

$$\left.\begin{array}{c} a = \dfrac{r + r_1}{2} \\ \sin A = \dfrac{\mu + \mu_1}{2a} \qquad \sin B = \dfrac{\mu - \mu_1}{2a} \end{array}\right\} \quad (334)$$

we find

$$r - r_1 = \frac{\mu^2 - \mu_1^2}{2a} = 2a \sin A \sin B$$
$$r^2 + r_1^2 = 2a^2 (1 + \sin^2 A \sin^2 B)$$
$$\mu^2 + \mu_1^2 = 2a^2 (\sin^2 A + \sin^2 B)$$

which, substituted in the above value of d^2, give

$$d^2 = a^2 \cos^2 A \cos^2 B$$

or

$$d = a \cos A \cos B \quad (335)$$

so that, A and B being found by (334), d is found by (335). The formulæ (334) for determining A and B may also be written as follows:

$$\sin A = \frac{15(\tau + \tau_1)\cos\delta}{4a} \qquad \sin B = \frac{15(\tau - \tau_1)\cos\delta}{4a}$$

in which $\tau = t_4 - t_1$ and $\tau_1 = t_3 - t_2$.

EXAMPLE.—On the 24th of June, 1850, at the Observatory of Bilk, PETERSEN'S comet and a star were observed with a double-ring micrometer, as follows:

Comet (N. of centre).		Star (S. of centre).	
t_1'	$18^h\ 15^m\ 54^s.$	t_1	$18^h\ 18^m\ 55^s.3$
t_2'	16 20	t_2	19 18.
t_3'	17 21	t_3	21 20.5
t_4'	17 48	t_4	21 37.5

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* BRÜNNOW'S Sphärische Astronomie, p. 549.

The approximate declination of the comet was $\delta' = +\ 59° 20'$, and the apparent place of the star was

$$\alpha = 14^h\ 53^m\ 30^s.75 \qquad \delta = +\ 59°\ 7'\ 12''.19$$

The radii of the rings were—

$$\text{Outer,}\quad r = 11'\ 21''.09$$
$$\text{Inner,}\quad r_1 = 9\ \ 26\ .29$$
whence
$$a = 10\ \ 23\ .69$$

Then we find:

	Comet.		Star.
τ'	$1^m\ 54^s.0$	τ	$2^m\ 42^s.2$
τ_1'	$1\ \ \ 1\ .0$	τ_1	$2\ \ \ 7\ .5$
$\log(\tau' + \tau_1')$	2.24304	$\log(\tau + \tau_1)$	2.46195
$\log(\tau' - \tau_1')$	1.72428	$\log(\tau - \tau_1)$	1.54033
$\log \frac{15 \cos \delta'}{4a}$	7.48667	$\log \frac{15 \cos \delta}{4a}$	7.48938
$\log \sin A'$	9.72971	$\log \sin A$	9.95133
$\log \sin B'$	9.21095	$\log \sin B$	9.02971
$\log \cos A'$	9.92623	$\log \cos A$	9.65137
$\log \cos B'$	9.99419	$\log \cos B$	9.99750
$\log d'$	2.71539	$\log d$	2.44384
$d' =$	$+\ 8'\ 39''.27$	$d =$	$-\ 4'\ 37''.87$

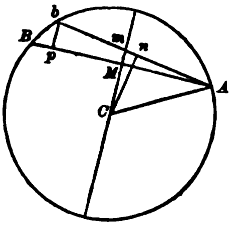
$$d' - d = +\ 13'\ 17''.14$$

and for the difference of right ascension,

$$\alpha' - \alpha = -\ 3^m\ 25^s.83$$

288. *To find the correction for the proper motion of one of the objects.*—The most common application of the ring micrometer is to the determination of the difference of right ascensions and declinations of a star, and a planet, or comet. But since a planet (or comet) changes both its right ascension and declination during the time of an observation, its path will not be at right angles to the declination circle drawn through the centre of the ring: so that the differences found by the preceding methods will require a correction.

Fig. 62.



Let Ab, Fig. 62, be the path of the planet across the ring; Cm the declination circle through C, the centre of the ring. Draw AB perpendicular to Cm, Cn perpendicular to Ab, bp perpendicular to AB. Put

$\Delta\alpha'$ = the increase of the planet's right ascension in one sidereal second,
$\Delta\delta'$ = the increase of the declination in one sid. second,
t_1', t_2' = the sid. times of ingress and egress of the planet at A and b,
$\tau' = t_2' - t_1'$,
x = the correction of the mean of t_1' and t_2', or of the right ascension of the planet found by the preceding methods,
$\frac{1}{2}(t_1' + t_2') + x$ = the sid. time of the planet's transit at m,
β = the angle $BAb = mCn$.

The arc bp may be regarded as a portion of the declination circle drawn through b. The angle at the pole included by this circle and the declination circle of A is the hour angle described by the planet in the time τ', which hour angle is $\tau' - \tau' . \Delta\alpha' = \tau'(1 - \Delta\alpha')$. Hence we have, very nearly,

$$Ap = 15\,\tau'(1 - \Delta\alpha')\cos\delta'$$

We have, also,

$$bp = \tau' . \Delta\delta'$$

whence

$$\tan\beta = \frac{\Delta\delta'}{15\cos\delta'(1 - \Delta\alpha')}$$

or, since the squares of $\Delta\delta'$ and $\Delta\alpha'$ or their product may be neglected,

$$\tan\beta = \frac{\Delta\delta'}{15\cos\delta'}$$

The correction x is the time in which the planet describes the line nm, and this time is found by the proportion

$$\tau' : x = Ab : nm = Ab : Cn\tan\beta$$

for which we can take

$$\tau' : x = 15\,\tau'\cos\delta' : d'\tan\beta$$

whence, substituting the value of $\tan\beta$,

$$x = \frac{d' . \Delta\delta'}{(15\cos\delta')^2} \tag{336}$$

Since $Ab = Ap\sec\beta$, and $\sec\beta$ differs from unity only by terms involving $(\Delta\delta')^2$, we may take $Ab = Ap$, and hence

$$An = \tfrac{1}{2}Ap = \frac{15\,\tau'\cos\delta'}{2}(1 - \Delta\alpha') = \mu'(1 - \Delta\alpha')$$

hat to compute $d' = Cn$ in this case we have

$$\sin\gamma' = \frac{\mu'}{r}(1 - \Delta\alpha') \qquad\qquad d' = r\cos\gamma' \qquad\qquad (337)$$

is, the computation by the preceding methods will give the e of d', corrected for the proper motion, if we employ — $\Delta\alpha'$) instead of μ'. In the method of Article 287, with a ble-ring micrometer, the logarithm of $1 - \Delta\alpha'$ may be added he logarithm of $\frac{15\cos\delta'}{4a}$.

XAMPLE.—In the example of the preceding article the comet's ion in one mean day was, in right ascension — $5^m\ 0^s$, and in ination — 1° 17′; and therefore, since one mean day contains 6 sidereal seconds,[*]

$$\Delta\alpha' = -\frac{300^s}{86636} \qquad\qquad \log(1 - \Delta\alpha') = 0.00150$$

$$\Delta\delta' = -\frac{4620''}{86636} \qquad\qquad \log\Delta\delta' = n8.72694$$

ce, in the computation of d' we have

$$\begin{array}{rl}
\log\dfrac{15\cos\delta'}{4a}(1 - \Delta\alpha') & 7.48817 \\
\log\sin A' & 9.73121 \\
\log\sin B' & 9.21245 \\
\log\cos A' & 9.92563 \\
\log\cos B' & 9.99415 \\
\log d' & 2.71475 \\
d' = & +\,8'\ 38''.50
\end{array}$$

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[*] he logarithm of $1 - \Delta\alpha'$ may be found at once from the change of right ascen- in 48 hours, as follows. Let this change be expressed in *minutes of arc*, and ted by $(\Delta\alpha')$, then we have

$$\Delta\alpha' = \frac{(\Delta\alpha') \times 60}{15 \times 2 \times 86636} = \frac{(\Delta\alpha')}{43318}$$

If M is the modulus of common logarithms, we have from the development of $1 - \Delta\alpha'$) in series, by neglecting the second and higher powers of $\Delta\alpha'$,

$$\log(1 - \Delta\alpha') = -M\Delta\alpha' = -\frac{0.43429\,(\Delta\alpha')}{43318}$$

ery nearly,

$$\log(1 - \Delta\alpha') = -0.00001\,(\Delta\alpha')$$

e, to correct for the proper motion in the computation of d, subtract from the ithm of μ' as many units of the 5th decimal place as there are minutes of arc in 8 hour increase of right ascension.

and, therefore,

$$d' - d = +\,13'\,16''.37$$

By (336) we find

$$x = \quad -\,0^s.47$$

whence

$$\alpha' - \alpha = -\quad 3^m 26^s.30$$

The correction of $d' - d$ for the curvature of the path is, in case, by (333), $+\,0''.78$; whence

$$\delta' - \delta = +\,13'\,17''.15$$

so that the corrections for curvature and proper motion h accidentally, almost destroy each other.

The apparent place of the comet (still affected, however parallax and planetary aberration, as well as the differe refraction) is, therefore,

$$\alpha' = \quad 14^h\,50^m\,\;4^s.45$$
$$\delta' = +\,59^\circ\,20'\,29''.34$$

at the sidereal time $18^h\,16^m\,50^s.75$.

289. It is yet to be shown under what conditions error observation or of the data will have the least effect upon results obtained with the ring micrometer. For the effect o error $\Delta\tau$ in the observed interval, we have, by differentiating (3

$$\Delta\gamma = \frac{15 \cos\delta\,.\,\Delta\tau}{2r\cos\gamma}$$

$$\Delta d = -\,r\sin\gamma\,.\,\Delta\gamma = -\,\tfrac{15}{2}\cos\delta\tan\gamma\,.\,\Delta\tau$$

which shows that the error in the observed time produces least effect upon d when $\tan\gamma$ is least, and, therefore, for the accurate determination of the declination, the chords descri by the two stars should be as far from the centre of the rin possible, or the difference of declination should be but little than the diameter of the ring. If d is not much less than will be advisable to let the stars pass across the field on opp sides of the centre, at nearly equal distances from it. But if very small, both stars should pass as far from the centr possible, on the same side of it.

For the effect of an error in r, we have

$$\Delta d = \frac{r}{d}\,\Delta r = \Delta r\,.\sec\gamma$$

is also least when the star is farthest from the centre of d.

the second star, we have also $\Delta d' = \Delta r \sec \gamma'$, and hence

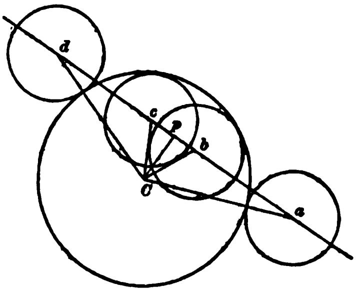
$$\Delta (d' - d) = \Delta r (\sec \gamma' - \sec \gamma)$$

if the stars are on the same parallel of declination (when the error in r has no effect upon the computed difference ination, as, indeed, is otherwise evident.

the accurate determination of the difference of right ion, it is plain that the stars should pass as near to the of the field as possible, since the immersions and emeran then be most accurately observed.

To find the radius of the ring.—First Method.—Observe the s of the sun's limb over the f the ring. Four contacts observed, the sun's centre at the points a, b, c, d (Fig. the times t_1, t_2, t_3, t_4. If the of the ring is denoted by r e sun's semidiameter by R, that the external contacts nd d) are observed at the at which the transit of the centre would be observed ring whose radius was $r + R$; while the internal contacts nd c) are observed at the times at which the transit of the centre would be observed over a ring whose radius was . Hence, putting δ = sun's declination, and

Fig. 63.



$$\tau = t_4 - t_1 \qquad \tau' = t_3 - t_2$$

re, by Art. 285, from the external contacts,

$$2(r + R) \sin \gamma = 15 \tau \cos \delta$$
$$2(r + R) \cos \gamma = d$$

m the inner contacts,

$$2(r - R) \sin \gamma' = 15 \tau' \cos \delta$$
$$2(r - R) \cos \gamma' = d$$

Eliminating γ and γ', we have

$$4(r+R)^2 = (15\,\tau \cos\delta)^2 + d^2$$
$$4(r-R)^2 = (15\,\tau' \cos\delta)^2 + d^2$$

and eliminating d^2, we obtain

$$r = \frac{(15\cos\delta)^2 (\tau+\tau')(\tau-\tau')}{16\,R} \qquad \text{(3}$$

In order to take into account the sun's motion in right ascensi the intervals τ and τ' should be expressed in apparent time.

It is easy to see that the formula (338) will still be applica when the sun's diameter is greater than that of the ring.

EXAMPLE.*—The sun was observed with a ring micrometer the Observatory of Bilk as follows:

External Contacts.	Internal Contacts.
$t_1 = 10^h\,31^m\;\;8^s.2$ Sid. time	$t_2 = 10^h\,32^m\,30^s.8$
$t_4 = 10\;\;34\;\;47.5$	$t_3 = 10\;\;33\;\;25.3$

The sun's declination was $\delta = +\,23°\,14'\,50''$; the semidiame $R = 15'\,45''.07$; and the sun's motion in right ascension w $4^m\,8^s.7$ in one day.

The sidereal intervals $3^m\,39^s.3$ and $54^s.5$ must be reduced intervals of apparent time by multiplying them by the factor

$$1 - \frac{248.7}{86636} = 0.99713$$

whence

$$\tau = 218^s.67 \qquad \tau' = 54^s.34$$

and hence, by (338),

$$r = 9'\,23''.57$$

Second Method.—Observe the transits of two stars the diff ence of whose declinations is accurately known. Then, τ and being, as before, the intervals between the ingress and egress the two stars respectively, we have

$$\mu = \tfrac{15}{2}\,\tau\cos\delta = r\sin\gamma \qquad d = \pm\, r\cos\gamma$$
$$\mu' = \tfrac{15}{2}\,\tau'\cos\delta' = r\sin\gamma' \qquad d' = \pm\, r\cos\gamma'$$

Since for determining r it will always be advisable to selec air of stars whose difference of declination is not much less ian the diameter of the ring, the stars will be observed on pposite sides of the centre: we shall, therefore, have

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* BRÜNNOW, Sphärische Astronomie, p. 561.

$$d' - d = r\,(\cos\gamma + \cos\gamma')$$

Let A and B be assumed, so that

$$A = \tfrac{1}{2}(\gamma' + \gamma) \qquad B = \tfrac{1}{2}(\gamma' - \gamma)$$

ien

$$\begin{aligned} d' - d &= r\,[\cos(A + B) + \cos(A - B)] = 2r\cos A\cos B \\ \mu' + \mu &= r\,[\sin(A + B) + \sin(A - B)] = 2r\sin A\cos B \\ \mu' - \mu &= r\,[\sin(A + B) - \sin(A - B)] = 2r\cos A\sin B \end{aligned}$$

Ience we derive

$$\left.\begin{aligned} \tan A = \frac{\mu' + \mu}{d' - d} \qquad \tan B &= \frac{\mu' - \mu}{d' - d} \\ r = \frac{d' - d}{2\cos A\cos B} \end{aligned}\right\} \quad (339)$$

We may also use any one of the following forms for r:

$$r = \frac{\mu' + \mu}{2\sin A\cos B} = \frac{\mu' - \mu}{2\cos A\sin B} = \frac{\mu'}{\sin(A + B)} = \frac{\mu}{\sin(A - B)}$$

In order to render this method exact, the atmospheric refraction should be taken into account. Its effect upon micrometric observations in general will be considered hereafter, but, since for determining the radius of the ring micrometer it will be advisable to take the observations near the meridian, the refraction may be allowed for in a very simple manner; for we may then neglect its effect upon the right ascensions of the stars. The effect upon the declinations is found by the formulæ (234) and ([illegible]) of Vol. I.; according to which, if δ and δ' are the true declinations, the apparent values are

$$\begin{aligned} &\delta + k'\cot(\delta + N) \\ &\delta' + k'\cot(\delta' + N) \end{aligned}$$

where $\tan N = \cot\varphi\cos\tau_0$, φ being the latitude of the place of observation, and τ_0 the hour angle of the centre of the ring. Hence the apparent difference of declination, which we will denote by $(\delta' - \delta)$,

$$(\delta' - \delta) = \delta' - \delta - \frac{k'\sin(\delta' - \delta)}{\sin(\delta + N)\sin(\delta' + N)}$$

for which we may take

$$(\delta' - \delta) = \delta' - \delta - \frac{k' \sin(\delta' - \delta)}{\sin^2 [\frac{1}{2}(\delta + \delta') + N]} \qquad ($$

which is to be used for $d' - d$ in (339). It will here gener suffice to take $k' = 57''$; but it may be accurately found Column B of Table II.

When the stars are not very near the equator, the correc for curvature must be applied. If r were given, the observati computed upon the supposition that the paths of the stars rectilinear, would give the approximate difference $d' - d$, hence in the inverse process we have only to use $d' - d$ inst of $(\delta' - \delta)$ in order to obtain the true value of r. Now, by (3

$$d' - d = (\delta' - \delta) - \tfrac{1}{2} \sin 1'' (d'^2 - d^2) \tan \tfrac{1}{2}(\delta' + \delta)$$

or, since $d'^2 - d^2 = -(\mu'^2 - \mu^2)$,

$$d' - d = (\delta' - \delta) + \tfrac{1}{2} \sin 1'' (\mu' + \mu)(\mu' - \mu) \tan \tfrac{1}{2}(\delta' + \delta) \qquad (3$$

in which $(\delta' - \delta)$ is the difference of declination corrected refraction.

Example.—The radius of the ring of the micrometer ployed in the example of Art. 285 was determined by the st *Asterope* and *Merope* of the *Pleiades*, the declinations of wh were

$$\delta' = +\ 24^\circ\ 4'\ 24''.26 \qquad \delta = +\ 23^\circ\ 28'\ 6''.85$$

and the observed intervals were

$$\tau' = 18^s.5 \qquad \tau = 56^s.2$$

In order to illustrate the use of (340), let us suppose the h angle of the centre of the ring to have been $\tau_0 = 1^h = 15^\circ$; th the latitude of Bilk being $\varphi = +\ 51^\circ\ 12'\ 25''$, we find

$N = 37^\circ\ 49'.6$
$\frac{1}{2}(\delta + \delta') + N = 61\ \ 35\ .9$
$\delta' - \delta = 36'\ 17''.41$
corr. $-\ \ 0\ .78$
$(\delta' - \delta) = 36'\ 16''.63$

$\log k' = \log 57''$ 1.7
$\log \operatorname{cosec}^2 [\frac{1}{2}(\delta + \delta') + N]$ 0.1
$\log \sin (\delta' - \delta)$ 8.0
log corr. $n9.8$

We find, in the next place,

$$\mu' = 126''.68 \qquad \mu = 386''.63$$

$$\log(\mu' + \mu) = 2.71038 \qquad \log(\mu' - \mu) = n2.41489$$

whence the correction for curvature is, by (341), $= -0''.14$, and therefore

$$d' - d = 36'\ 16''.48$$

with which we now find, by (339),

$$\begin{array}{llll}
\log\tan A = 9.37263 & & \log\tan B = 9.07714 & \\
 & \log\sec A & 0.01175 & \\
 & \log\sec B & 0.00308 & \\
 & \log(d'-d) & \underline{3.33775} & \\
 & \log 2r & 3.35258 & \\
 & & r = 18'\ 46''.03 &
\end{array}$$

Third Method.—Direct the telescope of a theodolite towards the objective of the telescope which carries the micrometer, and measure directly the angular diameter of the ring by either the vertical or the horizontal circle of the theodolite, as in the case of the filar micrometer, Art. 46.*

291. The filar micrometer, the heliometer, and the ring micrometer are now almost the only micrometers in use. I will, therefore, here only briefly mention two or three others, as it is not within the plan of this work to enter upon the history of the numerous instruments of this class which have been proposed.

The duplication of the images of objects, which is effected in the heliometer by dividing the objective, has also been effected by dividing the ocular, constituting what has been known as the *double-image eye-piece micrometer.* The principle of this instrument is identical with that of Ramsden's Dynameter, which is still used for measuring the magnifying power of telescopes (Art. 13). It is evident that by separating the two halves of a simple eye-lens until the image of one star coincides with that of another, the angular distance of the stars becomes known from the known angular value of a revolution of the screw by which the separation is produced. Amici, of Modena, is said to have produced the best micrometers of this kind.

The duplication of images is also effected by the use of a doubly refracting prism of rock crystal, originally proposed by Rochon. The difficulty of determining the relation between the given position of the crystal and the angular distance of two objects which have been brought into contact, is a considerable obstacle to its general use, to say nothing of the optical difficulties in obtaining well defined images free from color.*

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* Upon the ring micrometer, see also papers by Bessel in the *Monatliche Correspondenz*, Vols. XXIV. and XXVI.

STRUVE has proposed the use of a graduated plate of transparent mica placed in the focus of the equatorial, and this method has been employed by the Messrs. BOND in cataloguing small stars. Upon a plate of mica $\frac{1}{1000}$ of an inch in thickness are drawn two sets of parallel lines, one system representing declination circles, the other, at right angles to the first, representing parallels of declination. The relative declination of two stars which pass over the field is determined by merely observing the divisions of the graduated declination scale over which or near which they pass; and their relative right ascension is found from the observed times of their transits over the lines which represent declination circles, these times being noted by the aid of the electro-chronograph.†

An ingenious mode of employing a double eye piece micrometer (consisting of two complete eye pieces), apparently giving very precise results, is suggested by Mr. ALVAN CLARK, of Boston, in the Proceedings of the Am. Association for the Adv. of Science, 10th meeting, p. 108.

CORRECTION OF MICROMETRIC OBSERVATIONS FOR REFRACTION.

292. Since the position of each of the two observed stars is affected by the atmospheric refraction, their *relative* position, determined by the micrometer, is also affected by it. The object of the following investigations is to determine the correction of the micrometric measures themselves, without requiring a separate consideration of the absolute places of the two stars.‡

293. *To find the effect of refraction upon the observed angular distance of two stars and upon the angle which the great circle joining the stars makes with a vertical circle.*—This mode of observation is indeed not practised, but the investigation of the effect of refraction in

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* For a description of a number of double image micrometers, see PEARSON'S Practical Astronomy.

† See *Annals of the Astronomical Observatory of Harvard College*, Vol. I.

‡ I have followed BESSEL'S methods (*Astron. Untersuch.*, Vol. I.) in the investigation of the greater part of the formulæ. That portion of his article which relates to the ring micrometer is, however, considerably abridged and simplified.

this case is very simple, and will serve as the ground-work of the subsequent applications. Denote by

ζ, ζ', and z, z', the true and apparent zenith distances of the two stars S and S';
A, their difference of azimuth;
r, r', their refractions;
λ, λ', and l, l', the true and apparent angles which the great circle joining the stars makes with their respective vertical circles, all reckoned in the same direction;
σ, s, the true and apparent distances of the stars.

We have, in the triangle formed by the zenith and the apparent places of the stars, by the Gaussian equations of spherical trigonometry,

$$\begin{aligned} \sin \tfrac{1}{2}s \sin \tfrac{1}{2}(l + l') &= \sin \tfrac{1}{2}A \sin \tfrac{1}{2}(z + z') \\ \sin \tfrac{1}{2}s \cos \tfrac{1}{2}(l + l') &= \cos \tfrac{1}{2}A \sin \tfrac{1}{2}(z - z') \end{aligned}$$

and in the triangle formed by the zenith and the true places of the stars,

$$\begin{aligned} \sin \tfrac{1}{2}\sigma \sin \tfrac{1}{2}(\lambda + \lambda') &= \sin \tfrac{1}{2}A \sin \tfrac{1}{2}(\zeta + \zeta') \\ \sin \tfrac{1}{2}\sigma \cos \tfrac{1}{2}(\lambda + \lambda') &= \cos \tfrac{1}{2}A \sin \tfrac{1}{2}(\zeta - \zeta') \end{aligned}$$

If we put

$$l_0 = \tfrac{1}{2}(l + l') \qquad \lambda_0 = \tfrac{1}{2}(\lambda + \lambda') \qquad \zeta_0 = \tfrac{1}{2}(\zeta + \zeta')$$

and also substitute $\zeta - r$ and $\zeta' - r'$ for z and z', the elimination of A from the above equations gives

$$\begin{aligned} \sin \tfrac{1}{2}\sigma \sin \lambda_0 &= \sin \tfrac{1}{2}s \sin l_0 \cdot \frac{\sin \zeta_0}{\sin [\zeta_0 - \tfrac{1}{2}(r + r')]} \\ \sin \tfrac{1}{2}\sigma \cos \lambda_0 &= \sin \tfrac{1}{2}s \cos l_0 \cdot \frac{\sin \tfrac{1}{2}(\zeta - \zeta')}{\sin \tfrac{1}{2}[\zeta - \zeta' - (r - r')]} \end{aligned}$$

We may evidently, in the first equation, put r_0 for $\tfrac{1}{2}(r + r')$, which is equivalent to assuming that the mean of the refractions for the zenith distances ζ and ζ' is the same as the refraction for the mean of these zenith distances, an assumption producing here no sensible error in the factor $\sin [\zeta_0 - \tfrac{1}{2}(r + r')]$ or $\sin (\zeta_0 - r_0)$. We may also take

$$r - r' = \frac{dr_0}{d\zeta_0}(\zeta - \zeta')$$

in which the differential coefficient $\frac{dr_0}{d\zeta_0}$ expresses the rate of change of the refraction corresponding to ζ_0. Then, in the fraction

$$\frac{\sin \frac{1}{2}(\zeta - \zeta')}{\sin \frac{1}{2}[\zeta - \zeta' - (r - r')]}$$

which differs but little from unity, we may put the arcs for their sines: so that, denoting this fraction by b, we have

$$b = \frac{\zeta - \zeta'}{\zeta - \zeta' - (r - r')} = \frac{1}{1 - \frac{r - r'}{\zeta - \zeta'}} = \frac{1}{1 - \frac{dr_0}{d\zeta_0}}$$

If we also put

$$a = \frac{\sin \zeta_0}{\sin(\zeta_0 - r_0)}$$

and substitute $\frac{1}{2}\sigma$ and $\frac{1}{2}s$ for their sines, our formulæ become

$$\sigma \sin \lambda_0 = s \,.\, a \sin l_0$$
$$\sigma \cos \lambda_0 = s \,.\, b \cos l_0$$

From these we have

$$\tan \lambda_0 = \frac{a}{b} \tan l_0$$

which developed* gives

$$\lambda_0 - l_0 = -\frac{b - a}{b + a} \sin 2l_0 + \frac{1}{2}\left(\frac{b - a}{b + a}\right)^2 \sin 4l_0 - \&c. \qquad (342$$

From the same formulæ we derive

$$\sigma \cos(\lambda_0 - l_0) = s\,[a + (b - a) \cos^2 l_0]$$

and, dividing this by $\cos(\lambda_0 - l_0) = 1 - \frac{1}{2}(\lambda_0 - l_0)^2 + \&c.$, we obtain

$$\sigma - s = s\left[a - 1 + (b - a)\cdot\cos^2 l_0 + \frac{a}{2}\left(\frac{b - a}{b + a}\right)^2 \sin^2 2l_0 + \&c.\right] \qquad (343)$$

294. To facilitate the computation of (342) and (343) a convenient method of finding a and b is necessary. We have, for any indeterminate ζ,

$$a = \frac{\sin \zeta}{\sin(\zeta - r)} = \frac{\sin(z + r)}{\sin z} = \cos r + \frac{\sin r}{\tan z}$$
$$b = \frac{d\zeta}{d\zeta - dr} = \frac{d(z + r)}{dz} = 1 + \frac{dr}{dz}$$

Adopting for the refraction the form (Vol. I. Arts. 107 and 117)

$$r = k \tan z$$

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* By Pl. Trig. Art. 254.

in which

$$k = \alpha\beta^A\gamma^\lambda$$

we have, putting $\cos r = 1$,

$$a = 1 + k$$

$$b - a = k \tan^2 z + \frac{dk}{dz} \tan z$$

These quantities may therefore be found by the aid of Column A of Table II. But, as the argument is there the apparent zenith distance, while in micrometer observations it is generally the true zenith distance that is given, it is expedient to form a new table, by which a quantity x, depending upon the refraction, may be found with the argument ζ, such that

$$b - a = x \tan^2 \zeta$$

In order to obtain the value of x for any state of the air, **Bessel** gives it the same form as that already adopted for k, and assumes

$$x = \alpha'' \beta^{A''} \gamma^{\lambda''}$$

in which the factors β and γ, depending on the barometer and thermometer, have the same values as before.

The quantities $\log \alpha''$, A'', λ'', which are given in Column C of Table II., must be determined so as to satisfy the above definition of x for all values of β and γ. We have

$$x = k \frac{\tan^2 z}{\tan^2 \zeta} + \frac{dk}{dz} \frac{\tan z}{\tan^2 \zeta} = \left(k + \frac{dk}{dz} \cot z\right) \frac{\tan^2 z}{\tan^2 \zeta}$$

Taking the Napierian logarithms,

$$lx = l\alpha'' + A'' l\beta + \lambda'' l\gamma = l\left(k + \frac{dk}{dz} \cot z\right) + 2l\left(\frac{\tan z}{\tan \zeta}\right) \quad (344)$$

From the definition of k, we have

$$\frac{dk}{dz} = k\left[\frac{d\alpha}{\alpha\, dz} + l\beta \frac{dA}{dz} + l\gamma \frac{d\lambda}{dz}\right]$$

$$k + \frac{dk}{dz} \cot z = k\left[1 + \frac{d\alpha}{r\, dz} + \left(l\beta \frac{dA}{dz} + l\gamma \frac{d\lambda}{dz}\right) \cot z\right]$$

$$= k\left(1 + \frac{d\alpha}{r\, dz}\right)\left[1 + \frac{\left(l\beta \frac{dA}{dz} + l\gamma \frac{d\lambda}{dz}\right) \cot z}{1 + \frac{d\alpha}{r\, dz}}\right]$$

Since β and γ differ but little from unity, $l\beta$ and $l\gamma$ are so small that we may neglect their squares, so that the logarithm of the last factor of the above expression, under the form $l(1 + x)$, may be put $= x$, and hence

$$l\left(k + \frac{dk}{dz}\cot z\right) = l\alpha + Al\beta + \lambda l\gamma + l\left(1 + \frac{d\alpha}{r\,dz}\right) + \left[\frac{l\beta\frac{dA}{dz} + l\gamma\frac{d\lambda}{dz}}{1 + \frac{d\alpha}{r\,dz}}\right]\cot z \tag{345}$$

Now, let (z) denote that value of z which corresponds to the given ζ when $\beta = 1, \gamma = 1$, a value which can be found from Column A of the table, as in Art. 119, Vol. I. Let the corresponding values of α, A, λ, as found from that column, be denoted by (α), (A), (λ), and the corresponding refraction by (r); then, α', A', λ' being taken from Column B for the given ζ, we have, as in the article just referred to,

$$(r) = (\alpha)\tan(z) = \alpha'\tan\zeta$$
$$z = (z) - \alpha'\tan\zeta\,(A'\,l\beta + \lambda'\,l\gamma)$$

The second member of (345) is a function of z, which may be transformed into a function of (z). The small terms multiplied by $l\beta$ and $l\gamma$ will not be sensibly affected by substituting (z) for z, (A) for A, &c. The other terms may be developed by the formula

$$fz = f(z) + \frac{df(z)}{d(z)}\,y + \ldots.$$

in which

$$y = -\alpha'\tan\zeta\,(A'\,l\beta + \lambda'\,l\gamma) = -(\alpha)\tan(z)\,(A'\,l\beta + \lambda'\,l\gamma)$$

We find

$$l\left(k + \frac{dk}{dz}\cot z\right) = l(\alpha) + (A)\,l\beta + (\lambda)\,l\gamma + l\left(1 + \frac{d(\alpha)}{(r)\,d(z)}\right)$$
$$- \left[\frac{d(\alpha)}{d(z)}\tan(z) - \frac{\frac{(\alpha)}{(r)}\frac{d(\alpha)}{d(z)}\sec^2(z)}{1 + \frac{d(\alpha)}{(r)\,d(z)}}\right](A'\,l\beta + \lambda'\,l\gamma)$$
$$+ \left[\frac{l\beta\frac{d(A)}{d(z)} + l\gamma\frac{d(\lambda)}{d(z)}}{1 + \frac{d(\alpha)}{(r)\,d(z)}}\right]\cot(z)$$

We have, also,

$$\frac{\tan z}{\tan \zeta} = \frac{\tan [(z) + y]}{\tan \zeta} = \frac{\tan (z)}{\tan \zeta} - \frac{a'}{\cos^2 (z)} (A' l\beta + \lambda' l\gamma)$$

$$= \frac{a'}{(a)} \left[1 - \frac{(a)}{\cos^2 (z)} (A' l\beta + \lambda' l\gamma) \right]$$

$$2l\left(\frac{\tan z}{\tan \zeta}\right) = 2la' - 2l(a) - \frac{2(a)}{\cos^2 (z)} (A' l\beta + \lambda' l\gamma)$$

Hence, substituting in (344),

$$la'' + A'' l\beta + \lambda'' l\gamma = 2la' - l(a) + l\left(1 + \frac{d(a)}{(r)\,d(z)}\right)$$

$$+ l\beta \left[(A) - \frac{2(a)}{\cos^2 (z)} A' - \frac{d(a)}{d(z)} \tan (z) A' + \frac{\frac{(a)}{(r)} \frac{d(a)}{d(z)} \sec^2 (z) A' + \frac{d(A)}{d(z)} \cot (z)}{1 + \frac{d(a)}{(r)\,d(z)}} \right]$$

$$+ l\gamma \left[(\lambda) - \frac{2(a)}{\cos^2 (z)} \lambda' - \frac{d(a)}{d(z)} \tan (z) \lambda' + \frac{\frac{(a)}{(r)} \frac{d(a)}{d(z)} \sec^2 (z) \lambda' + \frac{d(\lambda)}{d(z)} \cot (z)}{1 + \frac{d(a)}{(r)\,d(z)}} \right]$$

Since this must be satisfied for all values of β and γ, the coefficients of $l\beta$ and $l\gamma$ in the two members must be equal, respectively. Now, we have found, in Vol. I. Art. 119,

$$(A) = A'\left(1 + \frac{d(r)}{d(z)}\right) = A' \left[1 + (a) \sec^2 (z) + \frac{d(a)}{d(z)} \tan (z) \right],$$

$$(\lambda) = \lambda'\left(1 + \frac{d(r)}{d(z)}\right) = \lambda' \left[1 + (a) \sec^2 (z) + \frac{d(a)}{d(z)} \tan (z) \right]$$

Substituting these values in the above equations, and comparing similar terms, we find

$$\left.\begin{aligned} la'' &= 2la' - l(a) + l\left(1 + \frac{d(a)}{(r)\,d(z)}\right) \\ \left(1 + \frac{d(a)}{(r)\,d(z)}\right)(A'' - A') &= -\frac{A'(a)}{\cos^2 (z)} + \frac{d(A)}{d(z)} \cot (z) \\ \left(1 + \frac{d(a)}{(r)\,d(z)}\right)(\lambda'' - \lambda') &= -\frac{\lambda'(a)}{\cos^2 (z)} + \frac{d(\lambda)}{d(z)} \cot (z) \end{aligned}\right\} \quad (346)$$

by which la'', A'', λ'' are computed. The quantities a and a' in Columns A and B of the table, are expressed in seconds, but a'' in Column C is in parts of the radius, so that we must add to the value found by the first of the equations (346), the constant $\log \sin 1'' = 4.685575$. In the second member of the other two equations we must also put $(\alpha) \sin 1''$ for (α), and $d(z) \sin 1''$ for $d(z)$.

295. With the table thus prepared, the computation of $\varkappa$ is precisely like that of k in finding the refraction. For example, to find $\log \varkappa$ for $\zeta = 80°$, Barom. 30.35 inches, Attached Therm. 40° F., Ext. Therm. 35° F; we have

$A'' =$	0.994	$\lambda'' =$	1.099	$\log \alpha'' =$	6.3947
$\log B =$	$+\ 0.01092$	$\log \gamma =$	$+\ 0.01185$	$A'' \log \beta =$	$+\ 0.0105$
$\log T =$	$-\ 0.00031$			$\lambda'' \log \gamma =$	$+\ 0.0130$
$\log \beta =$	$+\ 0.01061$			$\log \varkappa =$	6.4182

296. Our fundamental equations (342) and (343) may now be reduced to a much more simple form. It is evident that on account of the small value of $\varkappa$ we may omit the terms in $(b-a)^2$, &c. For the same reason, we may put $\frac{b-a}{2}$ for $\frac{b-a}{b+a}$, from which it differs only by terms involving $\varkappa^2$. In (343) we may put $a - 1 = \varkappa$ instead of its true value k, without sensible error; for even at the zenith distance 85° the difference of $\varkappa$ and k is only 0.00006, and consequently the error of substituting one for the other in this term will be less than $s \times 0.00006$, so that even if s were as great as $1000''$ the error would not amount to $0''.06$. We therefore adopt as fundamental the following simplified forms:

$$\left.\begin{aligned} \sigma - s &= s\varkappa\,(\tan^2\zeta \cos^2 l_0 + 1) \\ \lambda_0 - l_0 &= -\,\varkappa \tan^2\zeta \cos l_0 \sin l_0 \end{aligned}\right\} \quad (347)$$

In these equations ζ is the mean of the true zenith distances of the two stars, and $\varkappa$ the corresponding quantity in the refraction table. The quantity l_0 is that which would be given directly by the observation.

The mean zenith distance ζ will be found, by a single computation, from the mean of the hour angles of the two stars and the mean of their declinations. Denoting these by τ_0 and δ_0, and the latitude of the place of observation by φ, we have, by equations (20), Vol. I.,

$$\left.\begin{aligned} \tan N &= \cot \varphi \cos \tau_0 \\ \tan \zeta \sin q &= \frac{\tan \tau_0 \sin N}{\sin (\delta_0 + N)} \\ \tan \zeta \cos q &= \cot (\delta_0 + N) \end{aligned}\right\} \quad (348)$$

The parallactic angle q which these formulæ give at the same time with ζ will be required in the subsequent problems. In the observatory the computation is facilitated by a table, computed for the given latitude, which gives the value of N, and of og $n = \log(\tan \tau_0 \sin N)$, for every minute of the hour angle τ. Ve then have only to compute the equations

$$\left.\begin{aligned} \tan \zeta \sin q &= n \operatorname{cosec} (\delta_0 + N) \\ \tan \zeta \cos q &= \cot (\delta_0 + N) \end{aligned}\right\} \quad (348^*)$$

297. *Correction for refraction of micrometric observations of the istance and position angle between two stars.*—The observed position ngle p is the position angle at the middle point of the arc joining he two stars (Art. 260). Let π denote the true value of this ngle, q the true parallactic angle found by (348); then we have

$$\lambda_0 = \pi - q$$

and if q' is the apparent parallactic angle, we have

$$l_0 = p - q'$$

From the differential formula (47) of Vol. I. we find that if ζ varies by $d\zeta = r$, the angle q varies by the quantity

$$q' - q = r \sin q \tan \delta_0$$

and if we take for r the form (Vol. I. Art. 119)

$$r = k' \tan \zeta$$

we have

$$q' = q + k' \tan \zeta \sin q \tan \delta_0$$

and, consequently,

$$l_0 = p - q - k' \tan \zeta \sin q \tan \delta_0$$

This value of l_0 is to be substituted in (347); but in the terms already multiplied by $s\varkappa$ we may take $l_0 = p - q$. Hence we have

$$\begin{aligned} \sigma - s &= s\varkappa \left[\tan^2 \zeta \cos^2 (p - q) + 1\right] \\ \pi - p &= -\varkappa \tan^2 \zeta \cos (p - q) \sin (p - q) - k' \tan \zeta \sin q \tan \delta_0 \end{aligned}$$

Since the position angle cannot be determined within a num- r of seconds, the error of putting $\varkappa$ for k' in the last term of e formula for $\pi - p$ will be of no practical importance; and, oreover, since the terms of the series (342) have to be reduced to seconds by multiplying by the radius in seconds (= cosec 1″ we have, finally,

$$\sigma - s = s\varkappa[\tan^2\zeta\cos^2(p-q) + 1] \qquad (349$$
$$\pi - p = -\varkappa\,\text{cosec}\,1''[\tan^2\zeta\cos(p-q)\sin(p-q) + \tan\zeta\sin q\tan\delta_0$$

Having obtained σ and π by adding these corrections to s and the *true* difference of right ascension and declination of the sta may then be computed by Art. 264, employing σ and π for s a p; that is, by the formulæ

$$\left.\begin{aligned}\sin\tfrac{1}{2}(\alpha'-\alpha) &= \sin\tfrac{1}{2}\sigma\sin\pi\sec\delta_0\\ \sin\tfrac{1}{2}(\delta'-\delta) &= \sin\tfrac{1}{2}\sigma\cos\pi\sec\tfrac{1}{2}(\alpha'-\alpha)\end{aligned}\right\} \qquad (35$$

or by the approximate formulæ

$$\left.\begin{aligned}\alpha'-\alpha &= \sigma\sin\pi\sec\delta_0\\ \delta'-\delta &= \sigma\cos\pi\end{aligned}\right\} \qquad (350$$

298. If the apparent differences of right ascension and de nation have already been computed from s and p by Art. 2 and we wish to correct them for refraction, we have, by compar the formulæ (284) and (350*), and denoting the corrections wh the apparent values of $\alpha'-\alpha$ and $\delta'-\delta$ require by the symbol

$$\begin{aligned}\Delta(\alpha'-\alpha) &= (\sigma\sin\pi - s\sin p)\sec\delta_0\\ \Delta(\delta'-\delta) &= \sigma\cos\pi - s\cos p\end{aligned}$$

or,

$$\begin{aligned}\Delta(\alpha'-\alpha) &= [(\sigma-s)\sin p + \sigma(\sin\pi-\sin p)]\sec\delta_0\\ \Delta(\delta'-\delta) &= (\sigma-s)\cos p + \sigma(\cos\pi-\cos p)\end{aligned}$$

or, again, with sufficient accuracy,

$$\begin{aligned}\Delta(\alpha'-\alpha) &= [(\sigma-s)\sin p + s(\pi-p)\sin 1''\cos p]\sec\delta_0\\ \Delta(\delta'-\delta) &= (\sigma-s)\cos p - s(\pi-p)\sin 1''\sin p\end{aligned}$$

and, substituting the values of $\sigma - s$ and $\pi - p$ from (349),

$$\begin{aligned}\Delta(\alpha'-\alpha) &= s\varkappa[\tan^2\zeta\cos(p-q)\sin q - \tan\zeta\sin q\tan\delta_0\cos p + \sin p]\,\text{se}\\ \Delta(\delta'-\delta) &= s\varkappa[\tan^2\zeta\cos(p-q)\cos q + \tan\zeta\sin q\tan\delta_0\sin p + \cos p]\end{aligned} \qquad (3$$

These formulæ are somewhat abridged by introducing auxiliary u such that

$$\tan u = \tan\zeta\sin q\tan\delta_0$$

by which they become

$$\left.\begin{aligned}\Delta(\alpha'-\alpha) &= s\varkappa\,[\tan^2\zeta\cos(p-q)\sin q+\sec u\sin(p-u)]\sec\delta_0\\ \Delta(\delta'-\delta) &= s\varkappa\,[\tan^2\zeta\cos(p-q)\cos q+\sec u\cos(p-u)]\end{aligned}\right\}\quad(351^*)$$

Example.—In the example, Art. 264, we had the observed quantities $s = 316''.993$, $p = 169°\ 57'.7$. The latitude of the place of observation was $\varphi = 38°\ 53'.7$, and the sidereal time was $0^h\ 17^m\ 52^s$. The right ascension and declination of the middle point between the stars were, approximately,

$$\alpha_0 = 21^h\ 51^m\ 52^s \qquad \delta_0 = -13°\ 28'.5$$

The corrections for refraction being exceedingly small in the case of so small a value of s, the observer did not think it necessary to record the state of the atmosphere; but, for the sake of illustration, I shall assume Barometer 30.29 inches, Att. Therm. 49°, Ext. Therm. 41° Fahr.

We have, first, the hour angle of the middle point between the observed bodies, $\tau_0 = 2^h\ 26^m = 36°\ 30'$, with which and the above values of φ and δ_0 we find, by (348),

$$N = 44°\ 53'.9 \qquad \zeta = 62°\ 28'.5 \qquad q = 31°\ 28'.2$$

and by Column C of Table II.,

$$\log\varkappa = 6.4555$$

Then, by (349), we find

$$\sigma - s = +\ 0''.277 \qquad \pi - p = +\ 2'\ 1''.7$$

and hence

$$\sigma = 317''.270 \qquad \pi = 169°\ 59'.73$$

From these, by (350*), the true difference of right ascension and declination are found to be

$$(\alpha' - \alpha) = +\ 56''.68 \qquad (\delta' - \delta) = -\ 5'\ 12''.45$$

But, supposing the apparent differences to have been already computed as in Art. 264, namely,

$$\alpha' - \alpha = +\ 56''.82 \qquad \delta' - \delta = -\ 5'\ 12''.14$$

we should compute the corrections of these quantities by (351*), which give

$$\Delta(\alpha' - \alpha) = -\ 0''.136 \qquad \Delta(\delta' - \delta) = -\ 0''.306$$

which added to $\alpha' - \alpha$ and $\delta' - \delta$ give the same values $(\alpha' - \alpha)$ and $(\delta' - \delta)$ as above found.

299. *Correction for refraction of micrometer observations in wi the difference of right ascension has been obtained from the differenc the times of transit of the stars over threads lying in the direction circles of declination, and the difference of declination has been dire measured.* (2d Method, Art. 266.)

Let t and t' denote the observed sidereal times of transit of t two stars over the same declination circle. A star upon t same parallel of declination as the second star, but having t right ascension $\alpha' - (t' - t)$, would have been observed sim taneously with the first star, and would, therefore, have had t same apparent right ascension. The effect of refraction up the time of transit of this supposed star is evidently the same in the case of the real star; and the effect upon the differe of declination is also the same: so that this case is reduced the preceding by supposing the stars to have been observed w an apparent position angle $p = 0$, and apparent distance s $\delta' - \delta$. These substitutions in (351) give the required correcti

$$\Delta(\alpha' - \alpha) = \varkappa(\delta' - \delta)\,[\tan^2\zeta \cos q \sin q - \tan\zeta \sin q \tan\delta_0] \sec$$
$$\Delta(\delta' - \delta) = \varkappa(\delta' - \delta)\,[\tan^2\zeta \cos^2 q + 1]$$

These formulæ are simplified by introducing the auxiliary already used in the computation of ζ. Substituting the values $\tan\zeta \sin q$ and $\tan\zeta \cos q$ from (348) and (348*), they are reac reduced to the following:

$$\left.\begin{aligned} \Delta(\alpha' - \alpha) &= \frac{\varkappa(\delta' - \delta)}{\sin^2(\delta_0 + N)} \cdot \frac{n \cos(2\delta_0 + N)}{\cos^2\delta_0} \\ \Delta(\delta' - \delta) &= \frac{\varkappa(\delta' - \delta)}{\sin^2(\delta_0 + N)} \end{aligned}\right\} \quad (3$$

Example.—In the example, Art. 266, we have the obser difference of right ascension and declination of *Neptune* an known star,

$$\alpha' - \alpha = +\,1^m\,45^s.30 \qquad \delta' - \delta = -\,7'\,28''.22$$

and the place of the star,

$$\alpha = 21^h\,50^m\,8^s.99 \qquad \delta = -\,13°\,23'\,35''.11$$

The sidereal time of the star's transit being $23^h\ 26^m\ 43^s.4$, the common hour angle at which the objects were observed was

$$\tau_0 = 1^h\ 36^m\ 34^s.4 = 24°\ 8'.6$$

with which and $\varphi = 38°\ 53'.7$, $\delta_0 = -13°\ 27'.3$, we find, by (348),

$$N = 48°\ 31'.3 \qquad \log n = \log(\tan\tau_0 \sin N) = 9.5261$$
$$\zeta = 57\quad 0.1$$

and assuming Barom. 30.29 inches, Att. Therm. 49°, Ext. Therm. 41° Fahr., we find, by Column C of Table II.,

$$\log \varkappa = 6.4577$$

Hence, by (352),

$$\Delta(\alpha' - \alpha) = -0''.128 = -0^s.009 \qquad \Delta(\delta' - \delta) = -0''.389$$

The differences corrected for refraction are, therefore,

$$\alpha' - \alpha = +1^m\ 45^s.29 \qquad \delta' - \delta = -7'\ 28''.61$$

and hence the apparent place of *Neptune*, affected now only by parallax, was

$$\alpha' = 21^h\ 51^m\ 54^s.28 \qquad \delta' = -13°\ 31'\ 3''.72$$

on November 29, 1846, at $23^h\ 28^m\ 28^s.7$ sidereal time at Washington.

300. *Correction for refraction of observations made with the ring micrometer.*—At each transit of a star over the edge of the ring, its *apparent* distance from the centre, C, of the ring is equal to the radius r. If at the time t_1 of its first transit its true distance is σ_1, we shall have, by (349), putting r for s,

$$\sigma_1 = r\,[1 + \varkappa + \varkappa \tan^2 \zeta \cos^2 (p - q)] \qquad (353)$$

in which p is the position angle of the star referred to C. The zenith distance ζ and the parallactic angle q belong to the middle point between the star and C; but it is easily seen that it will produce no important error to assume them either for the point C or for a mean point between the stars compared. We shall, therefore, here suppose ζ and q to have the same values for all observations made in the same position of the ring. At the time t_2 of the star's second transit, the position angle, reckoned in the same direction as for the first transit from the declination circle through C, will be $360° - p$: so that, if σ_2 is then the true distance of the star from C, we have

$$\sigma_2 = r\,[1 + \varkappa + \varkappa \tan^2 \zeta \cos^2 (p + q)] \qquad (354)$$

Now, let

t_0 = the time of the star's transit over the *true* declination circle of C,
τ_1, τ_2 = the *true* hour angles of the star, reckoned from the declination circle of C, at the two observed transits,
δ, D = the declination of the star and of C;

then we have

$$t_0 = t_1 + \tau_1, \qquad t_0 = t_2 - \tau_2$$

and in the two triangles formed by the pole, the point C, and the two *true* places of the stars at the two observations, we have

$$\cos \sigma_1 = \sin D \sin \delta + \cos D \cos \delta \cos \tau_1$$
$$\cos \sigma_2 = \sin D \sin \delta + \cos D \cos \delta \cos \tau_2$$

From the difference of these equations, namely,

$$2 \sin \tfrac{1}{2} (\sigma_1 + \sigma_2) \sin \tfrac{1}{2} (\sigma_1 - \sigma_2) = 2 \cos D \cos \delta \sin \tfrac{1}{2} (\tau_1 + \tau_2) \sin \tfrac{1}{2} (\tau_1 - \tau_2)$$

we derive, approximately,

$$\tfrac{1}{2} (\tau_1 - \tau_2) = \left(\frac{\sigma_1 - \sigma_2}{2}\right)\left(\frac{\sigma_1 + \sigma_2}{2}\right) \frac{2 \sec D \sec \delta}{\tau_1 + \tau_2}$$

To reduce this expression to a practical form, we have first, from (353) and (354),

$$\tfrac{1}{2} (\sigma_1 - \sigma_2) = r \varkappa \tan^2 \zeta \sin p \cos p \sin 2q$$

in which we may use the approximate values of $\sin p$ and $\cos p$ given by (331), where γ is the same as p; namely,

$$\sin p = \frac{(t_2 - t_1) \cos \delta}{2r} \qquad \cos p = \frac{d}{r}$$

where d is the approximate value of $\delta - D$ found by neglecting the refraction.

For $\frac{1}{2} (\sigma_1 + \sigma_2)$ we may here use r; for, being only a multiplier of $\frac{1}{2} (\sigma_1 - \sigma_2)$, the remaining terms would give only terms in $\varkappa^2$

in the product. For $\tau_1 + \tau_2$ we put $t_2 - t_1$. These substitutions being made in the value of $\frac{1}{2}(\tau_1 - \tau_2)$, we have

$$\tfrac{1}{2}(\tau_1 - \tau_2) = d\varkappa \tan^2\zeta \sin 2q \sec D \qquad (855)$$

which is the correction to be added to the mean of the observed times, in order to obtain the true time t_0 of the star's transit over the declination circle of the centre of the ring; for we have

$$t_0 = \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}(\tau_1 - \tau_2)$$

To find the correction of d for refraction, we observe that if τ_1 and τ_2 were known, the true value of the difference $\delta - D$ would be found by the formulæ

$$(\delta - D)^2 = \sigma_1^2 - (\tau_1 \cos\delta)^2$$
$$(\delta - D)^2 = \sigma_2^2 - (\tau_2 \cos\delta)^2$$

In these formulæ, indeed, the path of the star is supposed to be rectilinear; but the correction for curvature has already been investigated, and is given by (333). The mean of these values may be expressed as follows:

$$(\delta - D)^2 = \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 + \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 - \left(\frac{\tau_1 + \tau_2}{2}\right)^2 \cos^2\delta - \left(\frac{\tau_1 - \tau_2}{2}\right)^2 \cos^2\delta$$

and, consequently, by neglecting terms in $\varkappa^2$,

$$(\delta - D)^2 = \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 - \left(\frac{\tau_1 + \tau_2}{2}\right)^2 \cos^2\delta.$$

The difference d is found from the formula

$$d^2 = r^2 - \left(\frac{t_2 - t_1}{2}\right)^2 \cos^2\delta$$

and therefore, observing that $\tau_1 + \tau_2 = t_2 - t_1$,

$$\begin{aligned}(\delta - D)^2 - d^2 &= \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 - r^2 \\ &= 2r^2\varkappa\,[1 + \tan^2\zeta\,(\sin^2 q + \cos^2 p \cos 2q)]\end{aligned}$$

Substituting d for $r\cos p$, and then dividing both members by $(\delta - D) + d$, (or by $2d$, since this will involve only errors of the order $\varkappa^2$), we find

$$(\delta - D) - d = \frac{r^2\varkappa}{d}(\tan^2\zeta \sin^2 q + 1) + d\varkappa \tan^2\zeta \cos 2q \qquad (856)$$

which is the required correction to be added to d.

For a second star, we have, in like manner,

$$\left.\begin{aligned} \tfrac{1}{2}(\tau_1' - \tau_2') &= d' \varkappa \tan^2 \zeta \sin 2q \sec D \\ t_0' &= \tfrac{1}{2}(t_1' + t_2') + \tfrac{1}{2}(\tau_1' - \tau_2') \end{aligned}\right\} \quad (357)$$

$$(\delta' - D) - d' = \frac{r^2 \varkappa}{d'}(\tan^2 \zeta \sin^2 q + 1) + d' \varkappa \tan^2 \zeta \cos 2q \quad (358)$$

The difference of right ascension of the stars found by neglecting the refraction is

$$\alpha' - \alpha = \tfrac{1}{2}(t_1' + t_2') - \tfrac{1}{2}(t_1 + t_2)$$

while the true value is $t_0' - t_0$: so that the correction for the refraction is

$$\Delta(\alpha' - \alpha) = \tfrac{1}{2}(\tau_1' - \tau_2') - \tfrac{1}{2}(\tau_1 - \tau_2)$$

or, by (355) and (357),

$$\Delta(\alpha' - \alpha) = (d' - d) \varkappa \tan^2 \zeta \sin 2q \sec \delta_0 \quad (359)$$

in which we have put $\delta_0 = \frac{1}{2}(\delta + \delta')$ instead of D. The correction of the difference of the declinations of the stars for refraction is, by (356) and (358),

$$\Delta(\delta' - \delta) = (d' - d) \varkappa \tan^2 \zeta \cos 2q - \frac{r^2 (d' - d)}{dd'} \cdot \varkappa (\tan^2 \zeta \sin^2 q + 1) \quad (360)$$

The values of ζ and q to be used in these formulæ will be found by (348), employing $\delta_0 = \frac{1}{2}(\delta + \delta')$ and the hour angle τ_0 of the centre of the ring, which will be found from the transit of one of the stars by the formula

$$\tau_0 = \tfrac{1}{2}(t_1 + t_2) - \alpha$$

EXAMPLE.—To correct the results in the example of Art. 285 for refraction.—We have there found

$$\begin{array}{rlrl} d' & = +\ 17'\ 51''.9 & \varphi & = +\ 51^\circ\ 12'.4 \\ d & = +\ 13\ \ 34\ .8 & \delta_0 & = +\ 24\ \ \ 3\ .3 \\ d' - d & = +\ \ 4\ \ 17\ .1 & \tau_0 & = \ \ 5^h\ 13^m\ 58^s. \\ \alpha' - \alpha & = -\ \ 1^m\ 49^s.50 & r & = \ \ 1126''.25 \end{array}$$

We find, by (348),

$$\begin{array}{rlrl} N & = \ \ 9^\circ\ \ 6'.7 & \log n & = 9.89083 \\ q & = 42\ \ 53\ .7 & \zeta & = 64^\circ\ 25'.0 \end{array}$$

The indications of the barometer and thermometer are not given; but, assuming a mean state of the air, the refraction table gives for this zenith distance log $\kappa = 6.4382$, with which we proceed to compute (359) and (360) as follows:

$\log (d' - d)$	2.4101		
$\log \kappa$	6.4382		
$\log \tan^2 \zeta$	0.6398	$\log \sec \delta_0$	0.0395
	9.4881		9.4881
$\log \cos 2q$	8.8658	$\log \sin 2q$	9.9988
1st term of (360) =	$+ 0''.02$	$\log \Delta(\alpha' - \alpha)$	9.5264
		$\Delta(\alpha' - \alpha) =$	$+ 0''.34 = + 0^s.02$
$\log \sin^2 q$	9.6658		
og $(\tan^2 \zeta \sin^2 q + 1)$	0.4802		
$\log (d' - d) \kappa$	8.8483		
$\log r^2$	6.1032		
	5.4317		
$\log dd'$	5.9412	The corrected values are then	
2d term of (360) =	$+ 0''.31$	$\alpha' - \alpha =$	$- 1^m\, 49^s.48$
$\Delta(\delta' - \delta) =$	$- 0''.29$	$\delta' - \delta =$	$+ 4'\, 16''.81$

The corrections for refraction are in this instance less than the probable errors of observation. Indeed, with the ring micrometer, it will seldom be worth while to consider the refraction unless the zenith distance is over 60° and the difference of declination over 10′.

CORRECTION OF MICROMETRIC OBSERVATIONS FOR PRECESSION, NUTATION, AND ABERRATION.

301. In most cases, micrometer observations of the difference of position of two celestial bodies have for their object the determination of the apparent place of one of these bodies from that of the other supposed to be given. The apparent place thus found is then usually to be reduced to the mean place for the beginning of the year, or any adopted epoch, by applying the corrections for precession, nutation, and aberration with reversed sign. Sometimes, also, it is desirable to reduce the data furnished by the micrometer on different dates to a common date. The only case of interest, however, is that in which the distance and position angle have been observed. I shall consider first the effect of aberration.

302. *To find the effect of aberration upon the angular distance of two stars.*—Let us denote by E the point of the ecliptic from which the earth is moving (as in Art. 387 of Vol. I.); by ϑ_1, ϑ_2, the true angular distances of the stars from E; by $\vartheta_1', \vartheta_2'$, the apparent distances from E affected by aberration; by σ and s, the true and apparent distances of the stars from each other; by γ_1, γ_2, the angles formed by σ with ϑ_1 and ϑ_2; by γ_1', γ_2', the angles formed by s with the same arcs. Then, since the aberration acts only in the great circle joining the star and the point E, the angle at E between the arcs ϑ_1 and ϑ_2 remains unchanged, and we have, precisely as in the investigation of the differential refraction in Art. 293,

$$\sin \tfrac{1}{2}\sigma \sin \tfrac{1}{2}(\gamma_1 + \gamma_2) = \sin \tfrac{1}{2} s \sin \tfrac{1}{2}(\gamma_1' + \gamma_2') \frac{\sin \frac{1}{2}(\vartheta_1 + \vartheta_2)}{\sin \frac{1}{2}(\vartheta_1' + \vartheta_2')}$$
$$\sin \tfrac{1}{2}\sigma \cos \tfrac{1}{2}(\gamma_1 + \gamma_2) = \sin \tfrac{1}{2} s \cos \tfrac{1}{2}(\gamma_1' + \gamma_2') \frac{\sin \frac{1}{2}(\vartheta_1 - \vartheta_2)}{\sin \frac{1}{2}(\vartheta_1' - \vartheta_2')}$$

If we write γ_0 and γ_0' for $\frac{1}{2}(\gamma_1 + \gamma_2)$ and $\frac{1}{2}(\gamma_1' + \gamma_2')$, we may put these equations under the form

$$\sigma \sin \gamma_0 = a\, s \sin \gamma_0'$$
$$\sigma \cos \gamma_0 = b\, s \cos \gamma_0'$$

in which we have put

$$a = \frac{\sin \vartheta_0}{\sin \vartheta_0'} \qquad b = \frac{\sin \frac{1}{2}(\vartheta_1 - \vartheta_2)}{\sin \frac{1}{2}(\vartheta_1' - \vartheta_2')}$$

Now, we have (Art. 385, Vol. I.)

$$\vartheta_0' - \vartheta_0 = k \sin \vartheta_0$$

in which $k = 20''.4451$; and hence

$$a = 1 - k \cos \vartheta_0$$
$$b = \frac{1}{1 + k \cos \vartheta_0} = 1 - k \cos \vartheta_0 + k^2 \cos^2 \vartheta_0 - \&c.$$

so that if we neglect k^2, as we may, we have $a = b$, and hence our equations give, simply,

$$\gamma_0 = \gamma_0'$$
$$\sigma = a s$$

Hence it follows, 1st, that the angle which σ makes with the arc ϑ_0 is not sensibly changed by the aberration; 2d, that the effect of aberration upon the distance σ is the same in whatever *direction* the arc σ may lie, and depends only on the distance (ϑ_0) of its middle point from the point E, or, in general, upon the right ascension and declination of this middle point. This latter principle suggests the most simple means of investigating a formula for computing the aberration in distance; we have only to assume the distance σ to coincide in direction with a declination circle, so that σ may be treated as the difference of declination of the stars. Then the effect of aberration upon σ will be found by differentiating the expression $Cc' + Dd'$, which expresses the correction for aberration (Art. 402, Vol. I.); thus,

$$\Delta\sigma = \sigma\left[C\cdot\frac{dc'}{d\delta} + D\cdot\frac{dd'}{d\delta}\right]$$

Taking the values of a' and b' for the middle point of σ, or for the right ascension α_0 and declination δ_0, we put

$$\gamma = \sigma\cdot\frac{dc'}{d\delta} = -\ \sigma(\tan\epsilon\sin\delta_0 + \sin\alpha_0\cos\delta_0)$$

$$\delta = \sigma\cdot\frac{dd'}{d\delta} = \sigma\,.\cos\alpha_0\cos\delta_0$$

and then for computing $\Delta\sigma$ we have the simple formula

$$\Delta\sigma = +\ C\gamma + D\delta \tag{361}$$

for which C and D can be taken from the Ephemeris for the given date. The correction thus found is to be added to the true distance to obtain the apparent distance.

The position angle p_0 at the middle point of σ is composed of the angle γ_0 and of the angle which the declination circle makes with the arc ϑ_0: so that the change in p_0 is the same as that in the latter angle, that is, it is the difference of directions of the declination circles drawn through the true and apparent places of the stars. This difference will be obtained at the same time with that produced by precession and nutation in the next article.

303. *To find the effect of precession, nutation, and aberration upon the position angle of two stars.*—Let α_0, δ_0, be the right ascension and declination of the middle point between the two stars. The change Δp_0 in the position angle is simply the change of direction of the declination circle drawn through this point: so that have

$$\tan \Delta p_0 = \Delta p_0 = \frac{d\alpha_0 \cos \delta_0}{d\delta_0}$$

or, taking $\alpha_0 = (\alpha_0) + Aa + Bb + Cc + Dd$ as the expression o the apparent right ascension at any time, where (α_0) is its mea value at the beginning of the given year (Vol. I. Art. 402), w have

$$\Delta p_0 = \cos \delta_0 \left[A \cdot \frac{da}{d\delta_0} + B \cdot \frac{db}{d\delta_0} + C \cdot \frac{dc}{d\delta_0} + D \cdot \frac{dd}{d\delta_0} \right]$$
$$= A.n \sin \alpha_0 \sec \delta_0 + B.\cos \alpha_0 \sec \delta_0 + C.\cos \alpha_0 \tan \delta_0 + D.\sin \alpha_0 \tan$$

Hence, putting

$$\left.\begin{array}{ll} \alpha' = n \sin \alpha_0 \sec \delta_0 & \gamma' = \cos \alpha_0 \tan \delta_0 \\ \beta' = \cos \alpha_0 \sec \delta_0 & \delta' = \sin \alpha_0 \tan \delta_0 \end{array}\right\} \quad (362$$

in which, for a given year 1800 + t (Vol. I. p. 617),

$$n = 20''.0607 - 0''.0000863\,t$$

we have

$$\Delta p_0 = A\alpha' + B\beta' + C\gamma' + D\delta' \qquad (363$$

The annual increase of p_0 is $n \sin \alpha_0 \sec \delta_0$. If we wish to reduc the mean value of p_0 from one given year 1800 + t to anothe 1800 + t', we must, therefore, add the quantity $(t' - t)\, n \sin \alpha_0 \sec \delta_0$ in which α_0 and δ_0 should be taken for the date $1800 + \frac{1}{2}(t + t')$ The mean value of p_0 being thus reduced to the beginning of th year 1800 + t', its apparent value for the day of the year wil then be found by adding the correction Δp_0 given by (363), A B, C, and D being taken for the day from the annual Ephemeri or the *Tabulæ Regiomontanæ.*

The precession and nutation, evidently, do not affect the app rent angular distance of two stars.

APPENDIX.

METHOD OF LEAST SQUARES.*

1. A NUMBER of observations being taken for the purpose of determining one or more unknown quantities, and these observations giving discordant results, it is an important problem to determine the *most probable* values of the unknown quantities. The method of least squares may be defined to be that method of treating this general problem which takes as its fundamental principle, that *the most probable values are those which make the sum of the squares of the residual errors a minimum.* But, to understand this definition, some degree of acquaintance with the method itself is necessary.

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* The first published application of the method is to be found in LEGENDRE, *Nouvelles méthodes pour la détermination des orbites des comètes*, Paris, 1806. The development, however, from fundamental principles is due to GAUSS, who declared that he had used the method as early as 1795. See his *Theoria Motus Corporum Cœlestium*, 1809, Lib. II. Sec. III.; *Disquisitio de elementis ellipticis Palladis*, 1811; *Bestimmung der Genauigkeit der Beobachtungen* (v. LINDENAU und BOHNENBERGER'S *Zeitschrift*, 1816, I. i. 185); *Theoria combinationis observationum erroribus minimis obnoxiæ*, 1828; *Supplementum theoriæ combinationis*, &c., 1826: all of which have been rendered quite accessible through a French translation by J. BERTRAND, *Méthode des moindres carrées. Mémoires sur la combinaison des observations*, par CH. FR. GAUSS, Paris, 1855.

For a digest of the preceding, together with the results of the labors of BESSEL and HANSEN, see ENCKE, *Ueber die Methode der kleinsten Quadrate*, Berliner Astron. Jahrbuch for 1834, 1835, 1836; in connection with which must be mentioned especially the practical work of GERLING, *Die Ausgleichungsrechnungen der practischen Geometrie*, Hamburg, 1843.

See also LAPLACE, *Théorie analytique des probabilités*, Liv. II. Chap. IV.; POISSON, *Sur la probabilité des résultats moyens des observations*, in the Connaissance des Temps for 1827; ENCKE, in the Berlin Jahrbuch for 1853; BESSEL, in *Astron. Nach.*, Nos. 358, 359, 399; HANSEN, in *Astron. Nach.*, Nos. 192, 202 et seq.; PEIRCE, in the *Astron. Journal* (Cambridge, Mass.), Vol. II. No. 21; LIAGRE, *Calcul des probabilités et théorie des erreurs*, Bruxelles, 1852.

ERRORS TO WHICH OBSERVATIONS ARE LIABLE.

2. Every observation which is a *measure*, however carefully it may be made, is to be regarded as subject to error; for experience teaches that repeated measures of the same quantity, *when the greatest precision is sought*,[*] do not give uniformly the same result. Two kinds of errors are to be distinguished.

Constant or *regular errors* are those which in all measures of the same quantity, made under the same circumstances, obtain the same magnitude; or whose magnitude is dependent upon the circumstances according to any determinate law. The causes of such errors must be the subject of careful preliminary search in all physical inquiries, so that their action may be altogether prevented or their effect removed by calculation. For example, among the constant errors may be enumerated refraction, aberration, &c.; the effect of the temperature of rods used in measuring a base line in a survey; the error of division of a graduated instrument when the same division is used in all the measures; any peculiarity of an instrument which affects a particular measurement always by the same amount, such as inequality of the pivots of a transit instrument, defective adjustment of the collimation, imperfections of lenses, defects of micrometer screws, &c., to which must be added constant peculiarities of the observer, who, for example, may always note the passage of a star over a thread of a transit instrument too soon, or too late, by a constant quantity, or who, in attempting to bisect a star with a micrometer thread, constantly makes the upper or the lower portion the greater; or who, in observing the contact of two images (in sextant measures, for instance), assumes for a contact a position in which the images are really at some constant small distance, or a position in which the images are really overlapped, &c. &c.

Thus, we have three kinds of constant errors:

1st. *Theoretical*, such as refraction, aberration, &c., whose effects, when their causes are once thoroughly understood, may be calculated *a priori*, and which thenceforth cease to exist as errors.

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[*] The qualification, "when the greatest precision is sought," is important; for if, *e.g.*, we were to determine the latitude of a place by repeated measures of the meridian altitude of the same fixed star with a sextant divided only to whole degrees, all our measures might give the same degree. The accordance of observations is, therefore, not to be taken as an infallible evidence of their accuracy. It is especially when we approach *the limits of our measuring powers* that we become sensible of the discrepancies of observations.

The detection of a constant error in a certain class of observations very commonly leads to investigations by which its cause is revealed, and thus our physical theories are improved.

2d. *Instrumental*, which are discovered by an examination of our instruments, or from a discussion of the observations made with them. These may also be removed when their causes are fully understood, either by a proper mode of using the instrument, or by subsequent computation.

3d. *Personal*, which depend upon peculiarities of the observer, and in delicate inquiries become the subject of special investigation under the name of "personal equations."

We are to assume that, in any inquiry, all the sources of constant error have been carefully investigated, and their effects eliminated as far as practicable. When this has been done, however, we find by experience that there still remain discrepancies, which must be referred to the next following class.

Irregular or *accidental errors* are those which have irregular causes, or whose effects upon individual observations are governed by no fixed law connecting them with the circumstances of the observations, and, therefore, can never be subjected *a priori* to computation. Such, for example, are errors arising from tremors of a telescope produced by the wind; errors in the refraction produced by anomalous changes of density of the strata of the atmosphere; from unavoidable changes in the several parts of an instrument produced by anomalous variations of temperature, or anomalous contraction and expansion of the parts of an instrument even at known temperatures; but, more especially, errors arising from the imperfection of the senses, as the imperfection of the eye in measuring very small spaces, of the ear in estimating small intervals of time, of the touch in the delicate handling of an instrument, &c.

This distinction between constant and irregular errors is, indeed, to a certain extent, rather relative than absolute, and depends upon the sense, more or less restricted, in which we consider observations to be of the *same nature* or made under the *same circumstances*. For example, the errors of division of an instrument may be regarded as constant errors when the same division comes into all measures of the same quantity, but as irregular when in every measure a different division is used, or when the same quantity is measured repeatedly with different instruments.

After a full investigation of the constant or regular errors, it is the next business of the observer to diminish as much as possible the irregular errors by the greatest care in the observations; and finally, when the observations are completed, there remains the important operation of combining them, so that the outstanding, unavoidable, irregular errors may have the least probable effect upon the results. For this combination we invoke the aid of the method of least squares, which may be said to have for its object the restriction of the effect of irregular errors within the narrowest limits according to the theory of probabilities, and, at the same time, to determine from the observations themselves the errors to which our results are probably liable. It is proper to observe here, however, to guard against fallacious applications, that the theory of the method is grounded upon the hypothesis that we have taken a large number of observations, or, at least, a number sufficiently large to determine the errors to which the observations are liable.

CORRECTION OF THE OBSERVATIONS.

3. When no more observations are taken than are sufficient to determine one value of each of the unknown quantities sought, we have no means of judging of the correctness of the results, and, in the absence of other information, are compelled to accept these results as true, or, at least, as the most probable. But when additional observations are taken, leading to different results, we can no longer unconditionally accept any one result as true, since each must be regarded as contradicting the others. The results cannot all be true, and are all probably, in a strict sense, false. The absolutely true value of the quantity sought by observation must, in general, be regarded as beyond our reach; and instead of it we must accept a value which may or may not agree with any one of the observations, but which is rendered *most probable* by the existence of these observations.

The condition under which such a probable value is to be determined, is that *all contradiction among the observations is to be removed.* This is a logical necessity, since we cannot accept for truth that which is contradictory or leads to contradictory results.

The contradiction is obviously to be removed by applying to the several observations (or conceiving to be applied) probable *corrections*, which shall make them agree with each other, and which we have reason to suppose to be equivalent in amount to the accidental errors severally. But let us here remark that we do not in this statement by any means imply that an observer is to *arbitrarily* assume a system of corrections which will produce accordance: on the contrary, the method we are about to consider is designed to remove, as far as possible, every arbitrary consideration, and to furnish a set of principles which shall always guide us to the most probable results. The conscientious observer, having taken every care in his observation, will set it down, however discrepant it may appear to him, as a portion of the testimony collected, out of which the truth, or the nearest approximation to it, is to be sifted.

Admitting, therefore, that the observations give us the best, as indeed the only, information we can obtain respecting the desired quantities, we must find a system of corrections which shall not only produce the desired accordance, but which shall also be the *most probable* corrections, and further *be rendered most probable by these observations themselves.*

THE ARITHMETICAL MEAN.

4. In order to discover a principle which may serve as a basis for the investigation, let us examine first the case of direct observations made for the purpose of determining a single unknown quantity.

Let the quantity to be determined by direct observation be denoted by x. (Suppose, for example, to fix our ideas, that this quantity is the linear distance between two fixed terrestrial points.) If but one measure of x is taken and the result is a, we must accept as the only and, therefore, the most probable value, $x = a$. Let a second observation, taken under the same or precisely equivalent circumstances, and with the same degree of care, so that there is no reason for supposing it to be more in error than the first, give the value b. Then, since there is no reason for preferring one observation to the other, the value of x must be so taken that the differences $x - a$, $x - b$ shall be numerically equal; and this gives

$$x = \tfrac{1}{2}(a + b)$$

This result must be regarded as the only one that can be inferred from the two observations consistently with our definition of accidental errors; for positive and negative accidental errors of equal absolute magnitude are to be regarded as equal errors and as equally probable, since, from the care bestowed on the observations and the supposed similarity of the circumstances under which they are made, there is no reason *a priori* for assuming either a positive or a negative error to be the more probable.

Now let a third observation be added, giving the value c. Since the three observations are of equal reliability, or, as we shall hereafter say, of *equal weight*, we must so combine a, b, and c that each shall have a like influence upon the result; in other words, x must be a symmetrical function of a, b, and c. If we first consider a and b alone, then a and c, then b and c, we shall find the values

$$\tfrac{1}{2}(a+b), \qquad \tfrac{1}{2}(a+c), \qquad \tfrac{1}{2}(b+c),$$

with each of which the additional observation c, b, or a is to be combined. Each combination must result in the same symmetrical function, which, whatever it may be, can be denoted by the functional symbol ψ. We must, therefore, have

$$\begin{aligned} x &= \psi\,[\tfrac{1}{2}(a+b), c] \\ &= \psi\,[\tfrac{1}{2}(a+c), b] \\ &= \psi\,[\tfrac{1}{2}(b+c), a] \end{aligned}$$

Introducing the sum of a, b, and c, or putting

$$s = a + b + c$$

these become

$$\begin{aligned} x &= \psi\,[\tfrac{1}{2}(s-c), c] = \psi\,[s, c] \\ &= \psi\,[\tfrac{1}{2}(s-b), b] = \psi\,[s, b] \\ &= \psi\,[\tfrac{1}{2}(s-a), a] = \psi\,[s, a] \end{aligned}$$

But s is already a symmetrical function of a, b, and c, and therefore these equations cannot all result in the same symmetrical function unless c, b, a, in the respective developments of the functions, disappear and leave only s. Hence we must have

$$x = \psi(s)$$

Now, to determine ψ, we observe that, as it must be general, its nature may be learned from any special but known case. Such a case is that in which the three observations give three equal values, or $a = b = c$; and in that case we have, as the only value, $x = a$, or

$$a = \psi(3a)$$

and, consequently, the symbol ψ signifies here the division by 3. Hence, generally,

$$x = \frac{a + b + c}{3}$$

In the same manner, if it had been previously shown that for m equally good observations the most probable value is

$$x = \frac{a + b + c + \dots + n}{m}$$

it would follow that for an additional observation p we must have

$$x = \frac{a + b + c + \dots + n + p}{m + 1}$$

for, putting $s = a + b + c + \dots + n + p$, we shall have

$$x = \psi \left[\frac{1}{m} (s - p), p \right] = \psi \, [s, p] = \psi \, (s), \text{ \&c.}$$

But we have shown that the form is true for three observed values: hence, it is true for four; and since it is true for four values it is true for five; and thus generally for any number.*

The principle here demonstrated, that the arithmetical mean of a number of equally good observations is the most probable value of the observed quantity, is that which has been universally adopted as the most simple and obvious, and might well be received as axiomatic. The above demonstration is chiefly valuable as exhibiting somewhat more clearly the nature of the assumption that underlies the principle, which is that, under trictly similar circumstances, positive and negative errors of the ame absolute amount are equally probable.

5. If now $n', n'', n''' \dots n^{(m)}$ are the m observed values of a quired quantity x, and if x_0 denotes their arithmetical mean, e assumption of x_0 as the most probable value of x gives $- x_0, n'' - x_0, n''' - x_0$, &c., as the most probable system of corctions (subtractive from the observed values) which produce e required accordance. But the equation

$$x_0 = \frac{n' + n'' + n''' + \dots + n^{(m)}}{m} \qquad (1)$$

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* ENCKE, Berliner Astron. Jahrbuch for 1884, p. 262.

may also be put under the form

$$(n' - x_0) + (n'' - x_0) + (n''' - x_0) + \dots\dots (n^{(m)} - x_0) = 0$$

that is, *the algebraic sum of the corrections is zero.*

This is, however, not the only characteristic of the system of corrections resulting from the use of the arithmetical mean. Let us examine the sum of the squares of the corrections. For brevity, let us denote the corrections, or, as they will be hereafter called, the *residuals*, by the symbol v: so that

$$v' = n' - x_0, \qquad v'' = n'' - x_0, \qquad v''' = n''' - x_0, \text{ \&c.}$$

and also denote the sums of quantities of the same kind by enclosing the common symbol in rectangular brackets: so that

$$[v] = v' + v'' + v''' + \text{\&c.}$$
$$[vv] = v'v' + v''v'' + v'''v''' + \text{\&c.}$$

a notation usually employed throughout the method of least squares. We have

$$[v] = 0 \tag{2}$$

and

$$\begin{aligned}[vv] &= (n' - x_0)^2 + (n'' - x_0)^2 + (n''' - x_0)^2 + \dots\dots \\ &= [nn] - 2\,[n]\,x_0 + mx_0^2 \quad .\end{aligned}$$

But since we have also

$$x_0 = \frac{[n]}{m}$$

this equation becomes

$$\left.\begin{aligned}[vv] &= [nn] - 2\,[n]\,\frac{[n]}{m} + m\,\frac{[n]^2}{m^2} \\ &= [nn] - \frac{[n]^2}{m}\end{aligned}\right\} \tag{3}$$

Let x_1 be any assumed value of x, giving the residuals

$$v_1 = n' - x_1 \qquad v_2 = n'' - x_1 \qquad v_3 = n''' - x_1, \text{ \&c.}$$

then, as above,

$$[v_1v_1] = [nn] - 2\,[n]\,x_1 + mx_1^2$$

Substituting in this the value of $[nn]$ given by (3), we find

$$\left.\begin{aligned}[v_1v_1] &= [vv] + \frac{[n]^2}{m} - 2\,[n]\,x_1 + mx_1^2 \\ &= [vv] + m\left(\frac{[n]}{m} - x_1\right)^2 \\ &= [vv] + m\,(x_0 - x_1)^2\end{aligned}\right\} \tag{4}$$

This equation determines the sum of the squares of the residuals for any assumed value of x. Since the last term is always positive, we see that this sum for any value of x differing from the arithmetical mean x_0 is always greater than $[vv]$. Hence it is a second characteristic of the arithmetical mean, that it makes *the sum of the squares of the residuals a minimum.*

6. Observations may be not only *direct*, that is, made directly upon the quantity to be determined, but also *indirect*, that is, made upon some quantity which is a function of one or more quantities to be determined. Indeed, the greater part of the observations in astronomy, and in physical science generally, belong to the latter class. Thus, let $x, y, z \ldots\ldots$ be the quantities to be determined, and M a function of them denoted by f, or

$$M = f(x, y, z \ldots .) \tag{5}$$

and let us suppose an observation to be made upon the value of M. We then have but a single equation between $x, y, z \ldots .$ and the observed quantity M, and the problem is as yet indeterminate. Various systems of values may be found to satisfy the equation, either exactly or approximately. Let us, however, suppose that the most probable system (as yet unknown) is expressed by $x = p, y = q, z = r \ldots .$, and let the value of the function, when these values are substituted in it, be denoted by V, or put

$$V = f(p, q, r \ldots .) \tag{6}$$

then $M - V$ is the residual error of the observation. In like manner, if a number of observations of the same kind be taken, in which the observed quantities $M', M'', M''' \ldots$ are functions determined by the same elements $p, q, r, \ldots .$, and if $V', V'', V''' \ldots .$ are the values of these functions when $p, q, r \ldots .$ are substituted in them, then $M' - V', M'' - V'', M''' - V''' \ldots .$ are the residual errors of the observations. If there are μ unknown quantities and also μ observations, and no more, there will be μ equations between the known and unknown quantities, which will fully determine the values of these unknown quantities: so that the probable values $p, q, r \ldots .$ are, in that case, those determinate values which *exactly* satisfy all the equations, and, consequently, reduce every one of the residuals $M' - V'$, $M'' - V''$, &c. to zero. But, if there are more than μ observations, the determinate values found from μ equations alone will not necessarily satisfy the remaining equations, in consequence of accidental errors in the observations. The problem, then, is *to determine from* ALL *the observations*, or *from all the equations, the most probable system of values of the unknown quantities*, or, which is the same thing, *the most probable system of residual errors*. In the case of direct observations, we have seen that the most probable value of the unknown quantity was that which made the algebraic sum of the residuals zero; but this principle followed from taking the arithmetical mean of the *same* quantity, and is obviously inapplicable in the present case. The second principle, that the most probable value is that which makes the sum of the squares of the residuals a minimum, is of a more general character, and might be assumed at once, as at least a *plausible* principle, to serve as the basis of the solution of our problem; but it will be more satisfactory to justify its adoption by the calculus of probabilities.

THE PROBABILITY CURVE.

7. Although accidental errors would seem at first sight to be of a capricious and irregular nature which would exclude them from the domain of mathematics, yet, upon examination from theoretical considerations, confirmed, as will be shown, by experience, we shall find that they are subject to remarkably precise laws. In the first place, we remark that they are subject to the following fundamental laws: 1st. Errors in excess and in defect —*i.e.* positive and negative, but of equal absolute value—are equally probable, and in a large number of observations are equally frequent. 2d. In every species of observations, there is a limit of error which the greatest accidental errors do not exceed: thus, if l denotes the absolute magnitude of this limit, all the positive errors are comprised between 0 and $+ l$, and all the negative errors between 0 and $- l$, and, consequently, all the errors are distributed over the interval $2l$. 3d. The errors are not distributed uniformly over this interval $2l$, but the smaller errors are more frequent than the larger ones.

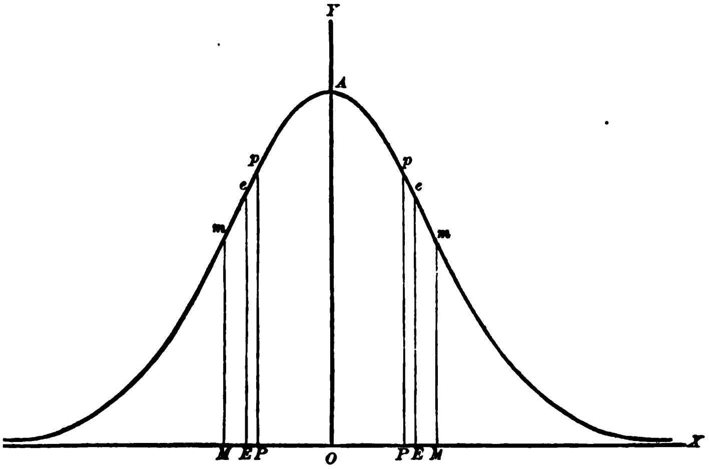
Thus the frequency of an error of a given magnitude may be regarded as a function of the error itself: so that, if we denote an error of a certain magnitude by Δ, and its relative frequency in a given large number of observations by $\varphi\Delta$, this function should obtain its maximum value for $\Delta = 0$, and become zero

. $\Delta = \pm l$. If, then, we denote the *probability** of an error y, or put

$$y = \varphi\Delta \tag{7}$$

ıay regard this as the equation of a curve, taking Δ as the ssa and y as the ordinate. The nature of this curve will be ately defined when we have discovered the form of the ion $\varphi\Delta$, but we can see in advance that a curve such as A is required to satisfy the conditions already imposed upon

Fig. A.



unction. For its maximum ordinate must correspond to); it must be symmetrical with reference to the axis of y, equal errors with opposite signs have equal probabilities; t must approach very near to the axis of abscissæ for values near the extreme limits, although the impossibility of asıg such extreme limits of error with precision must prevent ›m fixing the point at which the curve will finally meet the

The number of possible errors in any class of observations ictly speaking, finite; for there is always a limit of accuracy e observations, even when we employ the most refined ıments, in consequence of which there is a numerical sucın in our results. Thus, if 1″ is the smallest measure in a given case, the possible errors, arranged in their order of magnitude, can only differ by 1″ or an integral number of seconds. Hence, our geometrical representation should strictly consist of a number of isolated points; but, as these points will be more and more nearly represented by a continuous curve as we increase the accuracy of the observations, and thus diminish the intervals between the successive ordinates, we may, without hesitation, adopt such a continuous curve as expressing the law of error. We shall, therefore, regard Δ as a continuous variable, and $\varphi\Delta$ as a continuous function of it.

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* That is, if the error Δ occurs n times in m observations, $y = \varphi\Delta = \frac{n}{m}$.

Now, by the theory of probabilities, if $\varphi\Delta$, $\varphi\Delta'$, $\varphi\Delta''$....... are the respective probabilities of all the possible errors Δ, Δ', Δ''...... we have*

$$\varphi\Delta + \varphi\Delta' + \varphi\Delta'' + \ldots\ldots = 1$$

when the number of possible errors is finite. But the assumed continuity of our curve requires that we consider the difference between successive values of Δ as infinitesimal, and thus the number of values of $\varphi\Delta$ is infinite, and the probability of any one of these errors is an infinitesimal. To meet this difficulty, let us observe that if a finite series of errors Δ, Δ', Δ''.... be expressed in the smallest unit employed in the observations, these errors, arranged in the order of their magnitude, will be a series of consecutive integral numbers; the probability of the error Δ may be regarded as the same as the probability that the error falls between Δ and $\Delta + 1$; and the probability of an error between Δ and $\Delta + i$ will be the sum of the probabilities of the errors Δ, $\Delta + 1$, $\Delta + 2$, ... $\Delta + (i - 1)$. If i is small, the probability of each of the errors from Δ to $\Delta + i$ will be nearly the same as that of Δ: so that their sum will differ but little from $i\varphi\Delta$. As the interval between the successive errors diminishes, this expression becomes more accurate; and hence when we take $d\Delta$, the infinitesimal, instead of i, we have $\varphi\Delta \,.\, d\Delta$ as the rigorous expression of the probability that an error falls between Δ and $\Delta + d\Delta$. Hence, it follows, in general, that the probability that an error falls between any given limits a and b is the sum of all the elements of the form $\varphi\Delta . d\Delta$ between these limits, or the integral

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* For if there are n errors equal to Δ, n' equal to Δ', &c., and the whole number of errors is m, the probabilities of the errors are respectively $\varphi\Delta = \frac{n}{m}$, $\varphi\Delta' = \frac{n'}{m}$, &c., and the sum of these is $\frac{n + n' + \ldots\ldots}{m} = \frac{m}{m} = 1$.

$$\int_a^b \varphi\Delta . d\Delta$$

and this integral, taken between the extreme limits of error, and thus embracing all the possible errors, will be

$$\int_{-l}^{+l} \varphi\Delta . d\Delta = 1$$

We have heretofore assumed that the function $\varphi\Delta$ is to be zero for $\Delta = \pm l$. It must also be added that, since the probability of any error greater than $\pm l$ is also zero, we should have to determine this function in such a manner that it would be zero for all values of Δ from $+ l$ to $+ \infty$ and from $- l$ to $- \infty$. The obvious impossibility of determining such a function leads us to extend the limits $\pm l$ to $\pm \infty$, and to take

$$\int_{-\infty}^{+\infty} \varphi\Delta . d\Delta = 1 \qquad (8)$$

This will evidently be allowable if the integral taken from $\pm l$ to $\pm \infty$ is so small as to be practically insignificant. Besides, the extreme limits of error can never be fixed with precision, and it will suffice if the function $\varphi\Delta$ is such that it becomes very small for those errors which are regarded as very large.

9. Returning now to the general case of indirect observations, Art. 6, in which we suppose a quantity $M = f(x, y, z, \ldots.)$ to be observed, let Δ, Δ', Δ'' .... be the errors of the several observed values of M, and $\varphi\Delta$, $\varphi\Delta'$, $\varphi\Delta''$ .... their respective probabilities; then, the probability that these errors occur at the same time in the given series being denoted by P, we have, by a theorem of the calculus of probabilities,*

$$P = \varphi\Delta . \varphi\Delta' . \varphi\Delta'' \ldots\ldots \qquad (9)$$

The most probable system of values of the unknown quantities

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* If a single action of a cause can produce the effects a, a', a'', ... with the respective probabilities p, p', p'', ... the probability that two successive independent actions of the cause will produce the effects a and a' is pp'; and similarly for any number of effects. Thus, if an urn contains 2 white balls, 8 red ones, and 5 black ones, the probability that in two successive drawings (the original number of balls being the same at each drawing) one ball will be white and the other red is $\frac{2}{15} \times \frac{8}{15}$.

$x, y, z\ldots.$ will be that which makes the probability P a maximum. Consequently, since $x, y, z\ldots.$ are here supposed to be independent,* the derivative of P relatively to each of these variables must be equal to zero; or, since $\log P$ varies with P, the derivatives of $\log P$ must satisfy this condition, and we shall have

$$\frac{1}{P}\cdot\frac{dP}{dx} = 0, \qquad \frac{1}{P}\cdot\frac{dP}{dy} = 0, \text{ \&c.}$$

which, since

$$\log P = \log \varphi\Delta + \log \varphi\Delta' + \log \varphi\Delta'' + \ldots..$$

give the equations

$$\left.\begin{array}{l}
\varphi'\Delta\,\dfrac{d\Delta}{dx} + \varphi'\Delta'.\dfrac{d\Delta'}{dx} + \varphi'\Delta''.\dfrac{d\Delta''}{dx} + \ldots.. = 0 \\[2ex]
\varphi'\Delta.\dfrac{d\Delta}{dy} + \varphi'\Delta'.\dfrac{d\Delta'}{dy} + \varphi'\Delta''.\dfrac{d\Delta''}{dy} + \ldots.. = 0 \\[2ex]
\varphi'\Delta.\dfrac{d\Delta}{dz} + \varphi'\Delta'.\dfrac{d\Delta'}{dz} + \varphi'\Delta''.\dfrac{d\Delta''}{dz} + \ldots.. = 0 \\
\qquad\quad \text{\&c.} \qquad\quad \text{\&c.}
\end{array}\right\} \tag{10}$$

in which we have put

$$\varphi'\Delta = \frac{d\varphi\Delta}{\varphi\Delta d\Delta}, \qquad \varphi'\Delta' = \frac{d\varphi\Delta'}{\varphi\Delta' d\Delta'}, \text{ \&c.} \tag{11}$$

The number of equations in (10) being the same as that of the unknown quantities, these equations will serve to determine the unknown quantities when we have discovered the value of the function $\varphi'\Delta$, as will be shown hereafter.

Since the functions $\varphi\Delta$ and $\varphi'\Delta$ are supposed to be general, and therefore applicable whatever the number of unknown quantities, we may determine them by an examination of the special case in which there is but one unknown quantity, or that in which the observed values M, M', $M''\ldots.$ belong to the same quantity. In that case, the hypothesis that x is the value of this quantity gives the errors

$$\Delta = M - x, \qquad \Delta' = M' - x, \qquad \Delta'' = M'' - x\ldots..$$

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* That is, subject to no restrictions except that they shall satisfy the observations, or the equations $M = f(x, y, z, \ldots.)$. For the case of "conditioned" observations, see Art. 53 of this Appendix.

whence

$$\frac{d\Delta}{dx} = \frac{d\Delta'}{dx} = \frac{d\Delta''}{dx} \dots\dots = -1$$

and the first equation of (10) becomes

$$\varphi'(M - x) + \varphi'(M' - x) + \varphi'(M'' - x) + \dots\dots = 0 \qquad (12)$$

This being general for any number m of observations, and for any observed values M, M', M''...., let us suppose the special case

$$M' = M'' \dots\dots = M - mN$$

Since the arithmetical mean of the observed quantities is here the most probable value of x, we have

$$x = \frac{1}{m}(M + M' + M'' + \dots\dots)$$

$$= \frac{1}{m}[M + (m-1)(M - mN)]$$

$$= M - (m-1)N$$

whence

$$M - x = (m-1)N$$

$$M' - x = M'' - x \dots\dots = -N$$

and, consequently, (12) becomes

$$\varphi'[(m-1)N] + (m-1)\varphi'(-N) = 0$$

or,

$$\frac{\varphi'[(m-1)N]}{(m-1)N} = \frac{\varphi'(-N)}{-N}$$

That is, for all values of m, and therefore for all values of $(m-1)N$, we have $\frac{\varphi'[(m-1)N]}{(m-1)N}$ equal to the *same* quantity $\frac{\varphi'(-N)}{-N}$. Hence we have generally $\frac{\varphi'\Delta}{\Delta}$ equal to a constant quantity, and, denoting this constant by k, we have

$$\varphi'\Delta = k\Delta$$

or, by (11),

$$\frac{d\varphi\Delta}{\varphi\Delta} = k\Delta \, . \, d\Delta$$

Integrating,

$$\log \varphi\Delta = \tfrac{1}{2}k\Delta^2 + \log \varkappa$$

whence

$$\varphi\Delta = \varkappa \, e^{\frac{1}{2}k\Delta\Delta}$$

in which e is the base of the Napierian system of logarithms.

Since $\varphi\Delta$ must decrease as Δ increases, $\frac{1}{2}k$ must be essentially negative: representing it, therefore, by $-h^2$, our function becomes

$$\varphi\Delta = \varkappa\, e^{-hh\Delta\Delta}$$

To determine the constant $\varkappa$, let this value be substituted in (8), which gives

$$\int_{-\infty}^{+\infty} \varkappa\, e^{-hh\Delta\Delta} d\Delta = 1$$

Putting

$$t = h\Delta \tag{13}$$

this integral becomes

$$\frac{\varkappa}{h}\int_{-\infty}^{+\infty} e^{-tt} dt = 1$$

The known value of the definite integral in the first member is $\sqrt{\pi}$ (see Vol. I. p. 153); whence

$$\varkappa = \frac{h}{\sqrt{\pi}}$$

and the complete expression of $\varphi\Delta$ becomes

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}\, e^{-hh\,\Delta\Delta} \tag{14}$$

The constant h must depend upon the nature of the observations, and will be particularly examined hereafter. If we here take it as the unit of abscissæ in the curve of probability, the equation (7) becomes

$$y = \frac{1}{\sqrt{\pi}}\, e^{-\Delta\Delta}$$

by which the curve may be constructed. The values of y for a few values of Δ are as follows:

Δ	y	Diff.	Δ	y	Diff.
0.0	0.5642	— .0221	1.6	0.0436	— .0215
0.2	0.5421	— .0613	1.8	0.0221	— .0118
0.4	0.4808	— .0872	2.0	0.0103	— .0058
0.6	0.3936	— .0961	2.2	0.0045	— .0027
0.8	0.2975	— .0899	2.4	0.0018	— .0011
1.0	0.2076	— .0739	2.6	0.0007	— .0005
1.2	0.1337	— .0542	2.8	0.0002	— .0001
1.4	0.0795	— .0359	3.0	0.0001	
1.6	0.0436		∞	0.0000	

The curve, Fig. A, in Art. 7, is constructed from this table; but, to exhibit its character more distinctly, the scale of the ordinates is four times that of the abscissæ (which, indeed, corresponds to the case of $h = 2$). We see that the curve approaches very near to the axis for moderate values of Δ, and that the assumption of $\pm \infty$ instead of finite limits of Δ can involve no practical error. It is evident that the axis XX is an asymptote to the curve.

The differences in the above table indicate that the curve approaches the axis most rapidly at a point whose abscissa is between 0.6 and 0.8. The exact position of this point, which is a point of inflexion, is found by putting the second differential coefficient of y equal to zero, which gives

$$\frac{d^2y}{d\Delta^2} = -\frac{2}{\sqrt{\pi}} e^{-\Delta\Delta} + \frac{4\Delta\Delta}{\sqrt{\pi}} e^{-\Delta\Delta} = 0$$

whence

$$\Delta = \frac{1}{\sqrt{2}} = 0.7071$$

The ordinate Mm is drawn at this point. We shall have occasion to refer to it again hereafter.

THE MEASURE OF PRECISION.

10. The constant h requires special consideration. Since the exponent of e in (14) must be an abstract number, $\frac{1}{h}$ must be a concrete quantity of the same kind as Δ. In a class of observations in which Δ is small for a given probability $\varphi\Delta$, $\frac{1}{h}$ will be small, and h will be large. Thus, h will be the greater the more precise the nature of the observations, and is, therefore, called by GAUSS the *measure of precision*. If in one system of observations the probability of an error Δ is expressed by

$$\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}$$

and in another, more or less precise, by

$$\frac{h'}{\sqrt{\pi}} e^{-h'h'\Delta\Delta}$$

the probability that in one observation of the first system the error committed will be comprised between the limits $-\delta$ and $+\delta$ will be expressed by the integral

$$\int_{-\delta}^{+\delta} \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}\, d\Delta$$

and, in like manner, the probability that the error of an observation in the second system will be comprised between $-\delta'$ and $+\delta'$ will be expressed by

$$\int_{-\delta'}^{+\delta'} \frac{h'}{\sqrt{\pi}} e^{-h'h'\Delta\Delta}\, d\Delta$$

These integrals are evidently equal when we have $h\delta = h'\delta'$. If, for example, we have $h' = 2h$, the integrals will be equal when $\delta = 2\delta'$; that is, the double error will be committed in the first system with the same probability as the simple error in the second, or, in the usual mode of expression, the second system will be twice as precise as the first. We shall presently see how the value of h can be found for any given observations.

THE METHOD OF LEAST SQUARES.

11. The preceding discussion leads directly to important practical results. We have seen (Art. 9) that to find the most probable values of $x, y, z \ldots.$ from the observed values of $M = f(x, y, z, \ldots.)$ we are to render the probability $P = \varphi\Delta . \varphi\Delta' . \varphi\Delta'' \ldots.$ a maximum, that is, by (14),

$$P = h^m \pi^{-\frac{1}{2}m} e^{-hh(\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \ldots.)} \qquad (15)$$

must be a maximum; and this requires that the quantity $\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \ldots.$ should be a minimum. Thus, the principle that *the most probable values of the unknown quantities are those which make the sum of the squares of the residual errors a minimum*, is not limited to the case of direct observations, but is entirely general.

The principle is readily extended to observations of unequal precision. For if the degree of precision of the observations M, M', $M'' \ldots.$ be respectively h, h', $h'' \ldots.$, and we compare these observed quantities with the values V, V', $V'' \ldots.$, computed with the most probable values of $x, y, z \ldots.$, whereby we obtain the residual errors $M - V = \Delta$, $M' - V' = \Delta' \ldots.$, it is the same thing as if we had taken observations of equal precision (represented by 1) upon the quantities hM, $h'M'$, $h''M'' \ldots.$, and had compared them with the computed quantities $h V$, $h' V'$, $h'' V''\ldots$, whereby we should have found the errors $hM - h V = h\varDelta$, $h'M' - h' V' = h'\varDelta'\ldots$, in which case we should have to reduce to a minimum the quantity

$$h^2\varDelta^2 + h'^2\varDelta'^2 + h''^2\varDelta''^2 + \ldots$$

that is, *each error being multiplied by its measure of precision, and thereby reduced to the same degree of precision, the sum of the squares of the reduced errors must be a minimum.*

In what precedes is involved the whole theory of the method of least squares. I proceed to develop its practical features.

THE PROBABLE ERROR.

12. From the preceding articles it follows that the probability that the error of an observation falls between $\varDelta$ and $\varDelta + d\varDelta$ is expressed by

$$\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta} \, d\varDelta$$

and the probability that it falls between the limits 0 and a is expressed by

$$\frac{h}{\sqrt{\pi}} \int_{\Delta=0}^{\Delta=a} e^{-hh\Delta\Delta} \, d\varDelta$$

and this integral expresses the number of errors that we should expect to find between the limits 0 and a when the whole number of errors is put $= 1$ [equation (8)]. If we put $t = h\varDelta$, the integral takes the form

$$\frac{1}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-tt} \, dt$$

The whole number of errors, both positive and negative, whose numerical magnitude falls between the given limits is twice this integral, or

$$\frac{2}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-tt} \, dt \tag{16}$$

The value of this integral (which may be computed by the methods of Vol. I. Art. 113) is given in Table IX. The number of errors between any two given limits will be found by taking the difference between the tabular numbers corresponding to these limits. Since the total number of errors is taken as unity in the table, the required number of errors in any particular case is to be found by multiplying the tabular numbers by the actual number of observations. Thus, if there are 1000 observations, we find that

between $t = 0$ and $t = 0.5$ there are 520 errors.
" $t = 0.5$ " $t = 1.0$ " " 322 "
" $t = 1.0$ " $t = 1.5$ " " 123 "
" $t = 1.5$ " $t = 2.0$ " " 29 "
" $t = 2.0$ " $t = \infty$ " " 5 "

13. The degrees of precision of different series of observations may be compared together either by comparing the values of h, or by comparing the errors which are committed with equal facility in the two systems. The errors to be compared must occupy in the two systems a like position in relation to the extreme errors, and we may select for this purpose in each system *the error which occupies the middle place in the series of errors arranged in the order of their magnitude, so that the number of errors which are less than this assumed error is the same as the number of errors which exceed it.* The error which satisfies this condition is that for which the value of the integral (16) is 0.5. Denoting the corresponding value of t by ρ, we find, by interpolation from Table IX.,

$$\rho = 0.47694$$

and we have

$$\frac{2}{\sqrt{\pi}}\int_0^\rho e^{-tt}\,dt = \frac{1}{2} \qquad (17)$$

If then we denote by r the error which, in any system of observations whose degree of precision is h, corresponds to the value $t = \rho$, or put

$$\rho = hr \qquad h = \frac{\rho}{r} \qquad (18)$$

there will be a probability of $\frac{1}{2}$ that the error of any single observation in that system will be less than r, and the same probability that it will be greater than r; which is sometimes expressed by saying that *it is an even wager that the error will be less than r.* Hence r is called the *probable error.*

We may, therefore, compare different series of observations by comparing their probable errors, their degrees of precision being, by (18), inversely proportional to these errors.

14. In order to apply Table IX. in determining the number of errors in a given class of observations, we must know the

measure of precision h, or the probable error r: thus, if we wish the number of errors less than a, we enter the table with the argument $t = ah$, or $t = \frac{a\rho}{r}$

For greater convenience, we can employ Table IX.A, which gives the same function with the argument $\frac{a}{r}$. For example, if there are 1000 observations whose probable error is $r = 2''$, and we wish to know the number of errors less than $a = 1''$, we take from Table IX.A, with the argument $\frac{a}{r} = 0.5$, the number 0.26407, which multiplied by 1000 gives 264 as the required number.

The following example from the *Fundamenta Astronomiæ* of BESSEL will serve to show how far the preceding theory is sustained by experience. In 470 observations made by BRADLEY upon the right ascension of *Sirius* and *Altair*, BESSEL found the probable error of a single observation to be

$$r = 0''.2637$$

Hence, for the number of errors less than 0″.1 the argument of Table IX.A will be $\frac{0.1}{0.2637} = 0.3792$; and for 0.″2, 0″.3, &c., the successive multiples of 0.3792. Thus, we find from the table

for	0″.1	with arg.	0.3792	the number	0.20187
"	0 .2	"	0.7584	"	0.39102
"	0 .3	"	1.1376	"	0.55710
"	0 .4	"	1.5168	"	0.69372
"	0 .5	"	1.8960	"	0.79904
"	0 .6	"	2.2752	"	0.87511
"	0 .7	"	2.6544	"	0.92661
"	0 .8	"	3.0336	"	0.95926
"	0 .9	"	3.4128	"	0.97866
"	1 .0	"	3.7920	"	0.98946
			∞	"	1.00000

Subtracting each number from the following one, and multiplying the remainder by 470, the number of observations, there were found

Between	No. of errors by the theory.	No. of errors by experience.
0″.0 and 0″.1	95	94
0 .1 " 0 .2	89	88
0 .2 " 0 .3	78	78
0 .3 " 0 .4	64	58
0 .4 " 0 .5	50	51
0 .5 " 0 .6	36	36
0 .6 " 0 .7	24	26
0 .7 " 0 .8	15	14
0 .8 " 0 .9	9	10
0 .9 " 1 .0	5	7
over 1 .0	5	8

The agreement between the theory and experience, though not absolute, is remarkably close. The number of large errors by experience exceeds that given by the theory, and this has been found in other cases of a similar kind; which shows at least that the extension of the limits of error to ± ∞ has not introduced any error. The discrepancy rather indicates a source of error of an abnormal character, and calls for some criterion by which such abnormal observations may be excluded from our discussions and not permitted to vitiate our results. Such a criterion has been proposed by Prof. PEIRCE, and will be considered hereafter.

THE MEAN OF THE ERRORS, AND THE MEAN ERROR.

15. The selection of the probable error as the term of comparison between different series of observations is arbitrary, although it seems to be naturally designated by its middle position in the series of errors. There are two other errors which have been used for the same purpose.

The first is the *mean of the errors*, these being all taken with the positive sign. In order to find its relation to the probable error, let us first consider a finite series of errors

$$\Delta, \Delta', \Delta'', \ldots\ldots$$

with the respective probabilities

$$\frac{2a}{m}, \qquad \frac{2a'}{m}, \qquad \frac{2a''}{m}, \ldots$$

so that in m observations there will be $2a$ errors (numerically) equal to Δ, $2a'$ equal to Δ', &c., the probability of a positive error Δ being $\frac{a}{m}$. The mean of all these errors, each being repeated a number of times proportional to its probability, is

$$\frac{2a\Delta + 2a'\Delta' + 2a''\Delta'' + \ldots.}{m} = 2\Delta \cdot \frac{a}{m} + 2\Delta' \cdot \frac{a'}{m} + 2\Delta'' \cdot \frac{a''}{m} + \ldots.$$

When the number of errors is infinite, the probability of an error Δ is to be understood as the probability that it falls between Δ and $\Delta + d\Delta$, which is $\varphi\Delta \,.\, d\Delta$ (Art. 8), and the above formula for the mean of the errors becomes the sum of an infinite number of terms of the form $2\Delta\varphi\Delta \,.\, d\Delta$. Hence, putting

$$\eta = \text{the mean of the errors,}$$

we have

$$\eta = \int_0^\infty \frac{2h}{\sqrt{\pi}} \Delta e^{-hh\Delta\Delta} \, d\Delta = \frac{1}{h\sqrt{\pi}} \tag{19}$$

or, by (18),

$$\left.\begin{aligned} \eta &= \frac{r}{\rho\sqrt{\pi}} = 1.1829\, r \\ r &= 0.8453\, \eta \end{aligned}\right\} \tag{20}$$

Another error, very commonly employed in expressing the precision of observations, is that which has received the appellation of the *mean error* (*der mittlere Fehler* of the Germans), which is not to be confounded with the above mean of the errors. Its definition is, *the error the square of which is the mean of the squares of all the errors.* Hence, putting

$$\varepsilon = \text{the mean error,}$$

we have

$$\varepsilon^2 = \int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} \Delta^2 e^{-hh\Delta\Delta} \, d\Delta = \frac{1}{2h^2} \tag{21}$$

or, by (18),

$$\left.\begin{aligned} \varepsilon &= \frac{r}{\rho\sqrt{2}} = 1.4826\, r \\ r &= 0.6745\, \varepsilon \end{aligned}\right\} \tag{22}$$

When we put $h = 1$, we have $\varepsilon = \sqrt{\tfrac{1}{2}}$. The mean error is, therefore, the abscissa of the point of inflection of the curve of probability (Art. 9). In the figure, p. 479, OM is the mean error, *OP* the probable error, *OE* the mean of the errors, and *Mm*, *Ee*, their respective probabilities.

THE PROBABLE ERROR OF THE ARITHMETICAL MEAN.

16. The error above denoted by r is the probable error of one of the observed values of the unknown quantity x. We next to determine the relation between this and the proba error r_0 of the arithmetical mean of these values.

If Δ, Δ', Δ''.... are the errors of the observed values, most probable value of x is that which renders the probabilit

$$P = h^m \pi^{-\frac{1}{2}m} e^{-hh(\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \dots)}$$

a maximum (Art. 11), and, consequently, the sum $\Delta\Delta + \Delta$ $+ \dots$. a minimum. But this sum is rendered a minimum the assumption of the arithmetical mean x_0 as the most proba value (Art. 5), and hence the quantity P expresses the probabili of the arithmetical mean if Δ, Δ', Δ''.... are the errors of t observations when compared with this mean. The probabili of any other value of x, as $x_0 + \delta$, will be

$$\begin{aligned} P' &= h^m \pi^{-\frac{1}{2}m} e^{-hh\{(\Delta-\delta)^2 + (\Delta'-\delta)^2 + \dots\}} \\ &= h^m \pi^{-\frac{1}{2}m} e^{-hh\{[\Delta\Delta] - 2[\Delta]\delta + m\delta\delta\}} \end{aligned}$$

Since $[\Delta] = \Delta + \Delta' + \Delta'' + \dots = 0$ (Art. 5), and $[\Delta\Delta] = n$ (Art. 15), this expression may be put under the form

$$P' = h^m \pi^{-\frac{1}{2}m} e^{-mhh(\epsilon\epsilon + \delta\delta)}$$

and at the same time we have

$$P = h^m \pi^{-\frac{1}{2}m} e^{-mhh\epsilon\epsilon}$$

so that

$$P : P' = 1 : e^{-mhh\delta\delta}$$

that is, the probability of the error zero in the arithmetical me is to that of the error δ as $1 : e^{-mhh\delta\delta}$. For a single observati the probability of the error zero is to that of the error δ $1 : e^{-hh\delta\delta}$. Hence the measure of precision (Art. 10) of single observation being h, that of the arithmetical mean of such observations is $h\sqrt{m}$; from which follows the import orem that *the precision of the mean of a number of observations reases as the square root of their number.*[*]

If, then, r is the probable error of a single observation, and r_0
t of the arithmetical mean of several observations, we must have

$$r_0 = \frac{r}{\sqrt{m}}. \qquad (23)$$

d from the constant relation between the mean and the proba-
error (22),

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{m}} \qquad (24)$$

ETERMINATION OF THE MEAN AND PROBABLE ERRORS OF GIVEN OBSERVATIONS.

17. The principles now explained will enable us to determine e mean errors of any given series of directly observed quanti-
es. Let n, n', n''.... be the observed values; x_0 their arith-
etical mean; v, v', v''.... the residuals found by subtracting from each observed value: so that

$$v = n - x_0, \qquad v' = n' - x_0, \qquad v'' = n'' - x_0, \text{ \&c.}$$

f x_0 were certainly the true value of x, so that v, v', v''.... were e actual or (as we may say) the *true* errors, and, consequently, lentical with Δ, Δ', Δ''...., we should have, according to the bove, $m\varepsilon\varepsilon = [\Delta\Delta] = [vv]$, and hence

$$\varepsilon = \sqrt{\left(\frac{[vv]}{m}\right)}.$$

d this must always give a close approximation to the value of ε. ut the relation $m\varepsilon\varepsilon = [\Delta\Delta]$ was deduced from a consideration an infinite series of errors which would reduce the mean ror of x_0 to an infinitesimal, according to the principles assumed, d thus make v, v', v''.... identical with Δ, Δ', Δ''... A better proximation to the value of ε, where the series is limited, is to obtained by considering the mean error of x_0 itself, and conse-
ently, also, the mean errors of the residuals v, v', v''..... If n we suppose the true value of x to be $x_0 + \delta$, we shall have true errors

$$\Delta = v - \delta, \qquad \Delta' = v' - \delta, \qquad \Delta'' = v'' - \delta, \text{ \&c.}$$

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[*] See, in connection, Arts. 21 and 25.

whence, observing that $[v] = 0$,

$$[\Delta\Delta] = m\varepsilon\varepsilon = [vv] - 2\,[v]\,\delta + m\delta^2$$
$$= [vv] + m\delta^2$$

Thus the approximate value $m\varepsilon\varepsilon = [vv]$ requires the corre $m\delta^2$, the value of which depends upon the value we may as to δ. As the best approximation, we may assume it to b mean error ε_0: so that, by (24),

$$m\delta^2 = m\varepsilon_0^{\,2} = m\,\frac{\varepsilon\varepsilon}{m} = \varepsilon\varepsilon$$

which gives

$$m\varepsilon\varepsilon = [vv] + \varepsilon\varepsilon$$

whence

$$\varepsilon\varepsilon = \frac{[vv]}{m-1} \qquad \varepsilon = \sqrt{\left(\frac{[vv]}{m-1}\right)}$$

and consequently, also, by (22),

$$r = q\sqrt{\left(\frac{[vv]}{m-1}\right)} \qquad q = 0.6745$$

Thus from the actual residuals the mean and the probable e of a single observed value are found. Hence, by (23) and the mean and probable errors of the arithmetical mean wil found by the formulæ

$$\varepsilon_0 = \sqrt{\left(\frac{[vv]}{m\,(m-1)}\right)} \qquad r_0 = q\sqrt{\left(\frac{[vv]}{m\,(m-1)}\right)}$$

EXAMPLE.—Let us take the following measures of the o diameter of Saturn's ring observed by BESSEL at the Königs Observatory with the heliometer, in the years 1829–1831.* measures, denoted by n, are all reduced to the mean distanc Saturn from the sun, and are here assumed to have the s degree of precision.

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* *Astron. Nach.*, Vol. XII. p. 169.

n	*v*	*vv*
38″.91	− 0″.40	0.1600
39 .32	+ 0 .01	.0001
38 .93	− 0 .38	.1444
39 .31	0 .00	.0000
39 .17	− 0 .14	.0196
39 .04	− 0 .27	.0729
39 .57	+ 0 .26	.0676
39 .46	+ 0 .15	.0225
39 .30	− 0 .01	.0001
39 .03	− 0 .28	.0784
39 .35	+ 0 .04	.0016
39 .25	− 0 .06	.0036
39 .14	− 0 .17	.0289
39 .47	+ 0 .16	.0256
39 .29	− 0 .02	.0004
39 .32	+ 0 .01	.0001
39 .40	+ 0 .09	.0081
39 .33	+ 0 .02	.0004
39 .28	− 0 .03	.0009
39 .62	+ 0 .31	.0961
39″.41	+ 0″.10	0.0100
39 .40	+ 0 .09	.0081
39 .36	+ 0 .05	.0025
39 .20	− 0 .11	.0121
39 .42	+ 0 .11	.0121
39 .30	− 0 .01	.0001
39 .41	+ 0 .10	.0100
39 .43	+ 0 .12	.0144
39 .43	+ 0 .12	.0144
39 .36	+ 0 .05	.0025
39 .02	− 0 .29	.0841
39 .01	− 0 .30	.0900
38 .86	− 0 .45	.2025
39 .51	+ 0 .20	.0400
39 .21	− 0 .10	.0100
39 .17	− 0 .14	.0196
39 .60	+ 0 .29	.0841
39 .54	+ 0 .23	.0529
39 .45	+ 0 .14	.0196
39 .72	+ 0 .41	.1681

$x_0 = 39\ .308$ $[vv] = 1.5884$

Hence, since $m = 40$, we have, by (25) and (26),

$$\epsilon = \sqrt{\left(\frac{1.5884}{39}\right)} = 0''.202$$

$$r = 0''.202 \times 0.6745 = 0''.136$$

and consequently, by (28) and (24), or (27),

$$\epsilon_0 = \frac{0''.202}{\sqrt{(40)}} = 0''.032, \qquad r_0 = \frac{0''.136}{\sqrt{(40)}} = 0''.022$$

That is, the probable error of a single observation was 0″.136, and that of the final result $x_0 = 39''.308$ was only 0″.022.

18. The preceding method of finding the probable error from the squares of the residuals is that which is most commonly employed; but when the number of observations is very great, it is desirable to abridge the labor, if possible. A sufficient approximation can be obtained by the use of the first powers of the residuals as follows.

The number of observations being very great, we shall probably have as many positive as negative residuals. If v', v'',

v''' . . . are the positive and $v_1, v_2, v_3, \dots$ the negative residuals and if the true value of x is $x_0 + \delta$, the true errors will b $v' - \delta, v'' - \delta, v''' - \delta \dots$, and $-v_1 - \delta, -v_2 - \delta, -v_3 - \delta, \dots$ *If they are all taken with the positive sign only*, the errors are, therefore,

$$v' - \delta,\ v'' - \delta,\ v''' - \delta, \dots \quad \text{and} \quad v_1 + \delta,\ v_2 + \delta,\ v_3 + \delta, \dots$$

the mean of which, upon the hypothesis of an equal number o positive and negative residuals, is the same as that of the series

$$v', v'', v''' \dots \qquad v_1, v_2, v_3 \dots$$

Hence, denoting the sum of the *numerical values* of the residual by $[v]$, and the mean of the actual errors by η, as in Art. 15, w have

$$\eta = \frac{[v]}{m}$$

and hence, by (20),

$$r = 0.8453 \frac{[v]}{m} \qquad (28$$

and consequently, also, by (22),

$$\varepsilon = 1.2533 \frac{[v]}{m} \qquad (29)$$

In the example of the preceding article we find the mean of the residuals taken with the positive sign to be 0″.1555, which by (28) gives $r = 0''.1555 \times 0.8453 = 0''.131$, which is perhaps a sufficient approximation to the value found above. In this example, however, we have 22 positive residuals, 17 negative ones, and 1 zero: so that the hypothesis upon which the formula (28) was founded is not strictly applicable. In a larger number of observations we should expect a closer agreement with the hypothesis, and more accordant results.

We may, however, employ the first powers of the residuals more strictly according to the theory of probabilities. In a limited series each residual is to be regarded as liable to a probable error r', and their mean is to be regarded as the mean of the errors of the residuals themselves, rather than as the mean of the errors of the observations. Hence the formula

$$r' = 0.8453 \frac{[v]}{m}$$

gives the probable error of a residual. The relation between r' and r (= the probable error of an observed quantity n) may be found as follows. Each observed n may be supposed to be the result of observing the mean quantity x_0 increased by an observed error v. The probable error of $n = x_0 + v$ is, therefore (by a principle hereafter to be proved),

$$r = \sqrt{(r_0^2 + r'^2)} = \sqrt{\left(\frac{r^2}{m} + r'^2\right)}$$

whence

$$r = r' \sqrt{\frac{m}{m-1}}$$

or

$$r = 0.8453 \frac{[v]}{\sqrt{[m(m-1)]}} \tag{30}$$

which agrees with the formula given by C. A. F. PETERS.[*] According to this formula, we find in the above example $r = 0''.133$.

DETERMINATION OF THE MEAN AND PROBABLE ERRORS OF FUNCTIONS OF INDEPENDENT OBSERVED QUANTITIES.

19. Suppose, first, the most simple function of two independent observed quantities x and x_1, namely, their sum or difference

$$X = x \pm x_1$$

and let the given mean errors of x and x_1 be ε and ε_1. Although the number of observations by which x and x_1 have been found may not be given, we may assume it to have been any large number m, and the same for each of the quantities; the degrees of precision of the two series being inversely proportional to ε and ε_1. The true errors of the assumed observations may be assumed to be—

for x, $\Delta, \Delta', \Delta'' \ldots\ldots\ldots$
for x_1, $\Delta_1, \Delta_1', \Delta_1'' \ldots\ldots$

and the errors of X, consequently,

$$\Delta \pm \Delta_1, \quad \Delta' \pm \Delta_1', \quad \Delta'' \pm \Delta_1'', \ldots\ldots$$

Denoting the mean error of X by E, we have, by the definition,

$$\begin{aligned} mE^2 &= (\Delta \pm \Delta_1)^2 + (\Delta' \pm \Delta_1')^2 + (\Delta'' \pm \Delta_1'')^2 + \ldots\ldots \\ &= [\Delta\Delta] \pm 2\,[\Delta\Delta_1] + [\Delta_1\Delta_1] \end{aligned}$$

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[*] *Astron. Nach.*, Vol. XLIV. p. 82.

In a great number of observations there must be as many pos tive as negative products of the form $\Delta\Delta_1$, and such that we sha probably have $[\Delta\Delta_1] = 0$; and since we also have $m\varepsilon^2 = [\Delta\Delta$ $m\varepsilon_1^2 = [\Delta_1\Delta_1]$, this equation gives

$$E^2 = \varepsilon^2 + \varepsilon_1^2 \qquad (3$$

If we have

$$X = x \pm x_1 \pm x_2$$

and the mean errors of x, x_1, x_2 are ε, ε_1, ε_2, we have by the pr ceding equation the mean error of $x \pm x_1 = \sqrt{(\varepsilon^2 + \varepsilon_1^2)}$, and l a second application of the same equation, considering $x \pm x_1$: a single quantity, the mean error of X will be found by tl formula

$$E^2 = \varepsilon^2 + \varepsilon_1^2 + \varepsilon_2^2 \qquad (31$$

and the same principle may be thus extended to the algebra sum of any number of observed quantities.

In consequence of the constant relation (22), if r, r_1, $r_2 \ldots$ are the *probable* errors of x, x_1, $x_2 \ldots\ldots$ and R the probable err of $X = x \pm x_1 \pm x_2 \ldots\ldots$, we shall have

$$R^2 = r^2 + r_1^2 + r_2^2 + \ldots\ldots \qquad (32$$

EXAMPLE 1.—The zenith distance of a star observed in th meridian is

$$\zeta = 21^\circ\ 17'\ 20''.3 \quad \text{with the mean error } \varepsilon = 2''.3$$

and the declination of the star is given

$$\delta = 19^\circ\ 30'\ 14''.8 \quad \text{with the mean error } \varepsilon_1 = 0''.8$$

Required the mean error E of the latitude of the place of obse vation, found by the formula $\varphi = \zeta + \delta$. We have, by (31),

$$E = \sqrt{[(2.3)^2 + (0.8)^2]} = 2''.44$$

Hence

$$\varphi = 40^\circ\ 47'\ 35''.1 \quad \text{with the mean error } E = 2''.44$$

EXAMPLE 2.—The latitude of a place has been found with tl mean error $\varepsilon = 0''.25$, and the meridian zenith distance of sta observed at that place with a certain instrument has been fou to be subject to the mean error $\varepsilon_1 = 0''.62$: what is the me error E of the declinations of the stars deduced by the formula $\delta = \varphi - \zeta$? We have

$$E = \sqrt{[(0.25)^2 + (0.62)^2]} = 0''.67$$

20. Let us next consider the function

$$X = ax$$

and suppose x has been observed with the mean error ε, and a is a given constant. Every observation of x with the error $\pm \Delta$ gives X with the error $\pm a\Delta$: so that the mean error of X must be

$$E = a\varepsilon$$

In general, by combining this with the preceding principle, if we have

$$X = ax + a_1x_1 + a_2x_2 + \ldots .$$

and if the mean errors of x, x_1, x_2 .... are ε, ε_1, ε_2, ..., and E that of X, we shall have

$$E^2 = a^2\varepsilon^2 + a_1^2\varepsilon_1^2 + a_2^2\varepsilon_2^2 + \ldots . = [a^2\varepsilon^2] \tag{33}$$

and the same form may be used for probable errors.

Example.—As an example illustrating the application of both the preceding principles, suppose that in order to find the rate of a chronometer we find at the time t its correction $+ 12^m\ 13^s.2$ with the mean error $0^s.3$, and at the time t' the correction $+ 12^m\ 21^s.4$ with the same mean error $0^s.3$, and the interval $t' - t = 10$ days. The rate in the whole interval is

$$12^m\ 21^s.4 - 12^m\ 13^s.2 = + 8^s.2$$

with the mean error, according to Art. 19,

$$\sqrt{[(0.3)^2 + (0.3)^2]} = 0^s.42$$

The mean daily rate is then

$$+ \frac{8^s.2}{10} = + 0^s.82$$

with the mean error, according to Art. 20,

$$\frac{0^s.42}{10} = 0^s.042$$

21. If x, x_1, x_2 .... are the several observed values of the *same* quantity, their arithmetical mean being

$$x_0 = \frac{1}{m}(x + x_1 + x_2 + \ldots)$$

and if r is the probable error of each observation, what is the probable error r_0 of x_0? By Art. 19, the probable error of the sum $x + x_1 + x_2 + \ldots.$ is

$$\sqrt{(r^2 + r^2 + r^2 + \ldots)} = \sqrt{(mr^2)} = r\sqrt{m}$$

and the probable error of $\frac{1}{m}$th of the sum is, by Art. 20,

$$r_0 = \frac{1}{m} \times r\sqrt{m} = \frac{r}{\sqrt{m}}$$

as has been otherwise proved in Art. 16.

22. Let us now take the general case in which X is any function whatever of the observed quantities $x, x_1, x_2, \ldots.$ expressed by

$$X = f(x, x_1, x_2, \ldots.)$$

Let the variables be expressed in the form

$$x = a + x', \qquad x_1 = a_1 + x_1', \qquad x_2 = a_2 + x_2', \ldots.$$

a, a_1, a_2 ... being arbitrarily assumed very nearly equal to x, x_1, x_2 ... respectively, and such that x', x_1', x_2' ..... may be so small that their squares will be insensible. The given mean errors ε, ε_1, ε_2 ... may then be regarded as the mean errors of x', x_1', x_2' .... The function X developed by TAYLOR's theorem is

$$X = f(a, a_1, a_2 \ldots.) + \frac{dX}{dx}x' + \frac{dX}{dx_1}x_1' + \frac{dX}{dx_2}x_2' + \ldots.$$

and the mean error of X will be that of the quantity

$$\frac{dX}{dx}x' + \frac{dX}{dx_1}x_1' + \frac{dX}{dx_2}x_2' + \ldots.$$

or, by (33),

$$E^2 = \left(\frac{dX}{dx}\right)^2 \varepsilon^2 + \left(\frac{dX}{dx_1}\right)^2 \varepsilon_1^2 + \left(\frac{dX}{dx_2}\right)^2 \varepsilon_2^2 + \ldots \qquad (3$$

or, if r, r_1, r_2 ... are the probable errors of x, x_1, x_2 ..., and R that of X,

$$R^2 = \left(\frac{dX}{dx}\right)^2 r^2 + \left(\frac{dX}{dx_1}\right)^2 r_1^2 + \left(\frac{dX}{dx_2}\right)^2 r_2^2 + \dots \qquad (34^*)$$

This formula is, indeed, but approximative, since we have neglected the terms involving the higher powers in the development of X; but the mean errors of these small terms will be insensible if we suppose that the errors ϵ, ϵ_1, ϵ_2 ... are so small that the differences between the observed values x, x_1, x_2 ... and the true values are of the same order as the quantities x', x_1', x_2' ..., which will always be the case where proper care has been taken to reduce the accidental errors of observation to their smallest amount. If the given function is implicit, as

$$0 = f\,(X, x, x_1, x_2 \dots)$$

we should still by differentiation obtain the differential coefficients, and then find the mean error of X by (34).

EXAMPLE.—The local apparent time at a place in latitude $\varphi = 38°\ 58'\ 53''$ was found (Vol. I. Art. 145) from the sun's zenith distance $\zeta = 73°\ 12'\ 25''$, when the declination was $\delta = -\ 22°\ 50'\ 27''$, to be $t = 2^h\ 47^m\ 39^s.4$. What is the probable error of this result, supposing the probable errors of the data to be—

Probable error of $\varphi = r = 0''.5$
" " $\delta = r_1 = 0\ .6$
" " $\zeta = r_2 = 3\ .5$

The formula

$$0 = -\cos\zeta + \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t$$

expresses t as an implicit function of φ, δ, and ζ. We find (Vol. I. Art. 35)

$$\frac{dt}{d\varphi} = -\frac{1}{\cos\varphi\tan A}$$

$$\frac{dt}{d\delta} = \frac{1}{\cos\delta\tan q}$$

$$\frac{dt}{d\zeta} = \frac{1}{\cos\varphi\sin A}$$

where A is the azimuth and q the parallactic angle. We find from the data $A = +40° 1'$, $q = 32° 51'$, whence

$$\frac{dt}{d\varphi} = -1.532, \qquad \frac{dt}{d\delta} = 1.680, \qquad \frac{dt}{d\zeta} = +2.001$$

and the probable error of t is, by (34*)

$$R = \sqrt{[(0.5 \times 1.532)^2 + (0.6 \times 1.680)^2 + (3.5 \times 2.001)^2]} = 7''.12$$

or, in seconds of time,

$$R = 0^s.47$$

23. To complete this branch of our subject, it is to be observed that the preceding demonstrations apply only to the case where the quantities entering into combination are independent; but when they are merely different functions of the same observed quantities, the above formulæ are incomplete. Let us suppose that we have X and X', different functions of the same observed quantities $x, x_1, x_2, \ldots.$, or

$$X = f\,(x, x_1, x_2, \ldots..)$$
$$X' = f'(x, x_1, x_2, \ldots..)$$

the mean errors of $x, x_1, x_2 \ldots$ being $\varepsilon, \varepsilon_1, \varepsilon_2 \ldots.$; and that we wish to find the mean error E of the function,

$$Y = F(X, X')$$

If any single observation of $x, x_1, x_2 \ldots.$ is affected by an error $\delta, \delta_1, \delta_2, \ldots.$ respectively, the corresponding errors in X and X' will be—

$$\text{Error in } X, \quad \Delta = a\delta + a_1\delta_1 + a_2\delta_2 + \ldots.$$
$$\text{“} \quad X', \quad \Delta' = a'\delta + a_1'\delta_1 + a_2'\delta_2 + \ldots.$$

in which $a, a_1, a_2 \ldots.$ are the differential coefficients of X, and $a', a_1', a_2' \ldots.$ the differential coefficients of X', with reference to $x, x_1, x_2, \ldots.$ The corresponding error in Y will be

$$\Delta'' = A\Delta + A'\Delta'$$

in which A and A' are the differential coefficients of Y with reference to X and X'. The square of the mean error E will be the mean of the squares of all the values of Δ'' which result from all the possible values of δ, δ_1, $\delta_2 \dots$

Substituting the values of Δ and Δ', we have

$$\Delta'' = (Aa + A'a')\,\delta + (Aa_1 + A'a_1')\,\delta_1 + \dots$$

which we may briefly express as follows:

$$\Delta'' = \alpha\delta + \beta\delta_1 + \gamma\delta_2 + \dots$$

If the number of values of Δ'' is denoted by m, the mean of all the values of Δ''^2 will be

$$\frac{[\Delta''^2]}{m} = \alpha^2 \frac{[\delta^2]}{m} + \beta^2 \frac{[\delta_1^2]}{m} + \gamma^2 \frac{[\delta_2^2]}{m} + \dots$$

$$+ 2\alpha\beta \frac{[\delta\delta_1]}{m} + 2\alpha\gamma \frac{[\delta\delta_2]}{m} + \dots$$

In consequence of the various signs of $\delta\delta_1$, $\delta\delta_2$, &c., the mean value of each of these quantities will be zero; and the mean values of δ^2, δ_1^2, &c. are ε^2, ε_1^2, &c. Hence the formula becomes simply

$$E^2 = (Aa + A'a')^2\varepsilon^2 + (Aa_1 + A'a_1')^2\varepsilon_1^2 + \dots$$

or

$$\left.\begin{array}{l} E^2 = A^2(a^2\varepsilon^2 + a_1^2\varepsilon_1^2 + \dots) + A'^2(a'^2\varepsilon^2 + a_1'^2\varepsilon_1^2 + \dots) \\ \quad + 2AA'(aa'\varepsilon^2 + a_1a_1'\varepsilon_1^2 + \dots) \end{array}\right\} \quad (35)$$

To illustrate by a very simple example, let

$$X = 2x \qquad X' = 3x$$

and suppose $\varepsilon = 0.1$; then, to find the mean error E of

$$Y = X + X'$$

we cannot take $E = \sqrt{[(0.2)^2 + (0.3)^2]}$ as we should if X and X' were independent, but by the above formula we must take

$$E = \sqrt{[(0.2)^2 + (0.3)^2 + 2 \times 2 \times 3 \times (0.1)^2]} = 0.5$$

as in fact we find directly, in this simple case, by first substituting in Y the values of X and X'.

WEIGHT OF OBSERVATIONS.

24. Observations of the same kind are said to have the same or different weight according as they have the same or different mean (or probable) errors. We assume *a priori* that observations will have the same weight when they are made under precisely the *same circumstances*, including under this designation every thing that can affect the observations; but whether this condition has in any case been realized can only be learned, *a posteriori*, from the mean errors revealed by the observations themselves.

In order to obtain a numerical expression of the weight, let us suppose all our observations to be compared with a standard fictitious observation the mean error of which is any assumed quantity ϵ_1. Let the actual observations be subject to the mean error ϵ. Let it require a number p of standard observations to be combined in order to reduce the mean error of their arithmetical mean to that of an actual observation, that is, to ϵ; or, according to (24), let

$$\epsilon = \frac{\epsilon_1}{\sqrt{p}} \quad \text{or} \quad p\epsilon^2 = \epsilon_1^2 \tag{36}$$

then one of our actual observations is as good, that is, has the same weight, as p standard observations, and the number p may be used to denote that weight. If, in like manner, other observations of the same kind are subject to the mean error ϵ', and we have

$$p'\epsilon'^2 = \epsilon_1^2$$

one of these observations has the weight of p' standard observations, and the weights of the observations of the two actual series may be compared by means of the numbers p and p'. The weight of the fictitious observation is here the *unit of weight;* but this unit is altogether arbitrary, since it is only the *relative* weights of actual determinations that are to be considered.

It follows immediately, since we have

$$\epsilon_1^2 = p\epsilon^2 = p'\epsilon'^2$$

or

$$\frac{p}{p'} = \frac{\epsilon'^2}{\epsilon^2}$$ ([illegible])

hat *the weights of two observations are reciprocally proportional to the quares of their mean errors.*

The measure of precision (Art. 10) and the weight are to be listinguished from each other: the former varies inversely as he mean error, the latter inversely as the square of this error.

25. *To find the most probable mean of a number of observations of 'ifferent weights.*—Let n', n'', n''' .... be the given observed alues; p', p'', p''' .... their respective weights. By the preeding definition of the weight, the quantity n' may be considered s the mean of p' observations of the weight unity, n'' as the nean of p'' observations of the weight unity, &c. We may, herefore, conceive the given series of observed quantities reolved into a series of standard observations, all of equal weight, nd then apply to the latter series the principle of the arithmeical mean. The whole number of equivalent standard observaions will be $p' + p'' + p''' + \dots$; the sum of the p' standard bservations will be $p'n'$; the sum of the p'' standard observaions will be $p''n''$, &c.: hence the desired mean x_0 will be

$$x_0 = \frac{p'n' + p''n'' + p'''n''' + \dots}{p' + p'' + p''' + \dots} \tag{38}$$

r, more briefly,

$$x_0 = \frac{[pn]}{[p]} \tag{38*}$$

This formula shows that although the above demonstration mplies that p', p'', p''' .... are whole numbers, yet any numbers, whole or fractional, may be used which are in the same proporion; for f being any arbitrary factor, whole or fractional, we nay write for (38) the following:

$$x_0 = \frac{fp'n' + fp''n'' + fp'''n''' + \dots}{fp' + fp'' + fp''' + \dots}$$

nd then fp', fp'', fp''' .... may be regarded as the weights.

The value of x_0 is here an arithmetical mean only in the conentional sense implied in the substitution of fictitious observaons with uniform weights for the given observations. It may e called the *general mean* or the *probable mean.*

The weight of this general mean, referred to the unit of p', ', ... is $= p' + p'' + p''' + \dots$

The mean error of the general mean will be expressed by

$$\varepsilon_0 = \frac{\varepsilon_1}{\sqrt{(p' + p'' + p''' + \dots)}} = \frac{\varepsilon_1}{\sqrt{[p]}}$$

where ε_1 is the mean error corresponding to the unit of weight.

If ε_1 is not given, we shall have to find it from the observations themselves. Taking the difference between x_0 and each of the given quantities, we have the residuals

$$v' = n' - x_0, \qquad v'' = n'' - x_0, \qquad v''' = n''' - x_0, \dots$$

If $\varepsilon', \varepsilon'', \varepsilon''' \dots$ are respectively the mean errors of $n', n'', n''', \dots$, we shall have, as in Art. 17,

$$\varepsilon'^2 = v'v' + \varepsilon_0^2$$

whence

$$p'\varepsilon'^2 = \varepsilon_1^2 = p'v'v' + p'\varepsilon_0^2$$

and, in like manner,

$$\begin{gathered} \varepsilon_1^2 = p''v''v'' + p''\varepsilon_0^2 \\ \varepsilon_1^2 = p'''v'''v''' + p'''\varepsilon_0^2 \\ \text{\&c.} \end{gathered}$$

The number of given values $n', n'' \dots$ being $= m$, the sum of these equations is

$$m\varepsilon_1^2 = [pvv] + [p]\,\varepsilon_0^2$$

which combined with the above value of ε_0 gives

$$\varepsilon_1 = \sqrt{\left(\frac{[pvv]}{m-1}\right)} \tag{39}$$

and consequently, also,

$$\varepsilon_0 = \sqrt{\left(\frac{[pvv]}{(m-1)\,[p]}\right)} \tag{40}$$

EXAMPLE.—Let us suppose that the observations of Saturn's ring in Art. 17 had been given as in the following table, where the mean of the first seven observations of Art. 17 is given $= 39''.179$ with the weight $= 7$, the mean of the next following four $= 39''.285$ with the weight $= 4$, &c.

p	n	v	vv	pvv
7	39″.179	$-0''.129$	.016641	.1165
4	.285	-0 .023	529	21
5	.294	-0 .014	196	10
4	.407	$+0$ .099	9801	392
1	.410	$+0$ .102	10404	104
3	.320	$+0$ .012	144	4
3	.377	$+0$ .069	4761	143
4	.310	$+0$ .002	4	0
3	.127	-0 .181	32761	983
6	.448	$+0$ .140	19600	1176
$[p] = 40$	$x_0 = 39$ .308			$[pvv] = .3998$

Here the general mean x_0 found by (38) of course agrees with that found before. For the mean error corresponding to the unit of weight (which in this case is that of an observation as given in Art. 17), we have, by (39), since $m = 10$,

$$\varepsilon_1 = \sqrt{\left(\frac{.3998}{9}\right)} = 0''.211$$

and for the mean error of x_0, by (40),

$$\varepsilon_0 = \sqrt{\left(\frac{.3998}{9 \times 40}\right)} = 0''.033$$

which agree sufficiently well with the former values. A perfect agreement in the mean errors is not to be expected, since our formulæ are based upon the supposition that we have taken a sufficient number of observations to exhibit the several errors to which they are subject in the proportion of their respective probabilities; and this would require a very large number of bservations.

26. In the application of the preceding formulæ, it must be bserved that when the weights of different determinations of he same quantity are inferred from their mean errors, we must e certain that there are no constant errors (that is, constant uring the observations which compose a single determination) efore we can combine them together according to these weights, nless the constant errors are known to affect all the determinations equally and with the same sign. For example, if ten measures of the zenith distance of a star are made at one culmination, giving a mean error of 0″.4, and five measures at another, giving a mean error of 0″.8, the weights according to these errors would be as 4 to 1. But if it is known that the errors *peculiar to a culmination* (and affecting equally all the individual observations at that culmination) exceed 1″, it would be better to regard the observations as of the same weight, since there would be a greater probability of eliminating such peculiar errors by taking the simple arithmetical mean. If, however, the observer, from considerations independent of the observations, can estimate the weight of determinations made under different circumstances, then it is evident that these weights will serve for the combination, if the mean accidental errors of the several determinations are sensibly equal.

But if from the different circumstances we have deduced weights for the several determinations, and at the same time the mean errors (deduced from a discussion of the discrepancies of the observations composing each determination) are widely different, it is not easy to assign any general rule for reducing the weights which shall not be subject to some exceptions. In such cases, practical observers and computers have resorted to empirical formulæ, involving some arbitrary considerations, more or less plausible.

In many cases we can proceed satisfactorily as follows. Let

ε = the mean accidental error of a single observation,
η = the mean error peculiar to a determination which rests upon m such observations,
e = the total mean error of such a determination,

then, ε and η being supposed to be independent, we shall have

$$e^2 = \frac{\varepsilon^2}{m} + \eta^2 \qquad (41)$$

If then η can be obtained from independent considerations, this formula will give the value of e, and, consequently, the weight for each determination, and the combination may then be made by (38). For an example of a discussion according to these principles, see Vol. I. Art. 236.

INDIRECT OBSERVATIONS.

27. I proceed now to the application of the method of least uares to the solution of the general problem of determining e most probable values of any number of unknown quantities which the observed quantities are functions. The observa- ons are then said to be *indirect.* The particular case of direct bservations, already considered, is, however, included in this eneral problem; being the case in which the number of un- nown quantities is reduced to one, and this one is directly bserved.

The general problem embraces two classes of problems, which must be distinguished from each other. In the first class, the unknown quantities are *independent,* in the sense that they are subject to no conditions except those established by the observations: so that, *before taking the observations,* any assumed system of values of these quantities has the same probability as any other system. In the second class, there are assigned, *a priori,* certain conditions which the unknown quantities must satisfy at the same time that they satisfy (as nearly as possible) the conditions established by the observations. Thus, for example, if the three angles of a plane triangle are to be determined from observations of any kind, we have, *a priori,* the condition that the sum of hese angles must be equal to two right angles, and all the ystems of values which do not satisfy this condition are excluded t the outset. This class will be briefly considered hereafter, nder the head of "conditioned observations;" but our attention ill be chiefly directed to the first class, which includes most of e problems occurring in astronomical inquiries.

Again, the equations which the observations are to satisfy may *linear* or *non-linear;* the observed quantities may be *explicit* or *plicit* functions of the required quantities; but, for simplicity, e consider first the case of linear equations, to which all the hers may always be reduced.

EQUATIONS OF CONDITION FROM LINEAR FUNCTIONS.

28. Let us suppose the equations between the known and known quantities are of the form

$$ax + by + cz + \dots\dots + l = V$$

in which a, b, c, ... , l are known quantities given by theory each observation, V is the quantity observed, and x, y, z . . . . the quantities to be determined. For each observation, we a similar equation, and thus a system such as the following:

$$\left.\begin{array}{l} a'x\ + b'y\ + c'z\ + \ldots\ldots\ldots + l'\ = V' \\ a''x\ + b''y\ + c''z\ + \ldots\ldots\ldots + l''\ = V'' \\ a'''x + b'''y + c'''z + \ldots\ldots\ldots + l''' = V''' \\ \qquad \&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\}$$

the number of these equations being greater than that of unknown quantities (Art. 6). If our observations were perf all these equations would be satisfied by the same system values of x, y, z . . . ; but, being imperfect, let M', M'', M''' . . . . denote the values obtained by observation for V', V'', V'''. When these values are substituted in the second members of (there will, in general, be no system of values of x, y, z . . . . wh satisfies all the equations at the same time, and we can determine that system which is rendered most probable by observations. Let us therefore denote by N', N'', N''' . . . . values which the first members of our equations obtain w any hypothetical or assumed system of values of x, y, z . . . substituted in them; and put

$$v' = N' - M', \qquad v'' = N'' - M'', \qquad v''' = N''' - M''', \ldots.$$

then v', v'', v''' . . . . are the errors of the observations accor to this hypothesis. Finally, let us put

$$n' = l' - M', \qquad n'' = l'' - M'', \qquad n''' = l''' - M''', \ldots.$$

then our equations may be thus expressed:

$$\left.\begin{array}{l} a'x\ + b'y\ + c'z\ + \ldots. + n'\ = v' \\ a''x\ + b''y\ + c''z\ + \ldots. + n''\ = v'' \\ a'''x + b'''y + c'''z + \ldots. + n''' = v''' \\ \qquad \&c. \qquad\qquad\qquad \&c. \end{array}\right\}$$

If our observations were perfect, we should be able to values of x, y, z . . . . which would reduce all the quantities v', v''' . . . . to zero. It is usual, therefore, to write zero in the sec members:

$$\left.\begin{array}{l} a'x\ + b'y\ + c'z\ + \ldots. + n'\ = 0 \\ a''x\ + b''y\ + c''z\ + \ldots. + n''\ = 0 \\ a'''x + b'''y + c'''z + \ldots. + n''' = 0 \\ \qquad \&c. \qquad\qquad\qquad \&c. \end{array}\right\}$$

and these are called the *equations of condition*, since they express the conditions which the unknown quantities are required to satisfy as nearly as possible. We may, however, with more rigor regard (43) as our equations of condition, and treat them as expressing the general condition that the unknown quantities shall be such as to give the most probable system of errors $v', v'', v''' \dots$.

Now, according to Art. 11, the most probable system of values of $x, y, z \dots$ (and, consequently, the most probable system of errors) is that which makes the sum of the squares of the errors a minimum: thus, we are to reduce to a minimum the function

$$[vv] = v'v' + v''v'' + v'''v''' + \dots$$

Regarding $[vv]$ as a function of the variables $x, y, z \dots$ (which we must remember are here independent), the condition of minimum requires that its derivatives taken with reference to each variable shall each be zero; that is,

$$\frac{d[vv]}{dx} = 0, \quad \frac{d[vv]}{dy} = 0, \quad \frac{d[vv]}{dz} = 0, \dots$$

or

$$\left.\begin{array}{l} v'\dfrac{dv'}{dx} + v''\dfrac{dv''}{dx} + v'''\dfrac{dv'''}{dx} + \dots = 0 \\ v'\dfrac{dv'}{dy} + v''\dfrac{dv''}{dy} + v'''\dfrac{dv'''}{dy} + \dots = 0 \\ v'\dfrac{dv'}{dz} + v''\dfrac{dv''}{dz} + v'''\dfrac{dv'''}{dz} + \dots = 0 \\ \qquad \&c. \end{array}\right\} \quad (44)$$

(which we might have obtained directly from (10) by substituting $\varphi'\varDelta = k\varDelta = kv$, and dividing by the constant k). But, by differentiating the equations (43) with reference to $x, y, z \dots$ successively, we have

$$\begin{array}{lll} \dfrac{dv'}{dx} = a', & \dfrac{dv'}{dy} = b', & \dfrac{dv'}{dz} = c', \dots \\ \dfrac{dv''}{dx} = a'', & \dfrac{dv''}{dy} = b'', & \dfrac{dv''}{dz} = c'', \dots . \\ \&c. & \&c. & \&c. \end{array}$$

so that (44) are the same as the following:

$$\left.\begin{aligned} a'v' + a''v'' + a'''v''' + \ldots &= 0 \\ b'v' + b''v'' + b'''v''' + \ldots &= 0 \\ c'v' + c''v'' + c'''v''' + \ldots &= 0 \\ \&c. \end{aligned}\right\} \quad (44^*)$$

The number of these equations is the same as that of the unknown quantities; and if we now substitute in them the values of v', v'', v'''... from (43), we have the final or, as we shall call them, the *normal* equations, which determine the most probable values of x, y, z....

NORMAL EQUATIONS.

29. We see by (44*) that to form the first normal equation we multiply each of the equations of condition (43) or (43*) by the coefficient of x in that equation, and then form the sum of all the equations thus multiplied. The resulting equation is called the normal equation in x.* The sum of the equations of condition severally multiplied by the coefficients of y is the normal equation in y, &c. To abbreviate the expression of these sums, we put

$$\begin{aligned} [aa] &= a'a' + a''a'' + a'''a''' + \ldots \\ [ab] &= a'b' + a''b'' + a'''b''' + \ldots \\ [ac] &= a'c' + a''c'' + a'''c''' + \ldots \\ &\&c. \qquad\qquad \&c. \end{aligned}$$

then the normal equations are

$$\left.\begin{aligned} [aa]\,x + [ab]\,y + [ac]\,z + \ldots + [an] &= 0 \\ [ab]\,x + [bb]\,y + [bc]\,z + \ldots + [bn] &= 0 \\ [ac]\,x + [bc]\,y + [cc]\,z + \ldots + [cn] &= 0 \\ \&c. \qquad\qquad \&c. \end{aligned}\right\} \quad (45)$$

30. The formation of such normal equations is one of the most laborious parts of the computations involved in the method of least squares, especially when the number of equations is very great.† It is important to have a means of verification, or "control," to insure their accuracy, before proceeding with the next important process of elimination. A very simple and effective control is the following.

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* The "normal equation in x" is so called because it is the equation which determines the most probable value of x when the other variables are reduced to zero, or when x is the only unknown quantity; and so of the others.

† This labor may be abridged by the use of Dr. CRELLE'S *Rechentafeln*, Berlin, 1869.

Form the sums of the coefficients of the unknown quantities in the several equations, namely,

$$\left.\begin{aligned} a' + b' + c' + \ldots\ldots &= s' \\ a'' + b'' + c'' + \ldots\ldots &= s'' \\ a''' + b''' + c''' + \ldots\ldots &= s''' \\ \&c. & \end{aligned}\right\} \qquad (46)$$

If we multiply each of these by its n, and add the products, we have

$$[an] + [bn] + [cn] + \ldots\ldots = [sn] \qquad (47)$$

Also, multiplying each of (46) by its a, and adding, then each by its b, and adding, and so on, we have

$$\left.\begin{aligned} [aa] + [ab] + [ac] + \ldots\ldots &= [as] \\ [ab] + [bb] + [bc] + \ldots\ldots &= [bs] \\ [ac] + [bc] + [cc] + \ldots\ldots &= [cs] \\ \&c. & \end{aligned}\right\} \qquad (48)$$

The equations (47) must be satisfied when the absolute terms of the normal equations are correct, and (48) when the coefficients of the unknown quantities are correct.

31. The normal equations will give determinate values of $x, y, z \ldots$, provided they are really independent. If, however, any two of them become identical by the multiplication of either of them by a constant, the number of independent equations is, in fact, one less than that of the unknown quantities, and the problem becomes indeterminate. This difficulty does not arise from the method by which the normal equations are formed, but from the nature of the given equations of condition. In any such case, additional observations are necessary, for which the coefficients have such varied values as to lead to independent equations. Even when two equations cannot be reduced precisely to a single one by the introduction of a constant factor, if they can be made very nearly identical, the problem is still practically indeterminate. The indetermination will become evident in the actual elimination in practice when any one of the unknown quantities comes out with so small a coefficient that small errors in the observations would greatly change this coefficient. (See Art. 52.)

32. By whatever method the elimination is performed, we shall necessarily arrive at the same final values of the unknown quantities; but, when the number of equations is considerable, the method of substitution, with GAUSS's convenient notation, is universally followed; but, for the present, leaving the reader to choose his method, I proceed to explain the principles by which the mean errors of the values of $x, y, z \ldots .$ are determined.

MEAN ERRORS AND WEIGHTS OF THE UNKNOWN QUANTITIES.

33. Since we have put $n' = l' - M'$, $n'' = l'' - M''$, &c. (Art. 28), the mean error of $n', n'', n''' \ldots .$ is also that of $M', M'', M''', \ldots .$; that is, the mean error of $n', n'', n''' \ldots .$ is to be regarded as the mean error of an observation. If the elimination of the normal equations were fully carried out, each unknown quantity would be finally expressed as a linear function of $n', n'', n''', \ldots .$, and the mean errors of the latter being given, those of the unknown quantities would follow by the principle of Art. 20. It results, however, from the symmetry of the normal equations that several forms may be obtained for computing directly the weights of the unknown quantities, and from these weights the mean errors can afterwards be found.

34. *First method of computing the weights of the unknown quantities.*—For simplicity, let us first suppose all the observations to be of equal weight, or the mean errors of n', n'', n''' to be equal. Let

ε = the mean error of an observation,

ε_x = the mean error of the value of x found from the normal equations,

p_x = the weight of the value of x, the weight of an observation being unity;

then (Art. 24)

$$p_x = \frac{\varepsilon^2}{\varepsilon_x^2}$$

Now, let us suppose the elimination to be performed by the method of indeterminate coefficients. Let the first equation of (45) be multiplied by Q, the second by Q', the third by Q'', &c., and the products added. Then let the factors $Q, Q', Q'' \ldots$ (whose number is the same as that of the unknown quantities) be supposed to be determined so that in this final equation the coefficients of all the unknown quantities shall be zero, except that of x, which shall be unity. The conditions for determining these factors are, therefore,

$$\left.\begin{array}{l} [aa]\,Q + [ab]\,Q' + [ac]\,Q'' + \ldots\ldots = 1 \\ [ab]\,Q + [bb]\,Q' + [bc]\,Q'' + \ldots\ldots = 0 \\ [ac]\,Q + [bc]\,Q' + [cc]\,Q'' + \ldots\ldots = 0 \\ \quad\&c. \qquad\qquad\qquad \&c. \end{array}\right\} \quad (49)$$

and the final equation in x is

$$x + [an]\,Q + [bn]\,Q' + [cn]\,Q'' + \ldots\ldots = 0 \qquad (50)$$

Comparing (45) and (49), we see that the coefficients of $Q, Q', Q'' \ldots\ldots$ are the same as those of $x, y, z \ldots\ldots$, but that the absolute terms are -1 in (49) instead of $[an]$ in (45), and zero instead of $[bn]$, $[cn]$, &c. Hence, if the elimination of (45) were carried out, and the values of $x, y, z \ldots\ldots$ determined in terms of $n', n'', n''' \ldots\ldots$, the values of $Q, Q', Q'' \ldots\ldots$ would be found from these by merely putting $[an] = -1$, and $[bn] = [cn]$, &c. $= 0$. This is also evident from (50). I shall now show that Q is the reciprocal of the required weight of x.

The final value of x being a linear function of $n', n'', n''' \ldots\ldots$, the equation (50) may be supposed to be developed in the form

$$x + \alpha'n' + \alpha''n'' + \alpha'''n''' + \ldots\ldots = 0 \qquad (51)$$

in which $\alpha', \alpha'', \alpha''', \ldots\ldots$ are functions of $a', b', \ldots\ldots, a'', b'', \ldots\ldots$, &c.; and these functions are immediately found by developing $[an]$, $[bn]$, &c., in (50); for we then have, by comparing the coefficients of (50) and (51),

$$\left.\begin{array}{l} \alpha' = a'\,Q + b'\,Q' + c'\,Q'' + \ldots\ldots \\ \alpha'' = a''\,Q + b''\,Q' + c''\,Q'' + \ldots\ldots \\ \alpha''' = a'''Q + b'''Q' + c'''Q'' + \ldots\ldots \\ \qquad\&c. \qquad\qquad \&c. \end{array}\right\} \quad (52)$$

Multiplying each of these equations by its a, and adding all the products, we obtain, by (49),

$$a'\alpha' + a''\alpha'' + a'''\alpha''' + \ldots\ldots = 1$$

Multiplying each of (52) by its b, and adding, we obtain, by (49),

$$b'\alpha' + b''\alpha'' + b'''\alpha''' + \ldots\ldots = 0$$

and so on for as many equations as there are unknown quantities. These relations are briefly expressed thus:

$$[a\alpha] = 1 \qquad [b\alpha] = 0 \qquad [c\alpha] = 0,\ \&c. \qquad (53)$$

If, then, each of (52) is multiplied by its α, and the results added, we find, by (53),

$$[\alpha\alpha] = \alpha'^2 + \alpha''^2 + \alpha'''^2 + \ldots. = Q \qquad ($$

But, by Art. 20, when ε is the mean error of each of the qu tities $n', n'', n''', \ldots.$, the mean error of x found by (51) is

$$\varepsilon_x = \varepsilon \sqrt{[\alpha\alpha]}.$$

Hence

$$p_x = \frac{\varepsilon^2}{\varepsilon_x^2} = \frac{1}{[\alpha\alpha]} = \frac{1}{Q} \qquad ($$

as was to be proved.

Hence we have a first method of finding the weights. *In first normal equation write* -1 *for the absolute term* $[an]$, *and in other equations zero for each of the absolute terms* $[bn]$, $[cn]$, &c.; *value of* x *then found from these equations will be the reciprocal of weight of the value of* x *found by the general elimination.*

This rule is to be applied to each of the unknown quantit in succession, so that the reciprocal of the weight of y is tl value of y which will be found by putting $[bn] = -1$, a $[an] = [cn] =$ &c. $= 0$; the reciprocal of the weight of z is tl value of z which will be found by putting $[cn] = -1$, a $[an] = [bn]$, &c. $= 0$; &c.

It is evident, moreover, that although we have deduced t rule by the use of indeterminate multipliers, it must hold gc whatever method of elimination is adopted.

35. *Second method of computing the weights of the unknown qu tities.*—If we write the normal equations thus,

$$\begin{aligned}
[aa]\,x + [ab]\,y + [ac]\,z + \ldots. + [an] &= A \\
[ab]\,x + [bb]\,y + [bc]\,z + \ldots. + [bn] &= B \\
[ac]\,x + [bc]\,y + [cc]\,z + \ldots. + [cn] &= C \\
\text{\&c.} \qquad\qquad\qquad & \text{\&c.}
\end{aligned}$$

and perform the elimination, we shall obtain $x, y, z \ldots.$ in ter of $[aa]$, $[ab]$, &c., and of A, B, C, &c.; and if in the general val thus found we make $A = B = C$, &c. $= 0$, these values will reduced to those which would be found by carrying out elimination with zero in the second members of the norı equations. If we suppose the elimination performed by me of the indeterminate factors Q, Q', Q''.... already employed, the final equation for determining x will be

$$x + [an]Q + [bn]Q' + [cn]Q'' + \ldots. = QA + Q'B + Q''C + \ldots.$$

where the coefficient of A is the reciprocal of the required weight of x. But, whatever method of elimination is employed, the coefficient of A in this general value of x will necessarily be the same; and hence we derive the second method of determining the weights: *Write* A, B, C, &c., *instead of* 0, *in the second members of the normal equations, and carry out the elimination* (by any method at pleasure); *then the final values of* x, y, z.... *are those terms in the general values which are independent of* A, B, C....; *the weight of* x *is the reciprocal of the coefficient of* A *in the general value of* x; *the weight of* y *is the reciprocal of the coefficient of* B *in the general value of* y; &c.

36. *Third method of computing the weights of the unknown quantities.* —Let us suppose the elimination to be performed by the method of substitution, still retaining A, B, C.... in the second members, as in the preceding article. The final equation in x, according to this method, is found by substituting in the first normal equation the values of y, z.... given by the other equations. These substitutions do not affect the coefficient of A, which remains unity, so long as no reduction is made after the substitutions. Thus, the final equation in x is of the form

$$Rx = T + A + \text{terms in } B,\ C, \ldots.$$

in which T is the sum of all the absolute quantities resulting from the substitution, and is a function of $[aa]$, $[ab]$,....$[an]$. Hence the value of x is

$$x = \frac{T}{R} + \frac{A}{R} + \text{terms in } B,\ C, \ldots.$$

in which $\frac{T}{R}$ is the final value of x which results when $A = B = C \ldots. = 0$, and $\frac{1}{R}$ is necessarily the quantity denoted by Q in the preceding articles. Therefore R is the weight of x, and hence we have a third method of finding the weights: *Let the first normal equation* (the equation in x, Art. 29) *be taken as the final equation for determining* x, *and substitute in it the values of* y, z.... *in* *terms of x as found from the remaining equations; then, before freeing the equation of fractions or introducing any reduction factor, the coefficient of x in this equation is the weight of the value of x. In the same manner, substitute in the second normal equation* (the equation in y) *the values of $x, z \ldots.$ in terms of y as found from the other equations; the coefficient of y is then the weight of the value of y; and so proceed for each unknown quantity.*

According to this method we determine each unknown quantity, *together with its weight*, by a separate elimination carried through all the equations, in each case changing the order of elimination, until every unknown quantity has been made to come out the last. The algorithm of this process, with GAUSS's convenient system of notation, will be given hereafter (Art. 45).

37. *To find the mean error of observation.*—The weight of x being found, we have the ratio of ε_x to ε, but we have yet to determine ε, which, in general, cannot be assigned *a priori*, but must be deduced *a posteriori*, that is, from the observations, and consequently from the equations of condition. The residuals $v', v'', v''' \ldots.$, in (43), are those which result when the most probable values of $x, y, z \ldots.$ (namely, those resulting from the normal equations) are substituted in the first members. The actual or *true* errors (Art. 17) of observation are, however, those values of the first members of (43) which result when the *true* values of $x, y, z, \ldots.$ are substituted.

Let $x + \Delta x$, $y + \Delta y$, $z + \Delta z, \ldots.$ be the true values which, substituted in the equations of condition, give the true residuals $u', u'', u''' \ldots.$; so that we have

$$\left.\begin{array}{l} a'\,(x + \Delta x) + b'\,(y + \Delta y) + c'\,(z + \Delta z) + \ldots.\, n' = u' \\ a''\,(x + \Delta x) + b''\,(y + \Delta y) + c''\,(z + \Delta z) + \ldots.\, n'' = u'' \\ a'''(x + \Delta x) + b'''(y + \Delta y) + c'''(z + \Delta z) + \ldots.\, n''' = u''' \\ \qquad\qquad \&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\} \quad (56)$$

If these equations be multiplied by $a', a'', a''' \ldots.$, respectively, the sum of the products is

$$\left.\begin{array}{l} [aa]\,x \;\; + [ab]\,y \;\; + [ac]\,z \;\; + \ldots. + [an] \\ + [aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots. \end{array}\right\} = [au]$$

which by the first of (45) is reduced to

$$[aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots. - [au] = 0$$

the same manner, multiplying each of the equations (56) by b, c, &c., successively, we form the other equations of the owing group:

$$\left.\begin{array}{l} [aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots. - [au] = 0 \\ [ab]\,\Delta x + [bb]\,\Delta y + [bc]\,\Delta z + \ldots. - [bu] = 0 \\ [ac]\,\Delta x + [bc]\,\Delta y + [cc]\,\Delta z + \ldots. - [cu] = 0 \\ \quad\&\text{c.} \qquad\qquad\qquad \&\text{c.} \end{array}\right\} \tag{57}$$

se being of the same form as the normal equations (45), we that the value of Δx resulting from them will be of the same n as that of x resulting from (45), with only the substitution - u for n: hence, by (51),

$$\Delta x - \alpha'u' - \alpha''u'' - \alpha'''u''' - \ldots. = 0 \tag{58}$$

in, multiplying (56) by $v', v'', v'''\ldots.$, respectively, the sum he products is, by (44*), reduced to

$$[vn] = [vu]$$

. in the same manner, from (43),

$$[vn] = [vv]$$

nce

$$[vu] = [vv] = [vn] \tag{59}$$

e sum of the products obtained by multiplying the equations) respectively by $u', u'', u'''\ldots.$ is

$$[au]\,x + [bu]\,y + [cu]\,z + \ldots. + [nu] = [vu] = [vv]$$

l from (56), in the same manner,

$$\left.\begin{array}{l} [au]\,x \;\; + [bu]\,y \;\; + [cu]\,z \;\; + \ldots. + [nu] \\ + [au]\,\Delta x + [bu]\,\Delta y + [cu]\,\Delta z + \ldots. \end{array}\right\} = [uu]$$

ich two equations give

$$[uu] = [vv] + [au]\,\Delta x + [bu]\,\Delta y + [cu]\,\Delta z + \ldots. \tag{60}$$

r, $[uu]$ being the sum of the squares of the true errors of the ervations, its value is, as in Art. 17, $= m\varepsilon\varepsilon$, if we put

$m =$ the number of observations,
$=$ the number of equations of condition.

Consequently, if we could assume $\Delta x, \Delta y \ldots.$ to vanish, we should have

$$\varepsilon\varepsilon = \frac{[vv]}{m}$$

and this will usually give a close approximation to the value of ε, but it will give the true value only in the exceedingly improbable case in which the values of $x, y, z \ldots.$ are absolutely true, whereas they are to be regarded only as the most probable ones furnished by the observations. This formula, then, must always give too small a value of ε, since it ascribes too high a degree of precision to the observations. We must, therefore, add to $[vv]$ the quantities $[au]\,\Delta x$, $[bu]\,\Delta y$, &c., as in (60); but, as we cannot assign any other than approximate values of these quantities, let us assume for them their mean values as found by the theory of mean errors. The mean value of $[au]\,\Delta x$ will be found by multiplying together

$$[au] = a'u' + a''u'' + a'''u''' + \ldots.$$

and

$$\Delta x = \alpha'u' + \alpha''u'' + \alpha'''u''' + \ldots.$$

observing that the errors $u', u'', u''' \ldots.$, when we consider only their mean values, are to be regarded as having the double sign $\pm$; so that the mean value of the product will contain only the terms $a'\alpha'u'u'$, $a''\alpha''u''u''$, &c. Hence we take

$$[au]\,\Delta x = a'\alpha'u'u' + a''\alpha''u''u'' + a'''\alpha'''u'''u''' + \ldots.$$

and substituting in this the mean value of $u'u'$, $u''u''$, &c., which in each case is $\varepsilon\varepsilon$, we have

$$[au]\,\Delta x = (a'\alpha' + a''\alpha'' + a'''\alpha''' + \ldots.)\,\varepsilon\varepsilon$$

or, finally, by (53),

$$[au]\,\Delta x = \varepsilon\varepsilon$$

In the same manner, it must follow that $\varepsilon\varepsilon$ is the mean value of each of the terms $[bu]\,\Delta y$, $[cu]\,\Delta z$, &c. If then we put

$$\mu = \text{the number of unknown quantities,}$$

the equation (60) becomes

$$m\varepsilon\varepsilon = [vv] + \mu\varepsilon\varepsilon$$

whence

$$\epsilon\epsilon = \frac{[vv]}{m - \mu} \qquad \epsilon = \sqrt{\frac{[vv]}{m - \mu}} \qquad (61)$$

It is to be observed that when there is but one unknown quantity, or $\mu = 1$, this general form is reduced to the simple one (25), already given for direct observations.

Finally, $p_x, p_y, p_z, \ldots$ denoting the weights of $x, y, z \ldots$ found by any of the preceding methods, we have

$$\epsilon_x = \frac{\epsilon}{\sqrt{p_x}} \qquad \epsilon_y = \frac{\epsilon}{\sqrt{p_y}}, \text{ \&c.} \qquad (62)$$

88. **Example.**—Let us suppose the following very simple equations of condition to be given :*

$$\begin{aligned} x - y + 2z - 3 &= 0 \\ 3x + 2y - 5z - 5 &= 0 \\ 4x + y + 4z - 21 &= 0 \\ -x + 3y + 3z - 14 &= 0 \end{aligned}$$

If but the first three of these equations had been given, the problem would have been determinate. We should find from them $x = \frac{18}{7}$, $y = \frac{23}{7}$, $z = \frac{13}{7}$; and we should have to accept these values as final ones, with no means of judging of their accuracy, or of that of the observations upon which the equations are supposed to depend. A fourth observation having given us our fourth equation, we find that the values of x, y, z derived from the first three will not satisfy it, for when they are substituted in it the first member becomes $-\frac{8}{7}$, instead of zero. If we determine the values of x, y, and z from any three of the equations, and substitute these values in the fourth, we shall find a residual. Each one of the four systems of values of the unknown quantities thus found satisfies three equations exactly, and the fourth approximately; but, all the observations being subject to error, the most probable system of values can seldom satisfy any one of the equations exactly. Hence the necessity of a principle of computation which shall lead as directly as possible to such a probable system of values; and this principle is furnished by the method of least squares.

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* Gauss, *Theoria Motus*, Art. 184.

We are, then, by Art. 29, to deduce from these four equations three normal equations, and the values of x, y, z which exactly satisfy these are to be regarded as the most probable values.

To form the first normal equation, we multiply the first of the above equations of condition by 1 $(=a')$, the second by 3 $(=a'')$, the third by 4 $(=a''')$, and the fourth by -1 $(=a^{\text{iv}})$, and add the products. We thus find $[aa] = 27$, $[ab] = 6$, $[ac] = 0$, and $[an] = -88$.

To form the second normal equation, we multiply the first equation of condition by -1 $(= b')$, the second by 2 $(= b'')$, the third by 1 $(= b''')$, and the fourth by 3 $(= b^{\text{iv}})$, and add the products. We thus find $[ab] = 6$, $[bb] = 15$, $[bc] = 1$, $[bn] = -70$.

The third normal equation is formed by multiplying the first equation of condition by 2 $(= c')$, the second by -5 $(= c'')$, the third by 4 $(= c''')$, and the fourth by 3 $(= c^{\text{iv}})$, and adding the products. We find $[ac] = 0$, $[bc] = 1$, $[cc] = 54$, $[cn] = -107$.

Hence our normal equations are

$$\begin{array}{rrrr} 27x + & 6y & & - \ 88 = 0 \\ 6x + & 15y + & z & - \ 70 = 0 \\ & y + & 54z & - 107 = 0 \end{array}$$

the solution of which gives, as the most probable values,

$$x = \frac{49154}{19899} = 2.470$$

$$y = \frac{2617}{737} = 3.551$$

$$z = \frac{12707}{6633} = 1.916$$

In order to determine the mean, and hence also the probable, errors of these values, let us first determine their weights according to the preceding methods.

First. By the method of Art. 34, we first write $-1, 0, 0$, for the absolute terms of the three normal equations, and we have the three equations for determining the weight of x,

$$\begin{array}{rrrr} 27x' + & 6y' & - & 1 = 0 \\ 6x' + & 15y' + & & z' = 0 \\ & y' + & & 54z' = 0 \end{array}$$

in which accents are employed to distinguish the particular values from the above general ones. These give

$$x' = \frac{809}{19899}$$

which is the reciprocal of the required weight. Hence,

$$p_x = \frac{19899}{809} = 24.597$$

In a similar manner, to find the weight of y, we take the equations

$$\begin{aligned} 27x'' + 6y'' \qquad\quad &= 0 \\ 6x'' + 15y'' + z'' - 1 &= 0 \\ y'' + 54z'' \qquad &= 0 \end{aligned}$$

and find

$$y'' = \frac{54}{737}$$

whence

$$p_y = \frac{737}{54} = 13.648$$

And to find the weight of z, the equations

$$\begin{aligned} 27x''' + 6y''' \qquad\qquad &= 0 \\ 6x''' + 15y''' + z''' \qquad &= 0 \\ y''' + 54z''' - 1 &= 0 \end{aligned}$$

which give

$$z''' = \frac{41}{2211}$$

and

$$p_z = \frac{2211}{41} = 53.927$$

Secondly. By the method of Art. 35, we write our normal equations thus:

$$\begin{aligned} 27x + 6y \qquad - 88 &= A \\ 6x + 15y + z - 70 &= B \\ y + 54z - 107 &= C \end{aligned}$$

and, carrying out the elimination as if A, B, and C were known quantities, we find

$$\begin{aligned} 19899x &= 49154 + (809)A - 324\,B + 6\,C \\ 737y &= 2617 - 12\,A + (54)B - C \\ 6633z &= 12707 + 2A - 9\,B + (123)\,C \end{aligned}$$

and, therefore,

$$x = \frac{49154}{19899} \text{ with the weight } p_x = \frac{19899}{809}$$

$$y = \frac{2617}{737} \quad \text{" " "} \quad p_y = \frac{737}{54}$$

$$z = \frac{12707}{6633} \quad \text{" " "} \quad p_z = \frac{6633}{123}$$

the same as by the first method.

Thirdly. By the method of Art. 36, to find x and its w we eliminate y and z from the equation in x (the first n equation) by means of the other equations, employing succ substitutions. The last normal equation gives

$$z = -\frac{1}{54}y + \frac{107}{54}$$

which being substituted in the second gives

$$6x + \frac{809}{54}y - \frac{3673}{54} = 0$$

The value of y from this, namely,

$$y = -\frac{324}{809}x + \frac{3673}{809}$$

being substituted in the first normal equation, and no red being made, gives

$$\frac{19899}{809}x - \frac{49154}{809} = 0$$

where the coefficient of x is the weight, and the value of x same as before found.

To find y and its weight, we make the second the final tion. From the first and third we find

$$x = -\frac{6}{27}y + \frac{88}{27}$$

$$z = -\frac{1}{54}y + \frac{107}{54}$$

which substituted in the second give

$$\frac{737}{54}y - \frac{2617}{54} = 0$$

where the coefficient of y is its weight.

ally, to find z with its weight, we make the third normal ion the final one. From the first two we find

$$y = -\frac{9}{123}z + \frac{454}{123}$$

substituted in the third gives

$$\frac{6633}{123}z - \frac{12707}{123} = 0$$

the coefficient of z is its weight, and its value is the same before found.

a little attention, it will be perceived that the three methods e essentially the same numerical operations.

are next to find the mean errors of x, y, and z; for which se we must first find the mean error of an observation, ing here, for the sake of illustration, that the absolute terms given equations of condition are the observed quantities, at they are subject to the same mean error. Substituting se equations the above found values of x, y, and z, we the residuals as follows:

No.	v	vv
1	-0.249	0.0620
2	-0.068	.0046
3	$+0.095$	.0090
4	-0.069	.0048

$$m = 4,\ \mu = 3, \quad [vv] = 0.0804$$

$$\frac{[vv]}{m-\mu} = 0.0804$$

, by (61),

$$\varepsilon = \sqrt{0.0804} = 0.284$$

is the mean error of an observation, so far as this error e inferred from so small a number of observations. (See ext article.) Consequently, the mean errors of x, y, and z follows:

$$\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}} = 0.057$$

$$\varepsilon_y = \frac{\varepsilon}{\sqrt{p_y}} = 0.077$$

$$\varepsilon_z = \frac{\varepsilon}{\sqrt{p_z}} = 0.039$$

Multiplying these errors by the constant 0.6745, we shall have (Art. 15) the probable errors as follows:

Probable error of an observation $= 0.192$
" " x $= 0.038$
" " y $= 0.052$
" " z $= 0.026$

39. It has already been remarked in the foregoing pages, and the remark is especially important in the present connection, that the method of least squares supposes in general a great number of observations to have been taken, or a number sufficiently great to determine approximately the errors to which the observations are liable. Theoretically, the greater the number of observations the more nearly will the series of residuals express the series of actual errors, and, consequently, the more correct will be the value of ε inferred from these residuals. In practice, therefore, no dependence should be placed upon the mean or probable errors deduced from so small a number of observations as we have employed, for the sake of brevity and clearness, in the preceding example. Nevertheless, the method is, even in this case, the best adapted for determining the most probable values of the unknown quantities deducible from the given observations, and also their relative degree of precision. Thus, in this example, the degrees of precision (denoted by h, Art. 10) of x, y, and z, being inversely proportional to the mean errors, or directly proportional to the square roots of the weights, are nearly as the numbers 5, 3.7, and 7.3, so that from the four given observations z is about twice as accurately found as y, while the precision of x falls between that of y and z. But we can place but little dependence upon the result which assigns 0.284 as the mean error of observation, and 0.057, 0.077, 0.039 as the mean errors of x, y, and z, because this result is derived from too small a number of observations.

EQUATIONS OF CONDITION FROM NON-LINEAR FUNCTIONS.

40. Let the relation between the observed quantities V', V'', V'''. . . . and the unknown quantities X, Y, Z. . . . be, for the observations severally,

$$\left.\begin{array}{l} f'\;(V',\; X, Y, Z, \ldots\ldots) = 0 \\ f''\,(V'',\; X, Y, Z, \ldots\ldots) = 0 \\ f'''(V''', X, Y, Z, \ldots\ldots) = 0 \\ \qquad\qquad \&c. \end{array}\right\} \qquad (63)$$

the values of V', V'', V'''...., found by observation, be M'', M'''.... These values being substituted, we shall the equations

$$\left.\begin{array}{l} f'\ (M',\ X, Y, Z, \ldots.) = 0 \\ f''\ (M'',\ X, Y, Z, \ldots.) = 0 \\ f'''\ (M''', X, Y, Z, \ldots.) = 0 \\ \quad\quad \&c. \end{array}\right\} \qquad (64)$$

which the values of X, Y, Z.... are to be found. But, as annot effect the direct solution of these equations according e method of least squares so long as they are not linear, we t to the following indirect process, by which linear equations ndition are formed. Let *approximate* values of X, Y, Z.... be d, either by some independent method or from a sufficient ber of the equations (64) treated by any suitable process, and te these approximate values by X_0, Y_0, Z_0.... Let the most able values be

$$X = X_0 + x, \qquad Y = Y_0 + y, \qquad Z = Z_0 + z, \ldots\ldots$$

x, y, z.... are the corrections required to reduce our ap-imate values to the most probable values; in other words, z.... are the most probable corrections of the approximate s, and the method of least squares is now to be applied in ng these corrections.

bstitute the approximate values X_0, Y_0, Z_0... in (63), and by resolving the equations, the corresponding values of V''.... which denote by V_0', V_0''.... These will be func- which may be thus generally expressed:

$$\begin{array}{l} V_0' = F'\ (X_0, Y_0, Z_0 \ldots.) \\ V_0'' = F''\ (X_0, Y_0, Z_0 \ldots.) \\ \quad\quad \&c. \end{array}$$

, the values of V', V''.... which result when the most pro- values $X_0 + x$, $Y_0 + y$, $Z_0 + z$ are substituted, and which et unknown, being denoted by N', N''.... we have

$$\begin{array}{l} N' = F'\ (X_0 + x,\ Y_0 + y,\ Z_0 + z, \ldots.) \\ N'' = F''\ (X_0 + x,\ Y_0 + y,\ Z_0 + z, \ldots.) \\ \quad\quad \&c. \end{array}$$

by TAYLOR's Theorem, when we neglect the higher powers of $x, y, z \dots$ which are supposed to be very small quantities, we have

$$N' = V_0' + \frac{dV_0'}{dX_0}x + \frac{dV_0'}{dY_0}y + \frac{dV_0'}{dZ_0}z + \dots$$
$$N'' = V_0'' + \frac{dV_0''}{dX_0}x + \frac{dV_0''}{dY_0}y + \frac{dV_0''}{dZ_0}z + \dots$$
$$\&c. \qquad \&c.$$

where $\frac{dV_0'}{dX_0}, \frac{dV_0''}{dX_0}$, &c., $\frac{dV_0'}{dY_0}, \frac{dV_0''}{dY_0}$, &c. are simply the values of the derivatives of $V', V'' \dots$ found by differentiating (63) with reference to each of the variables, and afterwards substituting X_0, Y_0, &c. for $X, Y, \dots$ &c.

If now we denote the derivatives of $V', V'' \dots$ with reference to X by $a', a'' \dots$; their derivatives with reference to Y by b', $b'' \dots$ &c.: so that

$$N' = V_0' + a'x + b'y + c'z + \dots$$
$$N'' = V_0'' + a''x + b''y + c''z + \dots$$
$$\&c. \qquad \&c.$$

and then also put

$$v' = N' - M', \qquad v'' = N'' - M'', \ \&c.$$
$$n' = V_0' - M', \qquad n'' = V_0'' - M'', \ \&c.$$

our equations become

$$a'x + b'y + c'z + \dots + n' = v'$$
$$a''x + b''y + c''z + \dots + n'' = v''$$
$$a'''x + b'''y + c'''z + \dots + n''' = v'''$$
$$\&c. \qquad \&c.$$

in which $a', b' \dots a'', b'' \dots n', n'' \dots$ are all known quantities; and $v', v'' \dots$ are the residual errors of observation. These equations of condition are precisely like those already treated, and, being solved by the same method, give the most probable values of $x, y, z \dots$, and hence, also, the most probable values of $X, Y, Z \dots$

This process rests upon the assumption that the approximate values $X_0, Y_0, Z_0 \dots$ are already so nearly correct that the squares of $x, y, z \dots$ may be neglected. But should the values found for $x, y, z \dots$ show that this assumption was not admissible, the computation is to be repeated, starting with the last found values $X_0 + x$, $Y_0 + y$, $Z_0 + z \dots$ as the approximate values; and then the corrections which these last require will generally be so small that their higher powers may be neglected without sensible error. However, should this still not be the case, successive approximations, commencing always with the last found values, will at length lead to values which require only corrections suitably small.

Even when the given function is already linear, it is mostly expedient to follow the general method just given: namely, to substitute approximate values and form equations of condition to determine their corrections. This reduces x, y, z ... to small quantities, greatly simplifies the computations, and diminishes the chance of error.

TREATMENT OF EQUATIONS OF CONDITION WHEN THE OBSERVATIONS HAVE DIFFERENT WEIGHTS.

41. The process above explained assumes that all the observations are subject to the same mean error, and hence are all of the same weight. The more general case, in which the observations are of different weights, is easily reduced to this simple case. For, let

$$a'x + b'y + c'z + \ldots . + n' = v'$$

be an equation of condition of the weight p'; that is, one formed for an observation of the weight p'. The mean error of an observation of the weight unity being ϵ_1, the mean error of the actual observation, and, therefore, also of n', is $\epsilon' = \frac{\epsilon_1}{\sqrt{p'}}$. Hence the mean error of $n'\sqrt{p'}$ is, by Art. 20, equal to $\epsilon'\sqrt{p'}$, that is, equal to ϵ_1. If, therefore, we multiply the equation by $\sqrt{p'}$, so that we have

$$a'\sqrt{p'}.x + b'\sqrt{p'}.y + c'\sqrt{p'}.z + \ldots + n'\sqrt{p'} = v'\sqrt{p'}$$

it becomes an equation in which the mean error of the absolute term is the mean error of an observation of the weight unity. Hence we have only to multiply each equation of condition by the square root of its weight in order to reduce them all to the same unit of weight; after which the normal equations will be found as in other cases.

The mean error of observation, found by (61) from the equations of condition thus transformed, will be that of an observation of the weight unity, and the weights of the unknown quantities will come out with reference to the same unit.

ELIMINATION OF THE UNKNOWN QUANTITIES FROM THE NORMAL EQUATIONS BY THE METHOD OF SUBSTITUTION, ACCORDING TO GAUSS.

42. By means of a peculiar notation proposed by GAUSS, the elimination by substitution is carried on so as to preserve throughout the symmetry which exists in the normal equations. In order to explain this method, it will be expedient to suppose a limited number of unknown quantities. I shall take but *four*, but shall give the process in so general a form that it may readily be extended to any number.

The unknown quantities will be denoted by

$$x, y, z, w,$$

and their coefficients in the equations of condition by

$$a, b, c, d,$$

respectively, with sub-numerals denoting the number of the equation or observation upon which it depends, and by

$$n_1, n_2, n_3, \&c.$$

the absolute terms of the 1st, 2d, 3d, &c. equations respectively: so that the m equations of condition (here supposed to be reduced to the same weight by Art. 41) will be

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1w + n_1 = 0 \\ a_2x + b_2y + c_2z + d_2w + n_2 = 0 \\ a_3x + b_3y + c_3z + d_3w + n_3 = 0 \\ \quad \cdots \qquad \cdots \qquad \cdots \\ \quad \cdots \qquad \cdots \qquad \cdots \\ a_mx + b_my + c_mz + d_mw + n_m = 0 \end{array}\right\} \qquad (65)$$

and the four normal equations formed from these are

$$\left.\begin{array}{l} [aa]\,x + [ab]\,y + [ac]\,z + [ad]\,w + [an] = 0 \\ [ab]\,x + [bb]\,y + [bc]\,z + [bd]\,w + [bn] = 0 \\ [ac]\,x + [bc]\,y + [cc]\,z + [cd]\,w + [cn] = 0 \\ [ad]\,x + [bd]\,y + [cd]\,z + [dd]\,w + [dn] = 0 \end{array}\right\} \qquad (66)$$

e value of x from the first equation is

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[ad]}{[aa]}w - \frac{[an]}{[aa]}$$

this is substituted in the other three equations, we shall pre- erve the symmetry of the result by the following notation:

$$\begin{array}{l|l}
[bb] - \frac{[ab]}{[aa]}[ab] = [bb.1] & [dd] - \frac{[ad]}{[aa]}[ad] = [dd.1] \\
[bc] - \frac{[ab]}{[aa]}[ac] = [bc.1] & [bn] - \frac{[ab]}{[aa]}[an] = [bn.1] \\
[bd] - \frac{[ab]}{[aa]}[ad] = [bd.1] & [cn] - \frac{[ac]}{[aa]}[an] = [cn.1] \\
[cc] - \frac{[ac]}{[aa]}[ac] = [cc.1] & [dn] - \frac{[ad]}{[aa]}[an] = [dn.1] \\
[cd] - \frac{[ac]}{[aa]}[ad] = [cd.1] &
\end{array}$$

The three equations thus become

$$\left.\begin{array}{l}
[bb.1]y + [bc.1]z + [bd.1]w + [bn.1] = 0 \\
[bc.1]y + [cc.1]z + [cd.1]w + [cn.1] = 0 \\
[bd.1]y + [cd.1]z + [dd.1]w + [dn.1] = 0
\end{array}\right\} \qquad (67)$$

he presence of the numeral 1 is all that distinguishes these om original normal equations in y, z, and w. The elimination y will, therefore, be effected in the same manner as that of x. us, from the first, we have

$$y = -\frac{[bc.1]}{[bb.1]}z - \frac{[bd.1]}{[bb.1]}w - \frac{[bn.1]}{[bb.1]}$$

substitution of which in the other two equations leads to the lowing notation:

$$\begin{array}{l|l}
.1] - \frac{[bc.1]}{[bb.1]}[bc.1] = [cc.2] & [cn.1] - \frac{[bc.1]}{[bb.1]}[bn.1] = [cn.2] \\
.1] - \frac{[bc.1]}{[bb.1]}[bd.1] = [cd.2] & [dn.1] - \frac{[bd.1]}{[bb.1]}[bn.1] = [dn.2] \\
.1] - \frac{[bd.1]}{[bb.1]}[bd.1] = [dd.2] &
\end{array}$$

and the resulting equations are

$$\left.\begin{array}{l}[cc\,.\,2]\,z + [cd\,.\,2]\,w + [cn\,.\,2] = 0\\ [cd\,.\,2]\,z + [dd\,.\,2]\,w + [dn\,.\,2] = 0\end{array}\right\}\qquad(68)$$

From the first of these we have

$$z = -\frac{[cd\,.\,2]}{[cc\,.\,2]}\,w - \frac{[cn\,.\,2]}{[cc\,.\,2]}$$

which, substituted in the second, leads to the following notation:

$$[dd\,.\,2] - \frac{[cd\,.\,2]}{[cc\,.\,2]}[cd\,.\,2] = [dd\,.\,3] \quad\Bigg|\quad [dn\,.\,2] - \frac{[cd\,.\,2]}{[cc\,.\,2]}[cn\,.\,2] = [dn\,.\,3]$$

and the resulting equation is

$$[dd\,.\,3]\,w + [dn\,.\,3] = 0 \qquad (69)$$

whence

$$w = -\frac{[dn\,.\,3]}{[dd\,.\,3]}$$

Having thus found w, we substitute its value in the first of (68), and deduce z. Then the values of z and w being substituted in the first of (67), we deduce y; and finally, substituting the values y, z, and w in the first of (66), we deduce x. These latter substitutions are made in the numerical computation, but it is not necessary to write out here the formulæ which result from the literal substitutions, as it would not facilitate the computation.

It may be observed that all the auxiliaries $[bb\,.\,1]$, $[bc\,.\,1]$, $[cc\,.\,2]$, &c., may be expressed by the general formula

$$[\beta\gamma\,.\,\mu] - \frac{[\alpha\beta\,.\,\mu]}{[\alpha\alpha\,.\,\mu]}[\alpha\gamma\,.\,\mu] = [\beta\gamma\,.\,(\mu + 1)]$$

α, β, γ denoting any three letters, and μ any numeral.

For the convenience of reference, the final equations employed in the actual computation are brought together as follows, the coefficient of that unknown quantity which is found from each after the substitution of the values of the others being reduced to unity:

$$\left.\begin{aligned} x + \frac{[ab]}{[aa]}y + \frac{[ac]}{[aa]}z + \frac{[ad]}{[aa]}w + \frac{[an]}{[aa]} &= 0 \\ y + \frac{[bc.1]}{[bb.1]}z + \frac{[bd.1]}{[bb.1]}w + \frac{[bn.1]}{[bb.1]} &= 0 \\ z + \frac{[cd.2]}{[cc.2]}w + \frac{[cn.2]}{[cc.2]} &= 0 \\ w + \frac{[dn.3]}{[dd.3]} &= 0 \end{aligned}\right\} \quad (70)$$

As the number of unknown quantities increases, the number of auxiliaries to be found increases very rapidly. If we include the coefficients and absolute terms of the normal equations, the whole number of auxiliaries is shown in the following scheme:*

No. of unknown quantities	1	2	3	4	5	6	7	8
No. of auxiliaries	2	7	16	30	50	77	112	156

43. For the purpose of verification, it is expedient to repeat the elimination in inverse order, commencing with the last normal equation and ending with the first, which will bring out x. It will not be necessary to write out the formulæ for this inverse elimination, since when the form for computation has been once prepared, it suffices to place in it the coefficients of the normal equations in inverse order, and then to proceed with the numerical operations precisely as in the first elimination. The unknown quantities coming out in the first elimination in the order w, z, y, x, they will in the second come out in the order x, y, z, w.

This inversion has also the advantage of giving the weights of all the unknown quantities with the greatest facility, as will hereafter be shown.

44. A very complete final verification, or "control," is obtained as follows. Substitute the values of x, y, z, w in the equations of condition, and thus find the residuals v_1, v_2, $v_3 \ldots . v_m$, or the values which the first members assume. Form the sum

$$[vv] = v_1v_1 + v_2v_2 + v_3v_3 + \ldots . + v_mv_m$$

---

* The number of auxiliaries will be, in general,

$$\frac{i(i+1)(i+5)}{2.3}$$

where i denotes the number of unknown quantities.

which is also required in finding the mean error of observation by (61). Also form the following new auxiliaries:

$$[nn] = n_1n_1 + n_2n_2 + n_3n_3 + \cdots\cdots + n_mn_m$$

$$[nn] - \frac{[an]^2}{[aa]} = [nn.1] \qquad [nn.2] - \frac{[cn.2]^2}{[cc.2]} = [nn.3]$$

$$[nn.1] - \frac{[bn.1]^2}{[bb.1]} = [nn.2] \qquad [nn.3] - \frac{[dn.3]^2}{[dd.3]} = [nn.4]$$

then, if the whole computation, both of the normal equations themselves and of the subsequent elimination, is correct, we must have

$$[vv] = [nn.4] \tag{71}$$

To demonstrate this, we observe first that we have already, by (59),

$$[vv] = [vn]$$

If now we go back to the equations of condition, and multiply each by its n, the sum of the products is

$$[an]\,x + [bn]\,y + [cn]\,z + [dn]\,w + [nn] = [vn] = [vv]$$

If this equation be annexed as a fifth normal equation to the group (66), and the successive substitutions are made in it as in the others, beginning with x, it evidently becomes, successively,

$$\begin{aligned}
[bn.1]\,y + [cn.1]\,z + [dn.1]\,w + [nn.1] &= [vv] \\
[cn.2]\,z + [dn.2]\,w + [nn.2] &= [vv] \\
[dn.3]\,w + [nn.3] &= [vv] \\
[nn.4] &= [vv]
\end{aligned}$$

which last is the same as (71).

DETERMINATION OF THE WEIGHTS OF THE UNKNOWN QUANTITIES WHEN THE ELIMINATION HAS BEEN EFFECTED BY THE METHOD OF SUBSTITUTION.

45. By the general method explained in Art. 36, the elimination would have to be performed as many times as there are unknown quantities. It is desirable to have more direct methods. When there are but four unknown quantities, we can find their weights from the auxiliaries occurring in two successive eliminations in inverse order. In the first elimination, according to the order a, b, c, d, we find w by substitution in the last normal

equation, and, the coefficient of w being then $[dd.3]$, it follows, by Art. 36, that the weight of the value of w is

$$p_w = [dd.3]$$

In the inverse elimination, in the order d, c, b, a, the coefficient of x in the final equation, which would be denoted by $[aa.3]$, will be the weight of x, or

$$p_x = [aa.3]$$

Now, if a third elimination were carried out in the order x, y, w, z, or a, b, d, c (the third normal equation now taking the last place), we should have the same auxiliaries as in the first elimination, so far as those denoted by the numerals 1 and 2; and the equations (68) would still be the same, but in the following order:

$$[dd.2]\,w + [cd.2]\,z + [dn.2] = 0$$
$$[cd.2]\,w + [cc.2]\,z + [cn.2] = 0$$

The value of w given by the first of these is

$$w = -\frac{[cd.2]}{[dd.2]}z - \frac{[dn.2]}{[dd.2]}$$

which, substituted in the second, gives for the coefficient of z,

$$[cc.3] = [cc.2] - \frac{[cd.2]}{[dd.2]}[cd.2] = [dd.3] \times \frac{[cc.2]}{[dd.2]}$$

Therefore we have

$$p_z = [cc.2]\frac{[dd.3]}{[dd.2]}$$

In the fourth supposed elimination, in the order d, c, a, b, the auxiliaries denoted by 1 and 2 would be the same as in our actually performed second elimination; but in the final equation in y we should have for the coefficient of y the quantity

$$[bb.3] = [bb.2] - \frac{[ab.2]}{[aa.2]}[ab.2] = [aa.3] \times \frac{[bb.2]}{[aa.2]}$$

and, therefore,

$$p_y = [bb.2]\frac{[aa.3]}{[aa.2]}$$

Thus, when the elimination has been once inverted, we have found the weights of two of the unknown quantities directly, and the weights of the other two in terms of the auxiliaries previously used, and in a form adapted for logarithmic computation.

46. In order to give the above method greater generality, so that the reader may be enabled to extend it to a greater number of unknown quantities, we remark that the product of the form

$$P = [aa]\,[bb\,.\,1]\,[cc\,.\,2]\,[dd\,.\,3]\,\ldots\ldots$$

has the same value whatever order may be followed in the elimination. This is the same as saying that it is a symmetrical function of a, b, c, $d\ldots$ which is, consequently, not affected in value by the permutation of these letters.* Suppose, then, four orders of elimination, in which each unknown quantity in turn becomes the last, while the order of the remaining three quantities remains the same; and, to distinguish the auxiliaries which occur in each elimination, let the letter which occurs in the last auxiliary be annexed to each of the others; the above constant product may thus be expressed in the following four forms:

$$\begin{aligned} P &= [aa]_d\,[bb\,.\,1]_d\,[cc\,.\,2]_d\,[dd\,.\,3] \\ &= [aa]_c\,[bb\,.\,1]_c\,[dd\,.\,2]_c\,[cc\,.\,3] \\ &= [aa]_b\,[cc\,.\,1]_b\,[dd\,.\,2]_b\,[bb\,.\,3] \\ &= [bb]_a\,[cc\,.\,1]_a\,[dd\,.\,2]_a\,[aa\,.\,3] \end{aligned}$$

Now, it is evident that each time a new unknown quantity is made the last, we do not change *all* the auxiliaries, but only those which involve the letter which has become the last in the new order. It is readily seen, therefore, that if we annex a letter to those auxiliaries only which have a different value from that which is denoted by the same symbol in the first elimination, we shall have, simply,

$$\begin{aligned} P &= [aa]\,[bb\,.\,1]\,\;[cc\,.\,2]\,\;[dd\,.\,3] \\ &= [aa]\,[bb\,.\,1]\,\;[dd\,.\,2]\,\;[cc\,.\,3] \\ &= [aa]\,[cc\,.\,1]\,\;[dd\,.\,2]_b\,[bb\,.\,3] \\ &= [bb\,]\,[cc\,.\,1]_a\,[dd\,.\,2]_a\,[aa\,.\,3] \end{aligned}$$

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* The quantity P is, in fact, nothing more than the common denominator of the values of x, y, z, w, when these values are reduced to functions of the known quantities and in the form of simple fractions; and this common denominator must evidently have the same value whatever order of elimination is followed.

from which we deduce

$$\left.\begin{aligned} p_w &= [dd.3] \\ p_z &= [cc.3] = [cc.2]\cdot\frac{[dd.3]}{[dd.2]} \\ p_y &= [bb.3] = [bb.1]\cdot\frac{[cc.2]}{[cc.1]}\cdot\frac{[dd.3]}{[dd.2]}, \\ p_x &= [aa.3] = [aa]\cdot\frac{[bb.1]}{[bb]}\cdot\frac{[cc.2]}{[cc.1]}\cdot\frac{[dd.3]}{[dd.2]}. \end{aligned}\right\} \quad (72)$$

If this method is applied in the case of six unknown quantities, we shall in each of two eliminations have the weights of three of the unknown quantities by computing each time but one new auxiliary, and, therefore, the weights of all six when the second elimination is the inverse of the first. In the case of but four unknown quantities, by inverting the elimination we can find the weights of z and y twice, and thus verify our work.

47. If we have but three unknown quantities, the weights are determined at the same time with x, y, and z themselves, by a single elimination in the order a, b, c, in which z comes out first with the weight

$$p_z = [cc.2]$$

and then y and z, with the weights

$$p_y = [bb.2] = [bb.1]\cdot\frac{[cc.2]}{[cc.1]}$$

$$p_x = [aa.2] = [aa]\cdot\frac{[bb.1]}{[bb]}\cdot\frac{[cc.2]}{[cc.1]},$$

in which

$$[cc.1], = [cc] - \frac{[bc]}{[bb]}[bc]$$

INDEPENDENT DETERMINATION OF EACH UNKNOWN QUANTITY AND ITS WEIGHT, ACCORDING TO GAUSS.

48. Let the four equations (70) be multiplied respectively by 1, A', A'', A''', and let these factors be determined by the condition that in the sum of the products the coefficients of y, z, and w shall be zero. Also, let the last three equations of (70) be multiplied respectively by 1, B'', B''', and let these factors

be determined by the condition that in the sum of the products the coefficients of z and w shall be zero. Finally, let the last two equations of (70) be multiplied respectively by 1, C''', and let C''' be determined by the condition that in the sum of the products the coefficient of w shall be zero. The conditions which determine these factors are then

$$\left.\begin{aligned}
0 &= \frac{[ab]}{[aa]} + A' \\
0 &= \frac{[ac]}{[aa]} + \frac{[bc.1]}{[bb.1]} A' + A'' \\
0 &= \frac{[ad]}{[aa]} + \frac{[bd.1]}{[bb.1]} A' + \frac{[cd.2]}{[cc.2]} A'' + A''' \\
0 &= \frac{[bc.1]}{[bb.1]} + B'' \\
0 &= \frac{[bd.1]}{[bb.1]} + \frac{[cd.2]}{[cc.2]} B'' + B''' \\
0 &= \frac{[cd.2]}{[cc.2]} + C'''
\end{aligned}\right\} \quad (73)$$

and the final values of x, y, z, w, in terms of these factors, are given as follows:

$$\left.\begin{aligned}
-x &= \frac{[an]}{[aa]} + \frac{[bn.1]}{[bb.1]} A' + \frac{[cn.2]}{[cc.2]} A'' + \frac{[dn.3]}{[dd.3]} A''' \\
-y &= \frac{[bn.1]}{[bb.1]} + \frac{[cn.2]}{[cc.2]} B'' + \frac{[dn.3]}{[dd.3]} B''' \\
-z &= \frac{[cn.2]}{[cc.2]} + \frac{[dn.3]}{[dd.3]} C''' \\
-w &= \frac{[dn.3]}{[dd.3]}
\end{aligned}\right\} \quad (74)$$

49. As the equations (73) are above arranged, all the factors A are determined from the first system of three equations; the factors B from the second system of two equations, &c.; in each case, by successive substitution. This method then enables us to find each unknown quantity independently of the others.

Another form may be given to the computation of the auxiliary factors. Since in the formation of the equations (74) we have regarded [*an*], [*bn*], [*cn*], &c. as independent, we must still so d them when we invert the process and recompose the tions (70) from (74). If, then, we multiply the equations (74) ectively by 1, $\frac{[ab]}{[aa]}$, $\frac{[ac]}{[aa]}$, $\frac{[ad]}{[aa]}$, and add the products in order compose the first of (70), the coefficient of $[an]$ will be $\frac{1}{[aa]}$; he coefficients of $[bn.1]$, $[cn.2]$, &c. must severally be equal ro. The same principle will apply when we recompose the nd equation of (70) from the last three of (74), &c. Hence ave

$$\left.\begin{aligned}
0 &= A' + \frac{[ab]}{[aa]} \\
0 &= A'' + \frac{[ab]}{[aa]}B'' + \frac{[ac]}{[aa]} \\
0 &= A''' + \frac{[ab]}{[aa]}B''' + \frac{[ac]}{[aa]}C''' + \frac{[ad]}{[aa]} \\
0 &= B'' + \frac{[bc.1]}{[bb.1]} \\
0 &= B''' + \frac{[bc.1]}{[bb.1]}C''' + \frac{[bd.1]}{[bb.1]} \\
0 &= C''' + \frac{[cd.2]}{[cc.2]}
\end{aligned}\right\} \quad (75)$$

cording to this scheme, we first find A', B'', C''' from the ations in which they occur singly; then, with these factors, find the values of A'', B''', from the equations involving two ors, &c.

0. Again, let us write the 3d, 5th, and 6th equations of (75) he following order:

$$\begin{aligned}
A''' + \frac{[ab]}{[aa]}B''' + \frac{[ac]}{[aa]}C''' + \frac{[ad]}{[aa]} &= 0 \\
B''' + \frac{[bc.1]}{[bb.1]}C''' + \frac{[bd.1]}{[bb.1]} &= 0 \\
C''' + \frac{[cd.2]}{[cc.2]} &= 0
\end{aligned}$$

paring these with the first three of (70), we at once infer A''', B''', C''' are those values of x, y, z, respectively, which ould obtain from our first three normal equations by putting

$w = 1$ and omitting the terms in n; or, going back to (66), that A''', B''', C''' may be determined by the following conditions:

$$\begin{aligned}
[aa]\,A''' + [ab]\,B''' + [ac]\,C''' + [ad] &= 0 \\
[ab]\,A''' + [bb]\,B''' + [bc]\,C''' + [bd] &= 0 \\
[ac]\,A''' + [bc]\,B''' + [cc]\,C''' + [cd] &= 0
\end{aligned}$$

If now we multiply the normal equations (66) by A''', B''', C''', and 1, respectively, and add the products, the conditions just given will cause x, y, and z to disappear, and the resulting equation in w must be identical* with (69): so that A''', B''', C''' must also satisfy the following condition:

$$[an]\,A''' + [bn]\,B''' + [cn]\,C''' + [dn] = [dn\,.\,3] \tag{76}$$

The second and fourth equations of (75) being written as follows,

$$\begin{aligned}
A'' + \frac{[ab]}{[aa]}\,B'' + \frac{[ac]}{[aa]} &= 0 \\
B'' + \frac{[bc\,.\,1]}{[bb\,.\,1]} &= 0
\end{aligned}$$

and compared with the first two of (70), we infer that A'', B'' are those values of x and y which we obtain from the first two normal equations by putting $z = 1$, $w = 0$, and omitting the terms in n; that is, A'' and B'' must satisfy the conditions

$$\begin{aligned}
[aa]\,A'' + [ab]\,B'' + [ac] &= 0 \\
[ab]\,A'' + [bb]\,B'' + [bc] &= 0
\end{aligned}$$

Therefore, if we multiply the first three normal equations (66) by A'', B'', 1, respectively, and add the products, x and y will disappear, and, the resulting equation being identical with the first of (68), we must also have

$$[an]\,A'' + [bn]\,B'' + [cn] = [cn\,.\,2] \tag{77}$$

Lastly, it is evident that A' must also satisfy the condition

$$[an]\,A' + [bn] = [bn\,.\,1] \tag{78}$$

From these relations we readily infer general formulæ for the weights of the unknown quantities.

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* The equation (69) is the last normal equation, unchanged except by the substitution of *equivalents* for x, y, and z; and in the present article we eliminate x, y, and z by the use of factors, but do not change the last normal equation, since we multiply it by unity.

According to Art. 34, the reciprocal of the weight of x is that value which we obtain for x if we put $[an] = -1$ and $[bn] = [cn] = [dn] = 0$. But, under these conditions, the equations (76), (77), (78) give

$$[dn.3] = -A''', \qquad [cn.2] = -A'', \qquad [bn.1] = -A'$$

In order, therefore, that the value of x given by the first equation of (74) may become $\frac{1}{p_x}$, we have only to substitute $-A'''$, $-A''$, $-A'$, -1, respectively, for $[dn.3]$, $[cn.2]$, $[bn.1]$, $[an]$.

In the same manner, the weight of y being found by putting $[bn] = -1$ and $[an] = [cn] = [dn] = 0$, we have to put

$$[dn.3] = -B''', \qquad [cn.2] = -B'', \qquad [bn.1] = -1$$

in the second equation of (74), in order that we may put $\frac{1}{p_y}$ for y.

For the weight of z we have to put

$$[dn.3] = -C''', \qquad [cn.2] = -1$$

in the third equation of (74), and $\frac{1}{p_z}$ for z.

For the weight of w, we have to put

$$[dn.3] = -1$$

in the last equation of (74), and change w to $\frac{1}{p_w}$.

The final formulæ for the weights are, therefore,

$$\left.\begin{aligned}
\frac{1}{p_x} &= \frac{1}{[aa]} + \frac{A'A'}{[bb.1]} + \frac{A''A''}{[cc.2]} + \frac{A'''A'''}{[dd.3]} \\
\frac{1}{p_y} &= \frac{1}{[bb.1]} + \frac{B''B''}{[cc.2]} + \frac{B'''B'''}{[dd.3]} \\
\frac{1}{p_z} &= \frac{1}{[cc.2]} + \frac{C'''C'''}{[dd.3]} \\
\frac{1}{p_w} &= \frac{1}{[dd.3]}
\end{aligned}\right\} \qquad (79)$$

MEAN ERROR OF A LINEAR FUNCTION OF THE QUANTITIES x, y, z, w.

50. *To find the mean error of the function*

$$X = fx + gy + hz + iw + l \qquad (80)$$

when x, y, z, w are dependent upon the same observations.

The quantities x, y, z, w not being directly observed, their mean errors cannot be treated as independent, as was done in the case of directly observed quantities in Art. 22. We might proceed by the method of Art. 23; but, as we here suppose x, y, z, w to have been determined from the normal equations (66), we can obtain a more convenient method by the aid of the auxiliaries which have been introduced in the general elimination. The quantities x, y, z, w being functions of the directly observed quantities n', n'', n''', ... the mean error of X can be readily obtained by the principles of Art. 22, if we first reduce X to a function of these observed quantities. For this purpose, if the values of x, y, z, w deduced from (70) be substituted in X, we shall have an expression of the form

$$X = k_0\,[an] + k_1\,[bn.1] + k_2\,[cn.2] + k_3\,[dn.3] + l \qquad (81)$$

in which the coefficients k_0, k_1, k_2, k_3 are functions of $[aa]$, $[ab]$, &c. In order to determine these coefficients, let us substitute in this expression the values of $[an]$, $[bn.1]$, &c. given by (70). We find

$$\begin{aligned} X = -[aa]\,k_0x - [ab]\,k_0y - {} & [ac]\,k_0z - [ad]\,k_0w + l \\ - [bb.1]\,k_1y - {} & [bc.1]\,k_1z - [bd.1]\,k_1w \\ & - [cc.2]\,k_2z - [cd.2]\,k_2w \\ & \qquad\qquad - [dd.3]\,k_3w \end{aligned}$$

which becomes identical with (80) by assuming

$$\left.\begin{aligned} &[aa]\,k_0 = -f \\ &[ab]\,k_0 + [bb.1]\,k_1 = -g \\ &[ac]\,k_0 + [bc.1]\,k_1 + [cc.2]\,k_2 = -h \\ &[ad]\,k_0 + [bd.1]\,k_1 + [cd.2]\,k_2 + [dd.3]\,k_3 = -i \end{aligned}\right\} \quad (82).$$

These equations fully determine the coefficients. We find k_0 directly from the first, and then k_1, k_2, k_3, by successive substitutions in the others.

Now, to find the mean error of X under the form (81), let the mean error of each of the observed quantities n', n'', n''' .... be denoted by ε (these observed quantities being supposed of equal weight, or, rather, the equations of condition being supposed to have been reduced to the same weight), and let the corresponding mean errors of

$$[an], \qquad [bn.1], \qquad [cn.2], \qquad [dn.3], \qquad X,$$

be denoted by

$$E_0, \quad E_1, \quad E_2, \quad E_3, \quad (\varepsilon X).$$

Since we have

$$[an] = a'n' + a''n'' + a'''n''' + \ldots.$$

we have, by Art. 22,

$$E_0^2 = [aa]\,\varepsilon^2$$

Again, we have

$$[bn.1] = [bn] - \frac{[ab]}{[aa]}[an] = \sum\left[\left(b - \frac{[ab]}{[aa]}a\right)n\right]$$

and hence

$$\begin{aligned}
E_1^2 &= \varepsilon^2 \sum\left(b - \frac{[ab]}{[aa]}a\right)^2 \\
&= \varepsilon^2\left([bb] - \frac{2\,[ab]}{[aa]}[ab] + \frac{[ab]^2}{[aa]^2}[aa]\right) \\
&= \varepsilon^2\left([bb] - \frac{[ab]}{[aa]}[ab]\right) \\
&= [bb.1]\,\varepsilon^2
\end{aligned}$$

In a similar manner, we have, also,

$$E_2^2 = [cc.2]\,\varepsilon^2, \qquad E_3^2 = [dd.3]\,\varepsilon^2$$

The quantities x, y, z, w, being determined from the equations (70), their mean errors involve those of the quantities $[an]$, $[bn.1]$, $[cn.2]$, $[dn.3]$, precisely as if the latter had been independently observed quantities affected by the mean errors just determined. Hence also in (81) we regard $[an]$, $[bn.1]$, &c. as independent; and it then follows directly from the principles of Art. 22 that

$$(\varepsilon X)^2 = k_0^2 E_0^2 + k_1^2 E_1^2 + k_2^2 E_2^2 + k_3^2 E_3^2$$

or

$$(\varepsilon X)^2 = \left(k_0^2\,[aa] + k_1^2\,[bb.1] + k_2^2\,[cc.2] + k_3^2\,[dd.3]\right)\varepsilon^2 \quad (83)$$

51. From the preceding article we may easily find the formulæ (74) and (79). The function X becomes x when we assume $f = 1$, $g = h = i = l = 0$; and then (81) gives x while (83) gives ε_x^2, and hence the weight $= \frac{\varepsilon^2}{\varepsilon_x^2}$. This hypothesis gives in (82) $[aa]\,k_0 = -1$; and the remaining equations of (82) are identical with the first three of (73) if we put $[bb.1]\,k_1 = -A'$, $[cc.2]\,k_2 = -A''$, $[dd.3]\,k_3 = -A'''$; and then (81) becomes identical with the first of (74), and (83) with the first of (79). In a similar manner we may deduce the remaining equations of (74) and (79).

EXAMPLE.—In order to exhibit the numerical operations which the preceding method requires, in their proper order and within the limits of the page, I select an example involving but three unknown quantities. The following equations of condition were proposed by GAUSS (*Theoria Motus Corp. Coel.*, Art. 184) to illustrate his method:

$$\begin{array}{lr} (1) & x - y + 2z = 3 \\ (2) & 3x + 2y - 5z = 5 \\ (3) & 4x + y + 4z = 21 \\ (4) & -2x + 6y + 6z = 28 \end{array}$$

of which the first three are supposed to have the weight unity, while the last has the weight $\frac{1}{4}$. Multiplying the last by $\sqrt{\frac{1}{4}} = \frac{1}{2}$ (Art. 41), the equations of condition, reduced to the same weight, are—

$$\begin{array}{lr} (1) & x - y + 2z - 3 = 0 \\ (2) & 3x + 2y - 5z - 5 = 0 \\ (3) & 4x + y + 4z - 21 = 0 \\ (4) & -x + 3y + 3z - 14 = 0 \end{array}$$

The next step is to form the coefficients $[aa]$, $[ab]$, &c., of the normal equations. In the present example this can be done very easily without the aid of logarithms; but, in order to exhibit the work usually required in practice, I shall give the forms for logarithmic computation. The sums of the coefficients of the unknown quantities will be employed as checks, according to Art. 30. Their logarithms, together with those of a, b, c, n, are given in the following table:

	$\log a$	$\log b$	$\log c$	$\log s$	$\log n$
(1)	0.00000	n0.00000	0.30103	0.30103	n0.47712
(2)	0.47712	0.30103	n0.69897	$-\infty$	n0.69897
(3)	0.60206	0.00000	0.60206	0.95424	n1.32222
(4)	n0.00000	0.47712	0.47712	0.69897	n1.14613

It is important, where many operations are to be performed, to write down no more figures than are necessary for the clear prosecution of the work. Hence, in combining the preceding logarithms it will be found expedient to proceed as follows. Write each log a upon the lower edge of a slip of paper; then, placing this slip so that log a shall stand over log a, log b, log c, &c., of the same horizontal line, in succession, add together the two logarithms *mentally*, and, with the sum *in the head*, take from the logarithmic table the corresponding natural number (*aa*, *ab*, *ac*, *as*, or *an*), which place in a column appropriated for the purpose. Then write log *b* in the same manner, and form *bb*, *bc*, *bs*, *bn*, and so proceed to form all the coefficients of the normal equations, as in the following table:

	[*aa*]	[*ab*]		[*ac*]		[*as*]		[*an*]		[*bb*]	[*bc*]	
	+	+	−	+	−	+	−	+	−	+	+	−
(1)	1.0		1.0	2.0		2.0			3.0	1.0		2.0
(2)	9.0	6.0			15.0		0.0		15.0	4.0		10.0
(3)	16.0	4.0		16.0		36.0			84.0	1.0	4.0	
(4)	1.0		3.0		3.0		5.0	14.0		9.0	9.0	
		10.0	4.0	18.0	18.0	38.0	5·0	14.0	102.0		18.0	12.0
	+ 27.0	+ 6.0		0.0		+ 33.0		− 88.0		+ 15.0	+ 1.0	

	[*bs*]		[*bn*]		[*cc*]	[*cs*]		[*cn*]		[*sn*]		[*nn*]
	+	−	+	−	+	+	−	+	−	+	−	+
(1)		2.0	3.0		4.0	4.0			6.0		6.0	9.0
(2)	0.0			10.0	25.0		0.0	25.0		0.0		25.0
(3)	9.0			21.0	16.0	36.0			84.0		189.0	441.0
(4)	15.0			42.0	9.0	15.0			42.0		70.0	196.0
	24.0	2.0	3.0	73.0		55.0	0.0	25.0	182.0	0.0	265.0	
	+ 22.0		− 70.0		+ 54.0	+ 55.0		− 107.0		− 265.0		+ 671.0

Having ascertained that the results satisfy the test equations (48), we can write out the normal equations as follows:

$$\begin{aligned} 27x + 6y \qquad\quad - 88 &= 0 \\ 6x + 15y + \quad z - 70 &= 0 \\ y + 54z - 107 &= 0 \end{aligned}$$

We proceed to determine the values of x, y, z, according to our general formulæ, still carrying out the work with logarithms for the sake of illustration. Here, again, system and conciseness are indispensable. The whole computation is given below nearly in the form proposed by ENCKE. This form corresponds to the group of equations (70). It is divided into three principal compartments, corresponding, respectively, to the first three equations of (70), each beginning one column farther to the right. In the first compartment the first line of numbers contains the values of [*aa*], [*ab*], &c., the second line their logarithms, and the third line the logarithms of the coefficients of the first equation. The logarithms in this third line are formed by subtracting the first log. in the second line from each of the subsequent ones, for this purpose writing the first logarithm upon the lower edge of a slip of paper.

In the second compartment, the first line contains the values of $[bb]$, $[bc]$, &c.; the second line, the quantities subtractive from these, according to the formulæ in Art. 42. To form these subtractive quantities, write the logarithm of $\frac{[ab]}{[aa]}$ (which is here 9.34679) upon the lower edge of a slip of paper, and hold it successively over log $[ab]$ and each of the subsequent logarithms in the same line; add the two logarithms mentally in each case, take the corresponding natural number from the logarithmic table, and write it in its place below. Subtracting these numbers, we have the values of $[bb.1]$, $[bc.1]$, &c. The fourth line contains the logarithms of these quantities; the fifth, the logarithms of the coefficients of our second equation, formed by subtracting the first logarithm of the preceding line from each of the subsequent ones in that line.

In the third compartment we have—first, the values of $[cc]$, &c.; secondly, the values of the subtractive quantities formed from the last line of the first compartment as before; thirdly, the remainders which are the values of $[cc.1]$, &c. The fourth line contains the values of the quantities which are subtractive from the preceding and are formed from the last line of the second compartment by adding the first logarithm of that line to the logarithm immediately above it and to each of the subsequent logarithms in the same line; the fifth line contains the remainders which are the values of $[cc.2]$, &c.: the sixth line, the logarithms of these; and the last line, the logarithms of the coefficients of our third equation.

For control, we carry through the operations upon $[as]$, $[bs]$, &c., precisely as upon the other quantities; and then, according to the arrangement of the scheme, we should have, if we have computed correctly, each sum containing s equal to the sum of the quantities on its left in the same line, together with those of the same order in a vertical column over the first number in this line. Thus, we must have, in the present case,

$$\begin{array}{ll} [bs.1] = [bb.1] + [bc.1] & \qquad [sn.1] = [bn.1] + [cn.1] \\ [cs.1] = [cc.1] + [bc.1] & \qquad [sn.2] = [cn.2] \\ [cs.2] = [cc.2] & \end{array}$$

relations easily proved by means of the formulæ of Art. 42 combined with (48).

The columns $[sn]$ and $[nn]$ are added to the third compartment in order to form the quantity $[nn.3]$, from which the mean error of observation is to be deduced, as will be shown hereafter.

[aa]	[ab]	[ac]	[as]	[an]		
+ 27.000	+ 6.000	0.000	+ 33.000	− 88.000		
1.43136	0.77815	− ∞	1.51851	n1.94448		
	9.34679	− ∞	0.08715	n0.51312		
	[bb]	[bc]	[bs]	[bn]		
− 88.000						
0.000	+ 15.000	+ 1.000	+ 22.000	− 70.000		
+ 21.305	+ 1.333	0.000	+ 7.333	− 19.556		
− 66.695	+ 13.667	+ 1.000	+ 14.667	− 50.444		
n1.82409	1.13566	0.00000	1.16633	n1.70281		
log x = 0.39273		8.86434	0.03067	n0.56715		
		[cc]	[cs]	[cn]	[sn]	[nn]
	− 50.444	+ 54.000	+ 55.000	− 107.000	− 265.000	+ 671.000
	+ 1.916	0.000	0.000	0.000	− 107.555	+ 286.813
	− 48.528	+ 54.000	+ 55.000	− 107.000	− 157.445	+ 384.187
	n1.68599	+ 0.073	+ 1.073	− 3.691	− 54.135	+ 186.191
	log y = 0.55033	+ 53.927	+ 53.927	− 103.309	− 103.310	+ 197.996
		1.73181		n2.01414		+ 197.909
				log (− z) = n0.28233	[nn . 3] =	+ 0.087

After z has been found, its value is substituted in the second equation of (70), and y is deduced. Then, the values of y and z being substituted in the first equation, we find x. The numerical computations are given above in the margin.

Then, for the weights, by Art. 47, we have first to find the additional auxiliary

$$[cc.1]_a = [cc] - \frac{[bc]}{[bb]}[bc]$$

and by the formulæ of that article we have—

[bb]	[bc]	log [bb . 1]	1.13566	log [cc . 2]	1.73181
+ 15.000	+ 1.000	log [bb]	1.17609	log [cc . 1]	1.73239
1.17609	0.00000			log [cc . 1] a	1.73185
	8.82391				
	[cc]		1.43136	1.13566	1.73181
	+ 54.000		9.95957	9.99942	log p_z
	+ 0.067		9.99996	1.13508	
[cc . 1a] =	+ 53.933		1.39089	log p_y	
			log p_x		

The final result is then

$$\begin{array}{lll} x = +\,2.4702 & \text{with the weight} & 24.597 \\ y = +\,3.5508 & \text{"} \qquad \text{"} & 13.648 \\ z = +\,1.9157 & \text{"} \qquad \text{"} & 53.927 \end{array}$$

It only remains to substitute the values of x, y, and z in th original equations of condition, to form the residuals v, and fro these to determine the mean error of observation. Since he there are but three unknown quantities, we have, by (71),

$$[vv] = [nn\,.\,3]$$

and hence the mean error of an observation of the weight uni is, by (61), m being the number of equations of condition,

$$\varepsilon = \sqrt{\left(\frac{[nn\,.\,3]}{m-3}\right)} = 0.295$$

The direct computation of the residuals is, therefore, not necessa for determining ε: nevertheless, it is desirable in most cases t resort to the direct substitution also, not only for a final verific tion, but in order to examine the several observations, and t obtain the data for rejecting any doubtful one by the use o PEIRCE's Criterion, to be given hereafter. This direct substitu tion has already been carried out for this example on p. 52 where we have found $[vv] = 0.0804$, which agrees with the abov value of $[nn.3]$ as nearly as can be expected with the use of fiv decimal logarithms.

52. It not unfrequently happens that one of the unknow quantities is such that the given observations cannot determin it with accuracy. For example, in the reduction of a numb of observations of an eclipse, one of the unknown quantities is correction of the moon's parallax; but, unless the places of ol servation be remote from each other, the correction will be ve uncertain, and this uncertainty will affect all the other quantiti which enter into the equations of condition. In such a case, th unknown quantity will come out with a small coefficient, whic of itself will reveal the existence of the uncertainty when it not otherwise anticipated. In order that this uncertainty m not affect those quantities which are well defined by the obse vations, it is expedient to determine all the latter as functions c the uncertain quantity, which for that purpose must be made tl

last in the elimination. Thus, with four unknown quantities x, y, z, w, we proceed only as far as the auxiliaries denoted by the numeral 2; then, having found the factors A', A'', A''', B'', B''', C''', by (73) or (75), if we put

$$\left.\begin{aligned} -x' &= \frac{[an]}{[aa]} + \frac{[bn.1]}{[bb.1]}A' + \frac{[cn.2]}{[cc.2]}A'' \\ -y' &= \frac{[bn.1]}{[bb.1]} + \frac{[cn.2]}{[cc.2]}B'' \\ -z' &= \frac{[cn.2]}{[cc.2]} \end{aligned}\right\} \quad (84)$$

these will give the values of the unknown quantities which we should obtain from the first three normal equations if the last unknown quantity were disregarded or put = 0. Then, by (74), the final values of x, y, z, as functions of the uncertain quantity w, will be

$$\left.\begin{aligned} x &= x' + A'''w \\ y &= y' + B'''w \\ z &= z' + C'''w \end{aligned}\right\} \quad (85)$$

The values of x', y', z', will thus be well determined, and a subsequent independent determination of w will enable us to find the final values of x, y, z.*

Having found the weights of x', y', z' (which is done as if they were the only quantities under consideration), and their mean errors ε_x', ε_y', ε_z', then, when the quantity w is afterwards found, the mean errors of the final values will be

$$\left.\begin{aligned} \varepsilon^2_x &= \varepsilon_x'^2 + (A'''\varepsilon_w)^2 \\ \varepsilon^2_y &= \varepsilon_y'^2 + (B'''\varepsilon_w)^2 \\ \varepsilon^2_z &= \varepsilon_z'^2 + (C'''\varepsilon_w)^2 \end{aligned}\right\} \quad (86)$$

as we find from the equations (79), or by Art. 20.

CONDITIONED OBSERVATIONS.

53. In all that precedes, we have supposed that the several quantities to be found by observation, either directly or indirectly, were independent of each other. Although they were required to satisfy certain equations of condition as nearly as possible, yet they were so far independent that no contradiction was involved in supposing the values of one or more of them to be varied without

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* For an example in which three unknown quantities are thus determined as functions of *two* uncertain quantities, see Vol. I. p. 540.

varying the others. By such variations we should obtain systems of values *more or less probable*, but all *possible*.

There is a second class of problems, in which, besides the equations of condition which the unknown quantities are to satisfy approximately, there are also equations of condition which they must satisfy exactly: so that of all the systems of values which may be selected as approximately satisfying the first kind of equations, only those can be admitted as possible which satisfy exactly the equations of the second kind. The number of these rigorous equations of condition must be less than the number of unknown quantities; otherwise they would determine these quantities independently of all observations. These rigorous equations, then, may be satisfied by various possible systems of values, and we can therefore express the problem here to be considered as follows: *Of all the possible systems of values which exactly satisfy the rigorous equations of condition, to find the most probable, or that system which best satisfies the approximate equations of condition.*

The following are simple examples of conditioned observations. The sum of the three angles of a plane triangle must be 180°: so that if we observe each angle directly, and the sum of the observed values differs from 180°, these values must be corrected so as to satisfy this condition. The sum of the angles of a spherical triangle must be 180° + spherical excess. The sum of all the angles around a point, or the sum of all the differences of azimuth observed at a station upon a round of objects in the horizon, must be 360°.

The approximate conditions in these cases are expressed by the observations themselves; for the final values adopted must correspond as nearly as possible to the observed values. The corrections to be applied to the observed values are to be regarded as residual errors with their signs changed; and the solution of our problem is involved in the following statement: *Of all the systems of corrections which satisfy the rigorous equations, that system is to be received as the most probable in which the sum of the squares of the residuals in the approximate equations is a minimum.*

54. The general problem as above stated may be reduced to that of unconditioned observations, already considered. For let us suppose there are m' rigorous equations of condition, and m unknown quantities. From these m' equations let the values of m' unknown quantities be obtained in terms of the remaining

$m - m'$ quantities, and let these values be substituted in all the approximate equations of condition; then there will be left in the latter only $m - m'$ quantities, which may be treated as independent, so that, the approximate equations being now solved by the method of least squares, we have the values of the $m - m'$ quantities, with which we then find the values of the first m' quantities. This is a general solution of the problem; but it is not always the simplest in practice. I shall illustrate it by a simple example, before giving a method applicable to more complicated cases.

EXAMPLE.—At Pine Mount, a station of the U. S. Coast Survey, the angles between the surrounding stations 1, 2, 3, 4 were observed as follows:

			weight
1.2	Joscelyne—Deepwater.........	65° 11′ 52″.500	3
2.3	Deepwater—Deakyne..........	66 24 15 .553	3
3.4	Deakyne—Burden..............	87 2 24 .703	3
4.1	Burden—Joscelyne.............	141 21 21 .757	1

There are here four unknown quantities subjected to the single rigorous condition that their sum must be 360°. But, instead of taking the angles themselves as the unknown quantities, we shall assume approximate values of them, and regard the corrections which they require as the unknown quantities.

We assume

1.2	Joscelyne—Deepwater,	65° 11′ 52″.5 + w
2.3	Deepwater—Deakyne,	66 24 15 .5 + x
3.4	Deakyne—Burden,	87 2 24 .7 + y
4.1	Burden—Joscelyne,	141 21 21 .8 + z

the sum of which must satisfy the condition

$$359^\circ\ 59'\ 54''.5 + w + x + y + z = 360^\circ$$

or

$$w + x + y + z - 5''.5 = 0$$

The difference between the assumed value and the observed value in each case gives us a residual; and the approximate equations of condition are, therefore,

$$\begin{aligned} w - 0 \quad &= 0 \\ x - 0.053 &= 0 \\ y - 0.003 &= 0 \\ z + 0.043 &= 0 \end{aligned}$$

We have here but one rigorous condition (or $m' = 1$), and to eliminate this we have only to find from it the value of one unknown quantity in terms of the others, and substitute it in the approximate equations of condition: thus, substituting the value

$$w = -x - y - z + 5''.5$$

our equations of condition, containing now three independent unknown quantities, are

$$\begin{array}{rcl|c|}
 & & & \text{weight.} \\
-x - y - z + 5''.5 & = & 0 & 3 \\
x \qquad\qquad - 0\ .053 & = & 0 & 3 \\
y \qquad - 0\ .003 & = & 0 & 3 \\
z - 0\ .043 & = & 0 & 1
\end{array}$$

The normal equations, applying the weights, are then

$$\begin{aligned}
6x + 3y + 3z - 16.659 &= 0 \\
3x + 6y + 3z - 16.509 &= 0 \\
3x + 3y + 4z - 16.457 &= 0
\end{aligned}$$

which, being solved, give

$$\begin{aligned}
x &= +\ 0''.9675 \\
y &= +\ 0\ .9175 \\
z &= +\ 2\ .7005
\end{aligned}$$

whence also

$$w = +\ 0.9145$$

and the corrected values of the angles are

1.2	Joscelyne—Deepwater..........	65° 11′ 53″.4145
2.3	Deepwater—Deakyne...........	66 24 16 .4675
3.4	Deakyne—Burden................	87 2 25 .6175
4.1	Burden—Joscelyne..............	141 21 24 .5005
		360 0 0 .0000

55. When the number of unknown quantities is great, or when there are several rigorous conditions to be satisfied, the preceding method would lead to very tedious computations, since we are required to perform two eliminations, the first from our m' rigorous equations to find the first m' quantities in terms of the others, and the second from our normal equations involving all the remaining quantities. In order to obtain the general form

for a more condensed process, let the most probable values of a number (m) of directly observed quantities be

$$V', V'', V''', \&c. \ldots V^{(m)}$$

Let the observed values be

$$M', M'', M''', \&c. \ldots M^{(m)}$$

Let these observations have the weights

$$p', p'', p''', \&c. \ldots p^{(m)}$$

Let the equations which the most probable values are required to satisfy rigorously be expressed by

$$\left.\begin{array}{l} \varphi' = f' \ (V', V'', V''', \ldots) = 0 \\ \varphi'' = f'' \ (V', V'', V''', \ldots) = 0 \\ \varphi''' = f''' (V', V'', V''', \ldots) = 0 \\ \qquad \&c. \end{array}\right\} \quad (87)$$

and let

$$m' = \text{the number of these conditions.}$$

Let the most probable corrections of the observed values be

$$v', v'', v''', \&c. \ldots v^{(m)}$$

so that

$$V' = M' + v', \qquad V'' = M'' + v'', \qquad V''' = M''' + v''', \&c.$$

Let the values of φ', φ'', φ''' ... when the observed values are actually substituted be n', n'', n''' ... or

$$\left.\begin{array}{l} f' \ (M', M'', M''', \ldots) = n' \\ f'' \ (M', M'', M''', \ldots) = n'' \\ f''' (M', M'', M''', \ldots) = n''' \\ \qquad \&c. \end{array}\right\} \quad (88)$$

Let the differential coefficients $\frac{d\varphi'}{dV'}, \frac{d\varphi'}{dV''}$, &c., $\frac{d\varphi''}{dV'}, \frac{d\varphi''}{dV''}$, &c. be formed; substitute in them the values M', M'', M''' ... for V', V'', V''', and denote the resulting values by a', a'', &c., b', b'', &c.; that is, put

$$\begin{array}{lll} \frac{d\varphi'}{dV'} = a', & \frac{d\varphi'}{dV''} = a'', & \frac{d\varphi'}{dV'''} = a''', \&c. \\ \frac{d\varphi''}{dV'} = b', & \frac{d\varphi''}{dV''} = b'', & \frac{d\varphi''}{dV'''} = b''', \&c. \\ \frac{d\varphi'''}{dV'} = c', & \frac{d\varphi'''}{dV''} = c'', & \frac{d\varphi'''}{dV'''} = c''', \&c. \end{array}$$

These values of the differential coefficients will generally be sufficiently exact; but if M', M'', M''' . . . are found very greatly in error, a repetition of the computation might be necessary, in which the more exact values found by the first computation would be used.

The values of M', M'', M''' . . . being assumed to be so nearly correct that the second and higher powers of the corrections v', v'', v''' . . . may be neglected, we have at once, by TAYLOR'S Theorem, as in the similar case of Art. 40,

$$\left.\begin{aligned} \varphi' &= n' + a'v' + a''v'' + a'''v''' + \ldots . + a^{(m)}v^{(m)} = 0 \\ \varphi'' &= n'' + b'v' + b''v'' + b'''v''' + \ldots . + b^{(m)}v^{(m)} = 0 \\ \varphi''' &= n''' + c'v' + c''v'' + c'''v''' + \ldots . + c^{(m)}v^{(m)} = 0 \\ &\qquad\&\text{c.} \qquad\qquad \&\text{c.} \end{aligned}\right\} \quad (89)$$

which m' equations must be rigorously satisfied by the values of v', v'', v''' . . . .

The equations

$$V' - M' = 0, \qquad V'' - M'' = 0, \qquad V''' - M''' = 0, \text{ \&c.}$$

are the approximate equations of condition; or, more strictly,

$$V' - M' = v', \qquad V'' - M'' = v'', \qquad V''' - M''' = v''', \text{ \&c.}$$

are the equations of condition which are to be satisfied by the most probable system of residuals v', v'', v''' . . . . These, reduced to the unit of weight by Art. 41, become

$$(V' - M')\sqrt{p'} = v'\sqrt{p'}, \qquad (V'' - M'')\sqrt{p''} = v''\sqrt{p''}, \text{\&c.} \qquad (90)$$

and the most probable residuals $v'\sqrt{p'}$, $v''\sqrt{p''}$ are those the sum of whose squares is a minimum, or we must have

$$p'v'^2 + p''v''^2 + p'''v'''^2 + \text{\&c.} = \text{a minimum.}$$

Putting, then, the differential of this quantity equal to zero, we have

$$p'v'dv' + p''v''dv'' + p'''v'''dv''' + \text{\&c.} = 0 \qquad (91)$$

If v', v'', v''' . . . were independent of each other, each coefficient of this equation would necessarily be zero (as in Art. 28), and then the most probable values of V', V'', V''' . . . would be the directly observed values M', M'', M''' . . . But this minimum is here conditioned by the equations (89). If, then, we differentiate (89), the equations

$$\left.\begin{array}{l} a'dv' + a''dv'' + a'''dv''' + \ldots\ldots = 0 \\ b'dv' + b''dv'' + b'''dv''' + \ldots\ldots = 0 \\ c'dv' + c''dv'' + c'''dv''' + \ldots\ldots = 0 \\ \qquad\&c. \end{array}\right\} \quad (92)$$

must coexist with (91).

The number of the equations (92) is m', while the number of differentials is m; and since, by the nature of the case, we must have $m > m'$, we can, by elimination, find from (92) the values of m' differentials in terms of the remaining $m - m'$ differentials. Let us suppose this elimination to be performed, and that the values of the first m' differentials, found in terms of the others, are then substituted in (91); we shall thus have an equation in which the remaining $m - m'$ unknown quantities can be regarded as independent, and the coefficients of these $m - m'$ quantities in this final equation will then *severally* be equal to zero. We can arrive directly at the result of such an elimination and substitution as follows. Multiply the first equation of (92) by A, the second by B, the third by C, &c., and also the equation (91) by -1, and form the sum of all these products. Then, if A, B, C.... are determined so that m' differentials shall disappear from the sum (and they can be so determined, since it only requires m' conditions to determine m' quantities), the final equation obtained will contain only the $m - m'$ remaining differentials. But, the latter being independent, their coefficients must also be severally equal to zero: and hence we have, in all, the following m conditional equations:

$$\left.\begin{array}{l} a'A + b'B + c'C + \ldots\ldots - p'v' = 0 \\ a''A + b''B + c''C + \ldots\ldots - p''v'' = 0 \\ a'''A + b'''B + c'''C + \ldots\ldots - p'''v''' = 0 \\ \quad\&c. \qquad\qquad \&c. \end{array}\right\} \quad (93)$$

If we multiply the first of these by $\frac{a'}{p'}$, the second by $\frac{a''}{p''}$, &c., and add the products, we have, by comparison with the first equation of (89),

$$\left[\frac{aa}{p}\right]A + \left[\frac{ab}{p}\right]B + \left[\frac{ac}{p}\right]C + \ldots\ldots + n' = 0$$

in which the usual notation for sums is followed. In this way we can form m' normal equations containing m' quantities, namely,

$$\left.\begin{array}{l}\left[\frac{aa}{p}\right]A+\left[\frac{ab}{p}\right]B+\left[\frac{ac}{p}\right]C+\ldots.+n'=0\\ \left[\frac{ab}{p}\right]A+\left[\frac{bb}{p}\right]B+\left[\frac{bc}{p}\right]C+\ldots.+n''=0\\ \left[\frac{ac}{p}\right]A+\left[\frac{bc}{p}\right]B+\left[\frac{cc}{p}\right]C+\ldots.+n'''=0\\ \qquad\qquad\qquad\&c.\end{array}\right\}\quad(94)$$

If the observations are of equal weight, we have only to put $p=1$, or, in other words, omit p.

The factors A, B, C... are called by GAUSS the *correlatives* of the equations of condition.

The equations (94) being resolved by the usual method of elimination (Art. 42), the values of the correlatives found are then to be substituted in (93), whence we obtain directly the required corrections,

$$\left.\begin{array}{l}v'=\frac{1}{p'}\,(a'A+b'B+c'C+\ldots.)\\ v''=\frac{1}{p''}\,(a''A+b''B+c''C+\ldots.)\\ v'''=\frac{1}{p'''}\,(a'''A+b'''B+c'''C+\ldots.)\\ \qquad\&c.\qquad\qquad\&c.\end{array}\right\}\quad(95)$$

and hence, finally, the most probable values of the observed quantities, $V'=M'+v'$, $V''=M''+v''$, &c.

The comparative simplicity of this process will best be shown by applying it to the example of the preceding article. We there have given, by observation,

$$\begin{array}{llll}M'&=\ \ 65^\circ\ 11'\ 52''.500, & p'&=3\\ M''&=\ \ 66\ \ 24\ \ 15\ .553, & p''&=3\\ M'''&=\ \ 87\ \ \ 2\ \ 24\ .703, & p'''&=3\\ M^{\text{iv}}&=141\ \ 21\ \ 21\ .757, & p^{\text{iv}}&=1\end{array}$$

with the condition

$$V'+V''+V'''+V^{\text{iv}}-360^\circ=0$$

We have, first,

$$a' = a'' = a''' = a^{iv} = 1$$

and when M', M'', &c. are put for V', V'', &c., we have (88)

$$n' = -5''.487$$

As we have but one condition, we have also but one correlative A; the equation of condition is, by (89),

$$-5''.487 + v' + v'' + v''' + v^{iv} = 0$$

and the single normal equation may be constructed according to the following form:

p	a	$\frac{aa}{p}$
3	1	$\frac{1}{3}$
3	1	$\frac{1}{3}$
3	1	$\frac{1}{3}$
1	1	1

$$\left[\frac{aa}{p}\right] = 2$$

$$2A - 5''.487 = 0$$
$$A = +2''.7435$$

and hence, by (95),

	Corrected values.
$v' = +0.9145$	$V' = 65° 11' 53''.4145$
$v'' = +0.9145$	$V'' = 66\ 24\ 16\ .4675$
$v''' = +0.9145$	$V''' = 87\ 2\ 25\ .6175$
$v^{iv} = +2.7435$	$V^{iv} = 141\ 21\ 24\ .5005$
	$360\ 0\ 0$

agreeing with the result found by the much longer process of the preceding article.

56. The further prosecution of this branch of the subject belongs more especially to works on Geodesy. For more extended examples, see the special report of Mr. C. A. SCHOTT in the Report of the Superintendent of the U. S. Coast Survey for 1854, from which the above example has been drawn. Consult also BESSEL'S *Gradmessung in Ostpreussen in* 1838; ROSENBERGER, in the *Astronomische Nachrichten*, Nos. 121 and 122; BESSEL, *ibid.* No. 438; T. GALLOWAY, Application of the Method to a Portion

of the Survey of England, in the *Memoirs of the Royal Astronomical Society*, Vol. XV.; J. J. BÆYER's *Küstenvermessung;* FISCHER's *Geodæsie;* GERLING's *Ausgleichungs Rechnungen;* DIENGER's *Ausgleichung der Beobachtungsfehler;* LIAGRE, *Calcul des Probabilités;* and GAUSS, *Supplementum theoriæ combinationis*, &c.

CRITERION FOR THE REJECTION OF DOUBTFUL OBSERVATIONS.

57. It has been already remarked (p. 490) that the number of large errors occurring in practice usually exceeds that given by theory, and that this discrepancy, instead of invalidating the theory of purely "accidental" errors, rather indicates a source or sources of error of an abnormal character, and calls for a criterion by which such abnormal observations may be excluded. The criterion proposed by Prof. PEIRCE* will be given here with the investigation nearly in the words of its author, and with only some slight changes of notation.

58. "In almost every true series of observations, some are found which differ so much from the others as to indicate some abnormal source of error not contemplated in the theoretical discussions, and the introduction of which into the investigations can only serve, in the present state of science, to perplex and mislead the inquirer. Geometers have, therefore, been in the habit of rejecting those observations which appeared to them liable to unusual defects, although no exact criterion has been proposed to test and authorize such a procedure, and this delicate subject has been left to the arbitrary discretion of individual computers. The object of the present investigation is to produce an exact rule for the rejection of observations, which shall be legitimately derived from the principles of the Calculus of Probabilities.

"*It is proposed to determine in a series of m observations the limit of error, beyond which all observations involving so great an error may be rejected, provided there are as many as n such observations.*

"The principle upon which it is proposed to solve this problem is, *that the proposed observations should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making so many, and no more, abnormal observations.*

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* *Astronomical Journal* (Cambridge, Mass.), Vol. II. p. 161.

"In determining the probability of these two systems of errors, it must be carefully observed that, because observations are rejected in the second system, the corresponding observations of the first system must be regarded, not as being limited to their actual values, but only as surpassing the limit of rejection."

Let

μ = the number of unknown quantities,
m = the whole number of observations,
n = the number of observations proposed to be rejected,
$n' = m - n$, the number to be retained,
$\Delta, \Delta', \Delta'', \ldots \Delta^{(m)}$ = the system of errors when no observation is rejected,
$\Delta_1, \Delta_1', \Delta_1'', \ldots \Delta_1^{(n')}$ = the system of errors when n observations are rejected,
ϵ, ϵ_1 = the mean errors of the first and second system, respectively,
y = the probability, supposed unknown, of such an abnormal observation that it is rejected on account of its magnitude,
$y' = 1 - y$ = the probability that an observation is not of the abnormal character which involves its rejection,
x = the ratio of the required limit of error for the rejection of n observations to the mean error ϵ, so that $x\epsilon$ is the limiting error.

The probability of an error Δ in the first system will be, by (14) and (21),

$$\varphi\Delta = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{\Delta^2}{2\epsilon^2}}$$

and the same form will be used for the second system.

The probability of an error which exceeds the limit $x\epsilon$ will be expressed by the integral (Arts. 8 and 12)

$$2\int_{\Delta=x\epsilon}^{\Delta=\infty} \varphi\Delta d\Delta$$

or, denoting this by ψx,

$$\psi x = \frac{2}{\epsilon\sqrt{2\pi}} \int_{\Delta=x\epsilon}^{\Delta=\infty} e^{-\frac{\Delta^2}{2\epsilon^2}} d\Delta$$

which, by putting $t = \frac{\Delta}{\varepsilon\sqrt{2}}$ becomes

$$\psi\varkappa = \frac{2}{\sqrt{\pi}} \int_{t = \frac{\varkappa}{\sqrt{2}}}^{\infty} e^{-tt}\, dt$$

and this may be found directly from Table IX. by subtracting the tabular number corresponding to $t = \frac{\varkappa}{\sqrt{2}}$ from unity.

The probability of the first system of errors, embodying the condition that n observations exceed the limit $\varkappa\varepsilon$, is

$$P = \varphi\Delta \,.\, \varphi\Delta' \,.\, \varphi\Delta'' \ldots\ldots \left(\frac{\psi\varkappa}{\varphi(\varkappa\varepsilon)}\right)^{n}$$

$$= \frac{1}{\varepsilon^{n'}(2\pi)^{\frac{1}{2}n'}} e^{-\frac{\Sigma\Delta^2 - n\varkappa^2\varepsilon^2}{2\varepsilon^2}} (\psi\varkappa)^{n}$$

in which $\Sigma\Delta^2 = \Delta^2 + \Delta'^2 + \ldots . (\Delta^{(n)})^2$; and by (61) we have $\Sigma\Delta^2 = (m - \mu)\,\varepsilon^2$, whence

$$P = \frac{1}{\varepsilon^{n'}(2\pi)^{\frac{1}{2}n'}} e^{\frac{1}{2}(-m+\mu+n\varkappa^2)} (\psi\varkappa)^{n}$$

The probability of the second system of errors is

$$P_1 = y^n y'^{n'} . \varphi\Delta_1 . \varphi\Delta_1' . \varphi\Delta_1'' \ldots = \frac{y^n y'^{n'}}{\varepsilon_1^{n'}(2\pi)^{\frac{1}{2}n'}} e^{-\frac{\Sigma\Delta_1^2}{2\varepsilon_1^2}}$$

$$= \frac{y^n y'^{n'}}{\varepsilon_1^{n'}(2\pi)^{\frac{1}{2}n'}} e^{\frac{1}{2}(-n'+\mu)}$$

To authorize the proposed rejection of n observations, we must have

$$P < P_1$$

which gives at once

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^{n'} e^{\frac{1}{2}n(\varkappa^2-1)} (\psi\varkappa)^{n} < y^n y'^{n'}$$

The value of y must be determined by the condition that P_1 is a maximum, and therefore $y^n y'^{n'} = y^n\,(1-y)^{n'}$ is a maximum. Taking the logarithm of this quantity, and putting its differential equal to zero, we obtain for the maximum

$$\frac{y}{n} = \frac{y'}{n'} = \frac{1-y}{n'}$$

whence

$$y = \frac{n}{m} \qquad y' = \frac{n'}{m}$$

Putting then

$$\left.\begin{aligned} T^m &= y^n y'^{n'} = \frac{n^n n'^{n'}}{m^m} \\ R &= e^{\frac{1}{2}(x^2-1)} (\psi x) \end{aligned}\right\} \quad (96)$$

the limiting value of x, according to the above inequality, must be that which satisfies the equation

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^{n'} R^n = T^m$$

which gives the required criterion.

The relation of ε_1 to ε must depend on the nature of the equations which correspond to the rejected observations; but it will give a sufficient approximation to assume that the excess of $\Sigma\Delta^2$ over $\Sigma\Delta_1^2$ is only equal to the sum of the squares of the errors of the rejected observations, which gives the equation

$$(m - \mu)\varepsilon^2 - nx^2\varepsilon^2 = (m - \mu - n)\varepsilon_1^2$$

whence

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^2 = \frac{m - \mu - nx^2}{m - \mu - n}$$

which combined with the above equation gives

$$\frac{m - \mu - nx^2}{m - \mu - n} = \left(\frac{T}{R}\right)^{\frac{2n}{m-n}}$$

Putting, for brevity,

$$\lambda^2 = \left(\frac{T}{R}\right)^{\frac{2n}{m-n}} \qquad (97)$$

we find

$$x^2 - 1 = \frac{m - \mu - n}{n}(1 - \lambda^2) \qquad (98)$$

Table X.A gives the logarithms of T and R, computed by (96) with the aid of Table IX. We can, therefore, by successive approximations, find the value of x which satisfies the equations (97) and (98). Since R involves x, we must first assume an approximate value of x (which the observed residuals will suggest), with which λ^2 will be computed by (97), and hence x by (98).

With this first approximate value of $\varkappa$, a new value of log R be taken from the table, with which a second approximati $\varkappa$ will be found. Two or three approximations will usuall found sufficient.

In the application of this criterion, it is to be rememb that it must not be used to reject n observations unless it previously rejected $n-1$ observations. Hence we must first termine the limiting value of $\varkappa$ for the hypothesis of one doul observation, or $n = 1$, and if this rejects one or more obse tions, we can pass to the next hypothesis, $n = 2$, or $n = 3$, and so on until we arrive at the limit which excludes no observations.

The above arrangement of the tables is nearly the sam that given by Dr. B. A. GOULD,* who was the first to pre such tables and thus render the criterion available to prac computers. The only difference is in my table of Log. T, w I have found in practice to be more convenient than the c sponding one of Dr. GOULD.

EXAMPLE.—"To determine the limit of rejection of on two observations in the case of fifteen observations of the ve semidiameters of *Venus*, made by Lieut. HERNDON, with meridian circle at Washington, in the year 1846." In the re tion of these observations, Prof. PEIRCE assumed two unk quantities, and found the following residuals (v):

— 0″.30	— 0″.24	— 1″.40	+ 0″.18
— 0 .44	+ 0 .06	— 0 .22	+ 0 .39
+ 1 .01	+ 0 .63	— 0 .05	+ 0 .10
+ 0 .48	— 0 .13	+ 0 .20	

We have here $m = 15$, $\mu = 2$, $[vv] = 4.2545$, whence

$$\epsilon^2 = \frac{4.2545}{13} = 0.3273, \qquad \epsilon = 0''.572$$

We first try the hypothesis of *one* doubtful observatio $n = 1$. Assuming $\varkappa = 2$, the successive approximations be made as follows:

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* Report of the Superintendent of the U. S. Coast Survey for 1854, Appen 131*; also *Astron. Journal*, Vol. IV. p. 81.

		1st Approx.	2d Approx.
	Table X.A. $\log T$	8.404	8.4044
	" " $\log R$	9.309	9.3062
	$\log \frac{T}{R}$	9.095	9.0982
$\frac{2n}{m-n}=\frac{1}{7}$	$\log \lambda^2$	9.871	9.8712
	$\log(1-\lambda^2)$	9.410	9.4093
$\frac{m-\mu-n}{n}=12$	$\log 12$	1.079	1.0792
	$\log(x^2-1)$	0.489	0.4885
	$\log x^2$	0.610	0.6106
	x	2.02	2.020

Hence $x\epsilon = 1''.16$, which excludes the residual $1''.40$.

We may now try the hypothesis $n = 2$. Commencing again with the assumption $x = 2$, we have—

		1st Approx.	2d Approx.	3d Approx.	4th Approx.
	$\log T$	8.7210	8.7210	8.7210	8.7210
	$\log R$	9.309	9.3622	9.3544	9.3553
	$\log \frac{T}{R}$	9.412	9.3588	9.3666	9.3657
$\frac{2n}{m-n}=\frac{4}{13}$	$\log \lambda^2$	9.819	9.8027	9.8051	9.8048
	$\log(1-\lambda^2)$	9.531	9.5624	9.5582	9.5587
$\frac{m-\mu-n}{n}=\frac{11}{2}$	$\log \frac{11}{2}$	0.740	0.7404	0.7404	0.7404
	$\log(x^2-1)$	0.271	0.3028	0.2986	0.2991
	$\log x^2$	0.457	0.4783	0.4755	0.4758
	x	1.69	1.734	1.729	1.7295

Hence $x\epsilon = 0''.989$, which excludes the residuals $1''.40$ and $1''.01$.

If we now try the hypothesis $n = 3$, we shall find, in the same manner, $x\epsilon = 0''.887$, which does not exclude the residual $0''.63$: so that the residuals $1''.40$ and $1''.01$ are in this case the only abnormal ones. Rejecting these residuals, we shall now find $\epsilon_1 = 0''.339$.*

59. In order to facilitate the application of Peirce's Criterion

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* For another example, in which there were four unknown quantities, and in which the criterion was very useful, see p. 207 of this volume.

in the cases most commonly occurring in practice, Table X. (first given by Dr. GOULD) has been computed by the aid of the log T and log R, according to the preceding method.

The first page of this table is to be used when there is but one unknown quantity ($\mu = 1$), or for direct observations. It gives, by simple inspection, the value of x^2 for any number of observations from 3 to 60, and for any number of doubtful observations from 1 to 9.

The second page is used in the same manner when there are two unknown quantities ($\mu = 2$).

EXAMPLE.—Same as in the preceding article.—Having found, as above, $\varepsilon^2 = 0.3273$, we first take from Table X. for $\mu = 2$ the value of x^2 corresponding to $m = 15$ and $n = 1$, and find

$$x^2 = 4.080, \text{ whence } x^2\varepsilon^2 = 1.3354, \qquad x\varepsilon = 1''.16$$

which rejects the residual 1″.40.

Then, with $m = 15$, $n = 2$, we find, from the same page,

$$x^2 = 2.991, \qquad x^2\varepsilon^2 = 0.9790, \qquad x\varepsilon = 0''.989$$

which rejects the two residuals 1″.40 and 1″.01.

Passing, then, to the hypothesis $n = 3$, we find

$$x^2 = 2.403, \qquad x^2\varepsilon^2 = 0.7865, \qquad x\varepsilon = 0''.887$$

which does not exclude any more residuals.

60. The above investigation of the criterion involves some principles, derived from the theory of probabilities, which may seem obscure to those not familiar with that branch of science. Indeed, the possibility of establishing any criterion whatever for the rejection of doubtful observations, by the aid of the calculus of probabilities, has been questioned even by so distinguished an astronomer as AIRY.* It is easy, however, to derive an approximate criterion *for the rejection of one doubtful observation*, directly from the fundamental formula upon which the whole theory of the method of least squares is based.

We have seen that the function

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* Remarks upon PEIRCE's Criterion, *Astronomical Journal* (Cambridge), Vol. IV. p. 137. Professor WINLOCK's reply to the objections of the Astronomer Royal will be found in the same journal, Vol. IV. p. 145.

$$\Theta(\rho t') = \frac{2}{\sqrt{\pi}} \int_0^{\rho t'} e^{-tt}\, dt$$

(the value of which is given in Table IX.A) represents, in general, the number of errors less than $a = rt'$ which may be expected to occur in any extended series of observations when the whole number of observations is taken as unity, r being the probable error of an observation. If this be multiplied by the number of observations $= m$, we shall have the actual number of errors less than rt'; and hence the quantity

$$m - m \,.\, \Theta(\rho t') = m\,[1 - \Theta(\rho t')]$$

expresses the number of errors to be expected *greater* than the limit rt'. But if this quantity is less than $\frac{1}{2}$, it will follow that an error of the magnitude rt' will have a greater probability against it than for it, and may therefore be rejected. The limit of rejection *of a single doubtful observation*, according to this simple rule, is, therefore, obtained from the equation

$$\tfrac{1}{2} = m\,[1 - \Theta(\rho t')]$$

or

$$\Theta(\rho t') = \frac{2m - 1}{2m} \tag{99}$$

If we express the limiting error under the form $x\varepsilon$, ε being the mean error of an observation, we shall have

$$x = \frac{rt'}{\varepsilon} = 0.6745\,t' \tag{100}$$

With the value of $\Theta(\rho t')$ given by (99), we can find t' from Table IX.A, and hence x by (100).

EXAMPLE.—To find the limit of rejection of *one* of the observations given on p. 562. We there have $m = 15$, $\varepsilon = 0''.572$; and hence, by (99), $\Theta(\rho t') = 0.96667$, which in Table IX.A corresponds to $t' = 3.155$, whence, by (100), $x = 2.128$, $x\varepsilon = 1''.22$, which agrees very nearly with the limit found by PEIRCE's Criterion.

By the successive application of this rule (with the necessary modifications), it may be used for the rejection of two or more doubtful observations, and I have, by means of it, prepared a table which agrees so nearly with Table X. that, for practical purposes, it may be regarded as identical with that table. For the general case, however, when there are several unknown quantities and several doubtful observations, the modifications which the rule requires render it more troublesome than PEIRCE'S formula, and I shall, therefore, not develop it further in this place. What I have given may serve the purpose of giving the reader greater confidence in the correctness and value of PEIRCE'S Criterion.

TABLES.

For the explanation of the construction and use of these tables, consult the articles referred to below.

Table I. Mean Refraction. (Explanation, Vol. I. Art. 107.)

" II. A, B, C, D, E, and F, Bessel's Refraction Table. (Vol. I. Arts. 107, 117, 119; and Vol. II. Arts 294, 295.)

" III. Reduction of Latitude and Logarithm of the Earth's Radius. (Vol. I. Arts. 81, 82.)

" IV. Log A and Log B, for computing the Equation of Equal Altitudes. (Vol. I. Arts. 140, 141.)

" V. Reduction to the Meridian. Values of

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''} \quad \text{and} \quad n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}$$

(Vol. I. Arts. 170, 171.)

" VI. Logarithms of m and n. (Vol. I. Arts. 170, 171.)

" VII. A and VII. B. Limits of Circummeridian Altitudes. (Vol. I. Art. 175.)

" VIII. and VIII. A. For reducing transits over several threads to a common instant. (Vol. II. Arts. 173, 187.)

" IX. and IX. A. Probability of Errors. (Appendix, Arts. 12, 14.)

" X. and X. A. Peirce's Criterion for the Rejection of doubtful Observations. (Appendix, Arts. 58, 59.)

TABLES FOR CORRECTING LUNAR DISTANCES.

" XI. Dip of the Sea Horizon. (Vol. I. Art. 124.)

" XII. Augmentation of the Moon's Semidiameter. (Vol. I. Art. 130.)

" XIII. Correction of the Moon's Equatorial Parallax. (Vol. I. Art. 97.)

TABLE XIV. Mean Reduced Refraction for Lunars. (Vol. I. Art. 249.)

" XIV. A. Correction of the Mean Refraction for the Height of the Barometer. (Vol. I. Art. 249.)

" XIV. B. Correction of the Mean Refraction for the Height of the Thermometer. (Vol. I. Art. 249.)

" XV. Logarithms of A, B, C, D, for correcting Lunar Distances. (Vol. I. Art. 249.)

" XVI. Second Correction of the Lunar Distance. (Vol. I. Art. 249.)

" XVII. A and B. For finding the Correction of the Lunar Distance for the Contraction of the Moon's Semidiameter. (Vol. I. Art. 249.)

" XVIII. A and B. For finding the Correction of the Lunar Distance for the Contraction of the Sun's Semidiameter. (Vol. I. Art. 249.)

" XIX. For finding the value of *N* for correcting Lunar Distances for the Compression of the Earth. (Vol. I. Art. 249.)

" XX. Correction required on account of Second Differences of the Moon's Motion, in finding the Greenwich Time corresponding to a Corrected Lunar Distance. (Vol. I. Art. 66.)

TABLE I. Mean Refraction.

Barometer, 80 inches. Fahrenheit's Thermometer, 50°.

Mean Refraction	Apparent Zen. Dist.	Mean Refraction	Apparent Zen. Dist.	Mean Refraction	Apparent Zen. Dist.	Mean Refraction	Apparent Zen. Dist.	Mean Refraction.
′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
0 0.0	**48** 0	1 4.7	**65** 0	2 4.4	**75** 0	3 34.1	**80** 30	5 35.1
0 1.0	20	1 5.4	10	2 5.3	10	3 36.5	35	5 37.9
0 2.0	40	1 6.2	20	2 6.2	20	3 39.0	40	5 40.7
0 3.1	**49** 0	1 7.0	30	2 7.2	30	3 41.6	45	5 43.6
0 4.1	20	1 7.8	40	2 8.2	40	3 44.2	50	5 46.6
0 5.1	40	1 8.6	50	2 9.2	50	3 46.8	55	5 49.6
0 6.1	**50** 0	1 9.4	**66** 0	2 10.2	**76** 0	3 49.5	**81** 0	5 52.6
0 7.2	20	1 10.2	10	2 11.2	5	3 50.9	5	5 55.7
0 8.2	40	1 11.0	20	2 12.2	10	3 52.3	10	5 58.8
0 9.2	**51** 0	1 11.9	30	2 13.3	15	3 53.7	15	6 2.0
0 10.3	20	1 12.7	40	2 14.3	20	3 55.2	20	6 5.2
0 11.3	40	1 13.6	50	2 15.4	25	3 56.6	25	6 8.5
0 12.4	**52** 0	1 14.5	**67** 0	2 16.4	**76** 30	3 58.1	**81** 30	6 11.9
0 13.5	20	1 15.4	10	2 17.5	35	3 59.6	35	6 15.3
0 14.5	40	1 16.3	20	2 18.7	40	4 1.0	40	6 18.8
0 15.6	**53** 0	1 17.2	30	2 19.8	45	4 2.6	45	6 22.3
0 16.7	20	1 18.2	40	2 20.9	50	4 4.1	50	6 25.9
0 17.8	40	1 19.1	50	2 22.1	55	4 5.6	55	6 29.6
0 18.9	**54** 0	1 20.1	**68** 0	2 23.3	**77** 0	4 7.2	**82** 0	6 33.3
0 20.1	20	1 21.0	10	2 24.5	5	4 8.8	5	6 37.1
0 21.2	40	1 22.0	20	2 25.7	10	4 10.4	10	6 41.0
0 22.4	**55** 0	1 23.1	30	2 26.9	15	4 12.0	15	6 44.9
0 23.6	20	1 24.1	40	2 28.1	20	4 13.6	20	6 48.9
0 24.7	40	1 25.1	50	2 29.4	25	4 15.3	25	6 53.0
0 25.9	**56** 0	1 26.2	**69** 0	2 30.7	**77** 30	4 17.0	**82** 30	6 57.1
0 27.2	20	1 27.3	10	2 32.0	35	4 18.7	35	7 1.4
0 28.4	40	1 28.4	20	2 33.3	40	4 20.4	40	7 5.7
0 29.7	**57** 0	1 29.5	30	2 34.6	45	4 22.2	45	7 10.1
0 31.0	20	1 30.7	40	2 36.0	50	4 23.9	50	7 14.6
0 32.3	40	1 31.8	50	2 37.4	55	4 25.7	55	7 19.2
0 33.6	**58** 0	1 33.0	**70** 0	2 38.8	**78** 0	4 27.5	**83** 0	7 23.8
0 35.0	20	1 34.2	10	2 40.2	5	4 29.4	5	7 28.6
0 36.4	40	1 35.5	20	2 41.6	10	4 31.2	10	7 33.5
0 37.8	**59** 0	1 36.7	30	2 43.1	15	4 33.1	15	7 38.4
0 39.3	20	1 38.0	40	2 44.6	20	4 35.0	20	7 43.5
0 40.8	40	1 39.3	50	2 46.1	25	4 36.9	25	7 48.7
0 42.3	**60** 0	1 40.6	**71** 0	2 47.7	**78** 30	4 38.9	**83** 30	7 53.9
0 43.9	20	1 42.0	10	2 49.2	35	4 40.9	35	7 59.3
0 45.5	40	1 43.4	20	2 50.8	40	4 42.9	40	8 4.8
0 47.2	**61** 0	1 44.8	30	2 52.4	45	4 44.9	45	8 10.4
0 48.9	20	1 46.2	40	2 54.1	50	4 47.0	50	8 16.2
0 50.6	40	1 47.7	50	2 55.8	55	4 49.1	55	8 22.1
0 52.5	**62** 0	1 49.2	**72** 0	2 57.5	**79** 0	4 51.2	**84** 0	8 28.1
0 53.1	10	1 50.0	10	2 59.2	5	4 53.4	5	8 34.2
0 53.7	20	1 50.7	20	3 1.0	10	4 55.6	10	8 40.4
0 54.3	30	1 51.5	30	3 2.8	15	4 57.8	15	8 46.8
0 55.0	40	1 52.3	40	3 4.6	20	5 0.0	20	8 53.4
0 55.6	50	1 53.1	50	3 6.4	25	5 2.3	25	9 0.1
0 56.2	**63** 0	1 53.9	**73** 0	3 8.3	**79** 30	5 4.6	**84** 30	9 7.0
0 56.9	10	1 54.7	10	3 10.3	35	5 6.9	35	9 14.0
0 57.6	20	1 55.5	20	3 12.2	40	5 9.3	40	9 21.2
0 58.2	30	1 56.4	30	3 14.2	45	5 11.7	45	9 28.6
0 58.9	40	1 57.2	40	3 16.3	50	5 14.2	50	9 36.2
0 59.6	50	1 58.1	50	3 18.4	55	5 16.7	55	9 44.0
1 0.3	**64** 0	1 58.9	**74** 0	3 20.5	**80** 0	5 19.2	**85** 0	9 52
1 1.0	10	1 59.8	10	3 22.6	5	5 21.7	**86** 0	11 44
1 1.7	20	2 0.7	20	3 24.8	10	5 24.3	**87** 0	14 25
1 2.4	30	2 1.6	30	3 27.1	15	5 27.0	**88** 0	18 26
1 3.2	40	2 2.5	40	3 29.4	20	5 29.6	**89** 0	24 54
1 3.9	50	2 3.4	50	3 31.7	25	5 32.4	**90** 0	36 29
1 4.7	**65** 0	2 4.4	**75** 0	3 34.1	**80** 30	5 35.1		

TABLE II. Bessel's Refraction Table.

Zen. Dist.	A. Arg. App. Z. D.				B. Arg. True Z. D.				C. Arg. True Z. D.		
	Log a		A	λ	Log a'		A'	λ'	Log a''		A''
0° 0'	1.76156	2			1.76143	2			6.4458	0	
10 0	1.76154	5			1.76141	6			6.4458	2	
20 0	1.76149	10			1.76135	13			6.4456	4	
30 0	1.76139	9			1.76122	10			6.4452	3	
35 0	1.76130	11			1.76112	13			6.4449	3	
40 0	1.76119	15			1.76099	19			6.4446	5	
45 0	1.76104	4		1.0018	1.76080	5		1.0013	6.4441	2	
46 0	1.76100	4		1.0019	1.76075	5		1.0013	6.4439	2	
47 0	1.76096	4		1.0019	1.76070	5		1.0014	6.4437	1	
48 0	1.76092	5		1.0020	1.76065	6		1.0015	6.4436	2	
49 0	1.76087	5		1.0021	1.76059	6		1.0015	6.4434	1	
50 0	1.76082	5		1.0023	1.76053	6		1.0016	6.4433	2	
51 0	1.76077	6		1.0025	1.76047	7		1.0017	6.4431	2	
52 0	1.76071	6		1.0026	1.76040	8		1.0018	6.4429	1	
53 0	1.76065	7		1.0027	1.76032	8		1.0019	6.4428	3	
54 0	1.76058	8		1.0029	1.76024	10		1.0021	6.4425	3	
55 0	1.76050	8		1.0031	1.76014	10		1.0024	6.4422	3	
56 0	1.76042	9		1.0034	1.76004	11		1.0026	6.4419	3	
57 0	1.76033	10		1.0037	1.75993	12		1.0028	6.4416	4	
58 0	1.76023	11		1.0040	1.75981	14		1.0030	6.4412	4	
59 0	1.76012	11		1.0043	1.75967	14		1.0032	6.4408	4	
60 0	1.76001	13		1.0046	1.75953	16		1.0035	6.4404	4	
61 0	1.75988	15		1.0049	1.75937	18		1.0038	6.4400	5	
62 0	1.75973	16		1.0054	1.75919	20		1.0041	6.4395	5	
63 0	1.75957	18		1.0058	1.75899	22		1.0044	6.4390	6	
64 0	1.75939	20		1.0063	1.75877	25		1.0048	6.4384	6	
65 0	1.75919	22		1.0068	1.75852	28		1.0052	6.4378	8	
66 0	1.75897	26		1.0075	1.75824	31		1.0058	6.4370	9	
67 0	1.75871	29		1.0083	1.75793	36		1.0064	6.4361	10	
68 0	1.75842	33		1.0092	1.75757	40		1.0071	6.4351	12	
69 0	1.75809	38		1.0101	1.75717	47		1.0079	6.4339	13	
70 0	1.75771	45		1.0111	1.75670	55		1.0088	6.4326	15	
71 0	1.75726	51		1.0124	1.75615	63		1.0099	6.4311	19	
72 0	1.75675	60		1.0139	1.75552	74		1.0110	6.4292	21	
73 0	1.75615	72		1.0156	1.75478	88		1.0123	6.4271	25	
74 0	1.75543	86		1.0175	1.75390	106		1.0140	6.4246	28	
75 0	1.75457	16		1.0197	1.75284	19		1.0155	6.4218	4	
10	1.75441	16		1.0200	1.75265	20		1.0158	6.4214	4	
20	1.75425	17		1.0204	1.75245	20		1.0161	6.4210	5	
30	1.75408	17		1.0208	1.75225	21		1.0164	6.4205	5	
40	1.75391	18		1.0212	1.75204	22		1.0167	6.4200	6	
50	1.75373	18		1.0216	1.75182	23		1.0170	6.4194	6	
76 0	1.75355	19		1.0220	1.75159	23		1.0173	6.4188	7	
10	1.75336	20		1.0225	1.75136	24		1.0177	6.4181	7	
20	1.75316	21		1.0230	1.75112	25		1.0180	6.4174	7	
30	1.75295	21		1.0235	1.75087	27		1.0184	6.4167	7	
40	1.75274	22		1.0241	1.75060	27		1.0188	6.4160	7	
50	1.75252	23		1.0246	1.75033	28		1.0192	6.4153	8	
77 0	1.75229		1.0026	1.0252	1.75005		0.9975	1.0197	6.4145		0.997

.TABLE II. Bessel's Refraction Table.

en. ist.	A. Arg. App. Z. D.				B. Arg. True Z. D.				C. Arg. True Z. D.			
	Log α		A	λ	Log α′		A′	λ′	Log α″		A″	λ″
° 0′	1.75229	24	1.0026	1.0252	1.75005	29	0.9975	1.0197	6.4145	7	0.997	1.062
10	1.75205	25	1.0026	1.0258	1.74976	31	0.9974	1.0202	6.4138	8	0.997	1.064
20	1.75180	25	1.0027	1.0264	1.74945	31	0.9973	1.0208	6.4130	8	0.997	1.066
30	1.75155	26	1.0027	1.0272	1.74914	32	0.9972	1.0213	6.4122	8	0.996	1.067
40	1.75129	28	1.0028	1.0281	1.74882	34	0.9971	1.0219	6.4114	8	0.996	1.069
50	1.75101	29	1.0029	1.0290	1.74848	35	0.9970	1.0226	6.4106	9	0.996	1.071
0	1.75072	29	1.0030	1.0299	1.74813	36	0.9970	1.0234	6.4097	9	0.996	1.073
10	1.75043	30	1.0030	1.0308	1.74777	37	0.9969	1.0241	6.4088	10	0.996	1.075
20	1.75013	32	1.0031	1.0318	1.74740	39	0.9968	1.0249	6.4078	11	0.996	1.076
30	1.74981	34	1.0032	1.0328	1.74701	41	0.9967	1.0257	6.4067	11	0.996	1.078
40	1.74947	35	1.0033	1.0338	1.74660	43	0.9967	1.0265	6.4056	12	0.996	1.080
50	1.74912	36	1.0034	1.0347	1.74617	44	0.9966	1.0273	6.4044	12	0.995	1.082
0	1.74876	37	1.0035	1.0357	1.74573	46	0.9965	1.0281	6.4032	13	0.995	1.085
10	1.74839	40	1.0036	1.0367	1.74527	49	0.9964	1.0289	6.4019	14	0.995	1.087
20	1.74799	42	1.0037	1.0377	1.74478	50	0.9963	1.0296	6.4005	14	0.995	1.089
30	1.74757	43	1.0038	1.0387	1.74428	52	0.9962	1.0304	6.3991	15	0.995	1.091
40	1.74714	44	1.0039	1.0398	1.74376	55	0.9961	1.0312	6.3976	14	0.995	1.094
50	1.74670	47	1.0040	1.0409	1.74321	58	0.9960	1.0320	6.3962	15	0.994	1.096
0	1.74623	50	1.0041	1.0420	1.74263	60	0.9958	1.0329	6.3947	16	0.994	1.099
10	1.74573	52	1.0042	1.0431	1.74203	62	0.9957	1.0337	6.3931	17	0.994	1.102
20	1.74521	53	1.0043	1.0442	1.74141	66	0.9955	1.0346	6.3914	19	0.994	1.105
30	1.74468	56	1.0045	1.0454	1.74075	70	0.9954	1.0354	6.3895	19	0.993	1.108
40	1.74412	60	1.0046	1.0466	1.74005	72	0.9952	1.0363	6.3876	20	0.993	1.112
50	1.74352	64	1.0047	1.0479	1.73933	76	0.9951	1.0372	6.3856	20	0.993	1.115
0	1.74288	65	1.0049	1.0493	1.73857	80	0.9949	1.0382	6.3836	20	0.993	1.119
10	1.74223	68	1.0050	1.0508	1.73777	85	0.9948	1.0393	6.3816	21	0.992	1.123
20	1.74155	72	1.0052	1.0523	1.73692	87	0.9946	1.0404	6.3795	21	0.992	1.127
30	1.74083	76	1.0054	1.0540	1.73605	91	0.9944	1.0416	6.3774	22	0.992	1.132
40	1.74007	79	1.0056	1.0559	1.73514	97	0.9942	1.0429	6.3752	24	0.991	1.136
50	1.73928	83	1.0058	1.0579	1.73417	103	0.9940	1.0444	6.3728	26	0.991	1.141
0	1.73845	88	1.0060	1.0600	1.73314	107	0.9938	1.0459	6.3702	28	0.991	1.146
10	1.73757	94	1.0062	1.0622	1.73207	112	0.9936	1.0476	6.3674	31	0.990	1.151
20	1.73663	99	1.0065	1.0646	1.73095	121	0.9934	1.0493	6.3643	32	0.990	1.156
30	1.73564	105	1.0067	1.0671	1.72974	128	0.9931	1.0512	6.3611	33	0.989	1.161
40	1.73459	112	1.0070	1.0697	1.72846	135	0.9929	1.0531	6.3578	34	0.989	1.167
50	1.73347	118	1.0073	1.0725	1.72711	142	0.9926	1.0552	6.3544	36	0.988	1.172
0	1.73229	124	1.0075	1.0754	1.72569	151	0.9924	1.0573	6.3508	39	0.987	1.178
10	1.73105	131	1.0078	1.0784	1.72418	162	0.9920	1.0594	6.3469	42	0.986	1.183
20	1.72974	142	1.0081	1.0815	1.72256	173	0.9917	1.0617	6.3427	45	0.985	1.188
30	1.72832	151	1.0084	1.0846	1.72083	181	0.9913	1.0640	6.3382	48	0.984	1.193
40	1.72681	162	1.0088	1.0879	1.71902	194	0.9909	1.0664	6.3334	50	0.983	1.199
50	1.72519	173	1.0092	1.0914	1.71708	209	0.9905	1.0688	6.3284	53	0.982	1.204
0	1.72346	186	1.0096	1.0951	1.71499	223	0.9901	1.0715	6.3231	57	0.981	1.209
10	1.72160	199	1.0100	1.0992	1.71276	239	0.9897	1.0742	6.3174	59	0.980	1.214
20	1.71961	212	1.0105	1.1036	1.71037	255	0.9893	1.0771	6.3115	63	0.979	1.219
30	1.71749	227	1.0110	1.1082	1.70782	273	0.9888	1.0802	6.3052	65	0.977	1.224
40	1.71522	243	1.0115	1.1130	1.70509	293	0.9882	1.0834	6.2987	68	0.976	1.228
50	1.71279	259	1.0121	1.1178	1.70216	314	0.9876	1.0868	6.2919	72	0.974	1.232
0	1.71020		1.0127	1.1229	1.69902		0.9870	1.0903	6.2847		0.973	1.237

TABLE II. Bessel's Refraction Table.

D. *Factor depending upon the Barometer.*

Paris lines.	Log *B*	English inches.	Log *B*	French metres.	Log *B*	French metres.	Log
315	— 0.02445	**27.5**	— 0.03191	**0.725**	— 0.01560	**0.760**	+ 0.0
316	— 0.02307	**27.6**	— 0.03033	**0.726**	— 0.01500	**0.761**	+ 0.0
317	— 0.02170	**27.7**	— 0.02876	**0.727**	— 0.01440	**0.762**	+ 0.0
318	— 0.02033	**27.8**	— 0.02720	**0.728**	— 0.01380	**0.763**	+ 0.0
319	— 0.01897	**27.9**	— 0.02564	**0.729**	— 0.01321	**0.764**	+ 0.0
320	— 0.01761	**28.0**	— 0.02409	**0.730**	— 0.01261	**0.765**	+ 0.0
321	— 0.01625	**28.1**	— 0.02254	**0.731**	— 0.01202	**0.766**	+ 0.0
322	— 0.01490	**28.2**	— 0.02099	**0.732**	— 0.01142	**0.767**	+ 0.0
323	— 0.01356	**28.3**	— 0.01946	**0.733**	— 0.01083	**0.768**	+ 0.0
324	— 0.01221	**28.4**	— 0.01793	**0.734**	— 0.01024	**0.769**	+ 0.0
325	— 0.01088	**28.5**	— 0.01640	**0.735**	— 0.00965	**0.770**	+ 0.0
326	— 0.00954	**28.6**	— 0.01488	**0.736**	— 0.00906	**0.771**	+ 0.0
327	— 0.00821	**28.7**	— 0.01336	**0.737**	— 0.00847	**0.772**	+ 0.0
328	— 0.00689	**28.8**	— 0.01185	**0.738**	— 0.00788	**0.773**	+ 0.0
329	— 0.00556	**28.9**	— 0.01035	**0.739**	— 0.00729	**0.774**	+ 0.0
330	— 0.00425	**29.0**	— 0.00885	**0.740**	— 0.00670	**0.775**	+ 0.0
331	— 0.00293	**29.1**	— 0.00735	**0.741**	— 0.00612	**0.776**	+ 0.0
332	— 0.00162	**29.2**	— 0.00586	**0.742**	— 0.00553	**0.777**	+ 0.0
333	— 0.00032	**29.3**	— 0.00438	**0.743**	— 0.00494	**0.778**	+ 0.0
334	+ 0.00099	**29.4**	— 0.00290	**0.744**	— 0.00436	**0.779**	+ 0.0
335	+ 0.00228	**29.5**	— 0.00142	**0.745**	— 0.00378	**0.780**	+ 0.0
336	+ 0.00358	**29.6**	+ 0.00005	**0.746**	— 0.00319	**0.781**	+ 0.0
337	+ 0.00487	**29.7**	+ 0.00151	**0.747**	— 0.00261	**0.782**	+ 0.0
338	+ 0.00616	**29.8**	+ 0.00297	**0.748**	— 0.00203	**0.783**	+ 0.0
339	+ 0.00744	**29.9**	+ 0.00443	**0.749**	— 0.00145	**0.784**	+ 0.0
340	+ 0.00872	**30.0**	+ 0.00588	**0.750**	— 0.00087	**0.785**	+ 0.0
341	+ 0.00999	**30.1**	+ 0.00732	**0.751**	— 0.00029	**0.786**	+ 0.0
342	+ 0.01127	**30.2**	+ 0.00876	**0.752**	+ 0.00028	**0.787**	+ 0.0
343	+ 0.01253	**30.3**	+ 0.01020	**0.753**	+ 0.00086	**0.788**	+ 0.0
344	+ 0.01380	**30.4**	+ 0.01163	**0.754**	+ 0.00144	**0.789**	+ 0.0
345	+ 0.01506	**30.5**	+ 0.01306	**0.755**	+ 0.00201	**0.790**	+ 0.0
346	+ 0.01632	**30.6**	+ 0.01448	**0.756**	+ 0.00259	**0.791**	+ 0.0
347	+ 0.01757	**30.7**	+ 0.01589	**0.757**	+ 0.00316	**0.792**	+ 0.0
348	+ 0.01882	**30.8**	+ 0.01731	**0.758**	+ 0.00374	**0.793**	+ 0.0
349	+ 0.02007	**30.9**	+ 0.01871	**0.759**	+ 0.00431	**0.794**	+ 0.0
350	+ 0.02131	**31.0**	+ 0.02012	**0.760**	+ 0.00488	**0.795**	+ 0.0

E. *Factor depending upon the Attached Thermometer.*

(F.) Fahrenheit. (R.) Réaumur. (C.) Centigrade.

F.	Log *T*	R.	Log *T*	C.	Log *T*
— 30°	+ 0.00242	**— 35°**	+ 0.00308	**— 35°**	+ 0.00246
— 20	+ 0.00203	**— 30**	+ 0.00264	**— 30**	+ 0.00211
— 10	+ 0.00164	**— 25**	+ 0.00220	**— 25**	+ 0.00176
0	+ 0.00125	**— 20**	+ 0.00176	**— 20**	+ 0.00140
+ 10	+ 0.00086	**— 15**	+ 0.00132	**— 15**	+ 0.00105
20	+ 0.00047	**— 10**	+ 0.00088	**— 10**	+ 0.00070
30	+ 0.00008	**— 5**	+ 0.00044	**— 5**	+ 0.00035
40	— 0.00031	**0**	0.00000	**0**	0.00000
50	— 0.00070	**+ 5**	— 0.00044	**+ 5**	— 0.00035
60	— 0.00109	**10**	— 0.00088	**10**	— 0.00070
70	— 0.00148	**15**	— 0.00131	**15**	— 0.00105
80	— 0.00186	**20**	— 0.00175	**20**	— 0.00140
90	— 0.00225	**25**	— 0.00218	**25**	— 0.00175
100	— 0.00264	**30**	— 0.00262	**30**	— 0.00210
		35	— 0.00305	**35**	— 0.00244

$\text{Log } \beta = \log B + \log T.$

TABLE II. Bessel's Refraction Table.

F. *Factor depending upon the External Thermometer.*

(F.) Fahrenheit. (R.) Réaumur. (C.) Centigrade.

Log γ	F.	Log γ	R.	Log γ	C.	Log γ
+ 0.06279	35°	+ 0.01185	— 35°	+ 0.08990	— 35°	+ 0.07373
+ 0.06181	36	+ 0.01098	— 30	+ 0.07829	— 30	+ 0.06476
+ 0.06083	37	+ 0.01011	— 25	+ 0.06698	— 25	+ 0.05596
+ 0.05985	38	+ 0.00924	— 24	+ 0.06476	— 24	+ 0.05423
+ 0.05887	39	+ 0.00837	— 23	+ 0.06254	— 23	+ 0.05249
+ 0.05790	40	+ 0.00750	— 22	+ 0.06034	— 22	+ 0.05077
+ 0.05693	41	+ 0.00664	— 21	+ 0.05815	— 21	+ 0.04905
+ 0.05596	42	+ 0.00578	— 20	+ 0.05596	— 20	+ 0.04734
+ 0.05500	43	+ 0.00492	— 19	+ 0.05379	— 19	+ 0.04564
+ 0.05403	44	+ 0.00406	— 18	+ 0.05163	— 18	+ 0.04394
+ 0.05307	45	+ 0.00320	— 17	+ 0.04948	— 17	+ 0.04225
+ 0.05211	46	+ 0.00234	— 16	+ 0.04734	— 16	+ 0.04057
+ 0.05115	47	+ 0.00149	— 15	+ 0.04522	— 15	+ 0.03889
+ 0.05020	48	+ 0.00064	— 14	+ 0.04310	— 14	+ 0.03722
+ 0.04924	49	— 0.00021	— 13	+ 0.04099	— 13	+ 0.03556
+ 0.04829	50	— 0.00106	— 12	+ 0.03889	— 12	+ 0.03390
+ 0.04734	51	— 0.00191	— 11	+ 0.03681	— 11	+ 0.03225
+ 0.04640	52	— 0.00275	— 10	+ 0.03473	— 10	+ 0.03060
+ 0.04545	53	— 0.00360	— 9	+ 0.03266	— 9	+ 0.02896
+ 0.04451	54	— 0.00444	— 8	+ 0.03060	— 8	+ 0.02733
+ 0.04357	55	— 0.00528	— 7	+ 0.02855	— 7	+ 0.02570
+ 0.04263	56	— 0.00612	— 6	+ 0.02652	— 6	+ 0.02408
+ 0.04169	57	— 0.00696	— 5	+ 0.02449	— 5	+ 0.02247
+ 0.04076	58	— 0.00780	— 4	+ 0.02247	— 4	+ 0.02086
+ 0.03982	59	— 0.00863	— 3	+ 0.02046	— 3	+ 0.01926
+ 0.03889	60	— 0.00946	— 2	+ 0.01846	— 2	+ 0.01766
+ 0.03796	61	— 0.01029	— 1	+ 0.01646	— 1	+ 0.01607
+ 0.03704	62	— 0.01112	0	+ 0.01448	0	+ 0.01448
+ 0.03611	63	— 0.01195	+ 1	+ 0.01251	+ 1	+ 0.01290
+ 0.03519	64	— 0.01278	2	+ 0.01054	2	+ 0.01133
+ 0.03427	65	— 0.01360	3	+ 0.00859	3	+ 0.00976
+ 0.03335	66	— 0.01443	4	+ 0.00664	4	+ 0.00820
+ 0.03243	67	— 0.01525	5	+ 0.00470	5	+ 0.00664
+ 0.03152	68	— 0.01607	6	+ 0.00277	6	+ 0.00509
+ 0.03060	69	— 0.01689	7	+ 0.00085	7	+ 0.00354
+ 0.02969	70	— 0.01770	8	— 0.00106	8	+ 0.00200
+ 0.02878	71	— 0.01852	9	— 0.00297	9	+ 0.00047
+ 0.02787	72	— 0.01933	10	— 0.00486	10	— 0.00106
+ 0.02697	73	— 0.02015	11	— 0.00675	11	— 0.00259
+ 0.02606	74	— 0.02096	12	— 0.00863	12	— 0.00410
+ 0.02516	75	— 0.02177	13	— 0.01050	13	— 0.00562
+ 0.02426	76	— 0.02257	14	— 0.01236	14	— 0.00713
+ 0.02336	77	— 0.02338	15	— 0.01422	15	— 0.00863
+ 0.02247	78	— 0.02419	16	— 0.01607	16	— 0.01013
+ 0.02157	79	— 0.02499	17	— 0.01791	17	— 0.01162
+ 0.02068	80	— 0.02579	18	— 0.01974	18	— 0.01311
+ 0.01979	81	— 0.02659	19	— 0.02156	19	— 0.01459
+ 0.01890	82	— 0.02738	20	— 0.02338	20	— 0.01607
+ 0.01801	83	— 0.02819	21	— 0.02519	21	— 0.01754
+ 0.01713	84	— 0.02898	22	— 0.02699	22	— 0.01901
+ 0.01624	85	— 0.02978	23	— 0.02879	23	— 0.02047
+ 0.01536	86	— 0.03057	24	— 0.03057	24	— 0.02194
+ 0.01448	87	— 0.03136	25	— 0.03235	25	— 0.02338
+ 0.01360	88	— 0.03216	30	— 0.04114	30	— 0.03057
+ 0.01273	89	— 0.03294	35	— 0.04976	35	— 0.03765
+ 0.01185	90	— 0.03373				

Table III. Reduction of Latitude and Logarithm of the Earth's Radius.

Argument ϕ = Geographical Latitude. Compression = $\frac{1}{299.15}$

ϕ	$\phi - \phi'$	Diff.	log ρ	Diff.	ϕ	$\phi - \phi'$	Diff.	log ρ	Diff.
° ′	′ ″	″			° ′	′ ″	″		
0 0	0 0.00		0.000 0000		35 0	10 48.25		9.999 5248	
1 0	0 24.02	24.02	9.999 9996	4	10	49.63	1.38	5208	40
2 0	0 48.02	24.00	9982	14	20	50.98	1.35	5169	39
3 0	1 11.95	23.93	9961	21	30	52.31	1.33	5129	40
4 0	1 35.80	23.85	9930	31	40	53.62	1.31	5089	40
5 0	1 59.54	23.74	9891	39	50	54.90	1.28	5049	40
6 0	2 23.12	23.58	9.999 9843	48	36 0	10 56.16	1.26	9.999 5009	40
7 0	2 46.54	23.42	9786	57	10	57.41	1.25	4969	40
8 0	3 9.76	23.22	9721	65	20	58.63	1.22	4929	40
9 0	3 32.74	22.98	9648	73	30	59.82	1.19	4888	41
10 0	3 55.47	22.73	9566	82	40	11 1.00	1.18	4848	40
11 0	4 17.92	22.45	9476	90	50	2.15	1.15	4807	41
12 0	4 40.06	22.14	9.999 9377	99	37 0	11 3.28	1.13	9.999 4767	40
13 0	5 1.85	21.79	9271	106	10	4.39	1.11	4726	41
14 0	5 23.28	21.43	9157	114	20	5.47	1.08	4686	40
15 0	5 44.33	21.05	9035	122	30	6.54	1.07	4645	41
16 0	6 4.95	20.62	8905	130	40	7.58	1.04	4604	41
17 0	6 25.14	20.19	8768	137	50	8.59	1.01	4563	41
18 0	6 44.86	19.72	9.999 8624	144	38 0	11 9.59	1.00	9.999 4522	41
19 0	7 4.09	19.23	8472	152	10	10.56	0.97	4481	41
20 0	7 22.80	18.71	8314	158	20	11.51	.95	4440	41
21 0	7 40.99	18.19	8149	165	30	12.44	.93	4399	41
22 0	7 58.61	17.62	7977	172	40	13.34	.90	4358	41
23 0	8 15.66	17.05	7799	178	50	14.22	.88	4317	41
24 0	8 32.10	16.44	9.999 7614	185	39 0	11 15.08	.86	9.999 4276	41
25 0	8 47.93	15.83	7424	190	10	15.92	.84	4234	42
26 0	9 3.12	15.19	7228	196	20	16.73	.81	4193	41
27 0	9 17.65	14.53	7027	201	30	17.52	.79	4152	41
28 0	9 31.50	13.85	6820	207	40	18.29	.77	4110	42
29 0	9 44.66	13.16	6608	212	50	19.04	.75	4069	41
30 0	9 57.12	12.46	9.999 6392	216	40 0	11 19.76	.72	9.999 4027	42
10	9 59.12	2.00	6355	37	10	20.46	.70	3985	42
20	10 1.11	1.99	6319	36	20	21.13	.67	3944	41
30	3.07	1.96	6282	37	30	21.79	.66	3902	42
40	5.02	1.95	6245	37	40	22.42	.63	3860	42
50	6.94	1.92	6208	37	50	23.02	.60	3819	41
31 0	10 8.85	1.91	9.999 6171	37	41 0	11 23.61	.59	9.999 3777	42
10	10.73	1.88	6134	37	10	24.17	.56	3735	42
20	12.59	1.86	6096	38	20	24.70	.53	3693	42
30	14.44	1.85	6059	37	30	25.22	.52	3651	42
40	16.26	1.82	6021	38	40	25.71	.49	3609	42
50	18.06	1.80	5984	37	50	26.18	.47	3567	42
32 0	10 19.84	1.78	9.999 5946	38	42 0	11 26.62	.44	9.999 3525	42
10	21.60	1.76	5908	38	10	27.04	.42	3483	42
20	23.34	1.74	5870	38	20	27.44	.40	3441	42
30	25.05	1.71	5832	38	30	27.82	.38	3399	42
40	26.75	1.70	5794	38	40	28.17	.35	3357	42
50	28.43	1.68	5755	39	50	28.50	.33	3315	42
33 0	10 30.08	1.65	9.999 5717	38	43 0	11 28.80	.30	9.999 3273	42
10	31.71	1.63	5678	39	10	29.08	.28	3230	43
20	33.32	1.61	5640	38	20	29.34	.26	3188	42
30	34.91	1.59	5601	39	30	29.58	.24	3146	42
40	36.48	1.57	5562	39	40	29.79	.21	3104	42
50	38.03	1.55	5523	39	50	29.98	.19	3062	42
34 0	10 39.55	1.52	9.999 5484	39	44 0	11 30.14	.16	9.999 3019	43
10	41.06	1.51	5445	39	10	30.29	.15	2977	42
20	42.54	1.48	5406	39	20	30.41	.12	2935	42
30	44.00	1.46	5367	39	30	30.50	.09	2892	43
40	45.44	1.44	5327	40	40	30.57	.07	2850	42
50	46.86	1.42	5288	39	50	30.62	.05	2808	42
35 0	10 48.25	1.39	9.999 5248	40	45 0	30.65	.03	9.999 2766	42

le III. Reduction of Latitude and Logarithm of the Earth's Radius.

ϕ' = Geocentric Latitude. ρ = Earth's Radius.

	$\phi - \phi'$	Diff.	$\log \rho$	Diff.	ϕ	$\phi - \phi'$	Diff.	$\log \rho$	Diff.
′	′ ″	″			° ′	′ ″	″		
0	11 30.65		9.999 2766		55 0	10 49.74		9.999 0275	
10	30.65	0.00	2723	43	10	48.36	1.38	0235	40
20	30.63	.02	2681	42	20	46.97	1.39	0195	40
30	30.58	.05	2639	42	30	45.55	1.42	0155	40
40	30.51	.07	2596	43	40	44 11	1.44	0116	39
50	30.42	.09	2554	42	50	42.65	1.46	0076	40
0	11 30.31	.11	9.999 2512	42	56 0	10 41.16	1.49	9.999 0037	39
10	30.17	.14	2470	42	10	39.65	1.51	9.998 9998	39
20	30.01	.16	2427	43	20	38.13	1.52	9958	40
30	29.82	.19	2385	42	30	36.58	1.55	9919	39
40	29.61	.21	2343	42	40	35.01	1.57	9880	39
50	29.38	.23	2300	43	50	33.41	1.60	9841	39
0	11 29.12	.26	9.999 2258	42	57 0	10 31.80	1.61	9.998 9802	39
10	28.85	.27	2216	42	10	30.16	1.64	9764	38
20	28.54	.31	2174	42	20	28.50	1.66	9725	39
30	28.22	.32	2132	42	30	26.83	1.67	9686	39
40	27.87	.35	2089	43	40	25.13	1.70	9648	38
50	27.50	.37	2047	42	50	23 40	1.73	9610	38
8 0	11 27.10	.40	9.999 2005	42	58 0	10 21.66	1.74	9.998 9571	39
10	26.69	.41	1963	42	10	19.90	1.76	9533	38
20	26.24	.45	1921	42	20	18.11	1.79	9495	38
30	25.78	.46	1879	42	30	16.31	1.80	9457	38
40	25.29	.49	1837	42	40	14.48	1.83	9419	38
50	24.78	.51	1795	42	50	12.63	1.85	9382	37
9 0	11 24.24	.54	9.999 1753	42	59 0	10 10.77	1.86	9.998 9344	38
10	23.69	.55	1711	42	10	8.88	1.89	9307	37
20	23.11	.58	1669	42	20	6.97	1.91	9269	38
30	22.50	.61	1627	42	30	5.04	1.93	9232	37
40	21.87	.63	1586	41	40	3.08	1.96	9195	37
50	21.22	.65	1544	42	50	1.11	1.97	9158	37
50 0	11 20.55	.67	9.999 1502	42	60 0	9 59.12	1.99	9.998 9121	37
10	19.85	.70	1460	42	61 0	9 46.74	12.38	8902	219
20	19.13	.72	1419	41	62 0	9 33.65	13.09	8688	214
30	18.39	.74	1377	42	63 0	9 19.85	13.80	8479	209
40	17.63	.76	1335	42	64 0	9 5.36	14.49	8275	204
50	16.84	.79	1294	41	65 0	8 50.21	15.15	8077	198
51 0	11 16.02	.82	9.999 1252	42	66 0	8 34 40	15.81	9.998 7884	193
10	15.19	.83	1211	41	67 0	8 17.97	16.43	7697	187
20	14.33	.86	1170	41	68 0	8 0.92	17.05	7517	180
30	13.45	.88	1128	42	69 0	7 43.29	17.63	7342	175
40	12.55	.90	1087	41	70 0	7 25 08	18.21	7174	168
50	11.62	.93	1046	41	71 0	7 6.33	18.75	7013	161
52 0	11 10.67	.95	9.999 1005	41	72 0	6 47.06	19.27	9.998 6859	154
10	9.70	.97	0963	42	73 0	6 27.28	19.78	6713	146
20	8.71	.99	0922	41	74 0	6 7 03	20 25	6573	140
30	7.69	1.02	0881	41	75 0	5 46.33	20.70	6441	132
40	6.66	1.03	0840	41	76 0	5 25.20	21.13	6317	124
50	5.60	1.06	0800	40	77 0	5 3.67	21.53	6201	116
53 0	11 4.51	1.09	9.999 0759	41	78 0	4 41.77	21.90	9.998 6093	108
10	3.40	1.11	0718	41	79 0	4 19.53	22.24	5993	100
20	2.27	1.13	0677	41	80 0	3 56.96	22.57	5901	92
30	1.12	1.15	0637	40	81 0	3 34.10	22.86	5818	83
40	10 59 94	1.18	0596	41	82 0	3 10.98	23.12	5743	75
50	58.74	1.20	0556	40	83 0	2 47.63	23.35	5676	67
54 0	10 57.52	1.22	9.999 0515	41	84 0	2 24.07	23.56	9.998 5619	57
10	56.28	1.24	0475	40	85 0	2 0.33	23.74	5570	49
20	55.02	1.26	0435	40	86 0	1 36.44	23.89	5530	40
30	53.73	1.29	0395	40	87 0	1 12.43	24.01	5498	32
40	52 42	1.31	0355	40	88 0	0 48.34	24.09	5476	22
50	51.09	1.33	0315	40	89 0	0 24.18	24.16	5463	13
55 0	10 49.74	1.35	9.999 0275	40	90 0	0 0.00	24.18	9.998 5458	5

TABLE IV. Log A and Log B.

For Computing the Equation of Equal Altitudes.

For Noon, *A* —
For Midnight, *A* +

ARGUMENT = ELAPSED TIME.

For Noon
Midnight, *B*

Elapsed Time.	0ʰ		1ʰ		2ʰ		3ʰ		4ʰ		5ʰ
m	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*
0	9.4059	9.4059	9.4072	9.4034	9.4109	9.3959	9.4172	9.3828	9.4260	9.3635	9.4374
1	.4059	.4059	.4072	.4034	.4110	.3957	.4173	.3825	.4261	.3631	.4376
2	.4059	.4059	.4073	.4033	.4111	.3955	.4174	.3822	.4263	.3627	.4378
3	.4059	.4059	.4073	.4032	.4112	.3953	.4175	.3820	.4265	.3624	.4380
4	.4059	.4059	.4074	.4031	.4113	.3952	.4177	.3817	.4266	.3620	.4383
5	9.4059	9.4059	9.4074	9.4030	9.4113	9.3950	9.4178	9.3814	9.4268	9.3616	9.4385
6	.4060	.4059	.4074	.4029	.4114	.3948	.4179	.3811	.4270	.3612	.4387
7	.4060	.4059	.4075	.4028	.4115	.3946	.4181	.3809	.4272	.3608	.4389
8	.4060	.4059	.4075	.4027	.4116	.3944	.4182	.3806	.4273	.3604	.4391
9	.4060	.4059	.4076	.4026	.4117	.3943	.4183	.3803	.4275	.3600	.4393
10	9.4060	9.4059	4.4076	9.4025	9.4118	9.3941	9.4184	9.3800	9.4277	9.3596	9.4396
11	.4060	.4059	.4077	.4024	.4119	.3939	.4186	.3797	.4279	.3592	.4398
12	.4060	.4058	.4077	.4023	.4120	.3937	.4187	.3794	.4280	.3588	.4400
13	.4060	.4058	.4078	.4022	.4121	.3935	.4188	.3792	.4282	.3584	.4402
14	.4060	.4058	.4078	.4021	.4121	.3933	.4190	.3789	.4284	.3580	.4405
15	9.4060	9.4058	9.4079	9.4020	9.4122	9.3931	9.4191	9.3786	9.4286	9.3576	9.4407
16	.4060	.4058	.4079	.4019	.4123	.3929	.4193	.3783	.4288	.3572	.4409
17	.4060	.4057	.4080	.4018	.4124	.3927	.4194	.3780	.4289	.3568	.4411
18	.4061	.4057	.4080	.4017	.4125	.3925	.4195	.3777	.4291	.3564	.4414
19	.4061	.4057	.4081	.4016	.4126	.3923	.4197	.3774	.4293	.3559	.4416
20	9.4061	9.4057	9.4081	9.4015	9.4127	9.3921	9.4198	9.3771	9.4295	9.3555	9.4418
21	.4061	.4056	.4082	.4014	.4128	.3919	.4199	.3768	.4297	.3551	.4420
22	.4061	.4056	.4083	.4013	.4129	.3917	.4201	.3765	.4299	.3547	.4423
23	.4061	.4056	.4083	.4012	.4130	.3915	.4202	.3762	.4300	.3542	.4425
24	.4061	.4055	.4084	.4010	.4131	.3913	.4204	.3759	.4302	.3538	.4427
25	9.4062	9.4055	9.4084	9.4009	9.4132	9.3911	9.4205	9.3756	9.4304	9.3534	9.4430
26	.4062	.4055	.4085	.4008	.4133	.3909	.4207	.3752	.4306	.3530	.4432
27	.4062	.4054	.4086	.4007	.4134	.3907	.4208	.3749	.4308	.3525	.4434
28	.4062	.4054	.4086	.4006	.4135	.3905	.4209	.3746	.4310	.3521	.4437
29	.4062	.4054	.4087	.4004	.4136	.3903	.4211	.3743	.4312	.3516	.4439
30	9.4062	9.4053	9.4087	9.4003	9.4137	9.3900	9.4212	9.3740	9.4314	9.3512	9.4441
31	.4063	.4053	.4088	.4002	.4138	.3898	.4214	.3737	.4315	.3508	.4444
32	.4063	.4052	.4089	.4001	.4139	.3896	.4215	.3733	.4317	.3503	.4446
33	.4063	.4052	.4089	.3999	.4140	.3894	.4217	.3730	.4319	.3499	.4448
34	.4063	.4051	.4090	.3998	.4141	.3892	.4218	.3727	.4321	.3494	.4451
35	9.4064	9.4051	9.4091	9.3997	9.4142	9.3889	9.4220	9.3723	9.4323	9.3490	9.4453
36	.4064	.4050	.4091	.3995	.4144	.3887	.4221	.3720	.4325	.3485	.4456
37	.4064	.4050	.4092	.3994	.4145	.3885	.4223	.3717	.4327	.3480	.4458
38	.4064	.4049	.4093	.3993	.4146	.3882	.4224	.3713	.4329	.3476	.4460
39	.4065	.4049	.4093	.3991	.4147	.3880	.4226	.3710	.4331	.3471	.4463
40	9.4065	9.4048	9.4094	9.3990	9.4148	9.3878	9.4227	9.3707	9.4333	9.3467	9.4465
41	.4065	.4048	.4095	.3988	.4149	.3875	.4229	.3703	.4335	.3462	.4468
42	.4065	.4047	.4095	.3987	.4150	.3873	.4231	.3700	.4337	.3457	.4470
43	.4066	.4047	.4096	.3985	.4151	.3871	.4232	.3696	.4339	.3453	.4473
44	.4066	.4046	.4097	.3984	.4152	.3868	.4234	.3693	.4341	.3448	.4475
45	9.4066	9.4045	9.4097	9.3982	9.4154	9.3866	9.4235	9.3690	9.4343	9.3443	9.4477
46	.4067	.4045	.4098	.3981	.4155	.3863	.4237	.3686	.4345	.3438	.4480
47	.4067	.4044	.4099	.3979	.4156	.3861	.4238	.3683	.4347	.3433	.4482
48	.4067	.4043	.4100	.3978	.4157	.3859	.4240	.3679	.4349	.3429	.4485
49	.4068	.4043	.4100	.3976	.4158	.3856	.4242	.3675	.4351	.3424	.4487
50	9.4068	9.4042	9.4101	9.3975	9.4159	9.3854	9.4243	9.3672	9.4353	9.3419	9.4490
51	.4068	.4041	.4102	.3973	.4161	.3851	.4245	.3668	.4355	.3414	.4492
52	.4069	.4041	.4103	.3972	.4162	.3849	.4246	.3665	.4357	.3409	.4494
53	.4069	.4040	.4103	.3970	.4163	.3846	.4248	.3661	.4359	.3404	.4497
54	.4069	.4039	.4104	.3969	.4164	.3843	.4250	.3657	.4361	.3399	.4500
55	9.4070	9.4038	9.4105	9.3967	9.4165	9.3841	9.4251	9.3654	9.4363	9.3394	9.4503
56	.4070	.4038	.4106	.3965	.4167	.3838	.4253	.3650	.4366	.3389	.4505
57	.4071	.4037	.4107	.3964	.4168	.3836	.4255	.3646	.4368	.3384	.4508
58	.4071	.4036	.4107	.3962	.4169	.3833	.4256	.3643	.4370	.3379	.4510
59	.4071	.4035	.4108	.3960	.4170	.3830	.4258	.3639	.4372	.3374	.4513
60	9.4072	9.4034	9.4109	9.3959	9.4172	9.3828	9.4260	9 3635	9.4374	9.3369	9.4515

ble III. Reduction of Latitude and Logarithm of the Earth's Radius.

ϕ' = Geocentric Latitude. ρ = Earth's Radius.

ϕ	$\phi - \phi'$	Diff.	log ρ	Diff.	ϕ	$\phi - \phi'$	Diff.	log ρ	Diff.
° ′	′ ″	″			° ′	′ ″	″		
45 0	11 30.65		9.999 2766		55 0	10 49.74		9.999 0275	
10	30.65	0.00	2723	43	10	48.36	1.38	0235	40
20	30.63	.02	2681	42	20	46.97	1.39	0195	40
30	30.58	.05	2639	42	30	45.55	1.42	0155	40
40	30.51	.07	2596	43	40	44.11	1.44	0116	39
50	30.42	.09	2554	42	50	42.65	1.46	0076	40
46 0	11 30.31	.11	9.999 2512	42	56 0	10 41.16	1.49	9.999 0037	39
10	30.17	.14	2470	42	10	39.65	1.51	9.998 9998	39
20	30.01	.16	2427	43	20	38.13	1.52	9958	40
30	29.82	.19	2385	42	30	36.58	1.55	9919	39
40	29.61	.21	2343	42	40	35.01	1.57	9880	39
50	29.38	.23	2300	43	50	33.41	1.60	9841	39
47 0	11 29.12	.26	9.999 2258	42	57 0	10 31.80	1.61	9.998 9802	39
10	28.85	.27	2216	42	10	30.16	1.64	9764	38
20	28.54	.31	2174	42	20	28.50	1.66	9725	39
30	28.22	.32	2132	42	30	26.83	1.67	9686	39
40	27.87	.35	2089	43	40	25.13	1.70	9648	38
50	27.50	.37	2047	42	50	23.40	1.73	9610	38
48 0	11 27.10	.40	9.999 2005	42	58 0	10 21.66	1.74	9.998 9571	39
10	26.69	.41	1963	42	10	19.90	1.76	9533	38
20	26.24	.45	1921	42	20	18.11	1.79	9495	38
30	25.78	.46	1879	42	30	16.31	1.80	9457	38
40	25.29	.49	1837	42	40	14.48	1.83	9419	38
50	24.78	.51	1795	42	50	12.63	1.85	9382	37
49 0	11 24.24	.54	9.999 1753	42	59 0	10 10.77	1.86	9.998 9344	38
10	23.69	.55	1711	42	10	8.88	1.89	9307	37
20	23.11	.58	1669	42	20	6.97	1.91	9269	38
30	22.50	.61	1627	42	30	5.04	1.93	9232	37
40	21.87	.63	1586	41	40	3.08	1.96	9195	37
50	21.22	.65	1544	42	50	1.11	1.97	9158	37
50 0	11 20.55	.67	9.999 1502	42	60 0	9 59.12	1.99	9.998 9121	37
10	19.85	.70	1460	42	61 0	9 46.74	12.38	8902	219
20	19.13	.72	1419	41	62 0	9 33.65	13.09	8688	214
30	18.39	.74	1377	42	63 0	9 19.85	13.80	8479	209
40	17.63	.76	1335	42	64 0	9 5.36	14.49	8275	204
50	16.84	.79	1294	41	65 0	8 50.21	15.15	8077	198
1 0	11 16.02	.82	9.999 1252	42	66 0	8 34.40	15.81	9.998 7884	193
10	15.19	.83	1211	41	67 0	8 17.97	16.43	7697	187
20	14.33	.86	1170	41	68 0	8 0.92	17.05	7517	180
30	13.45	.88	1128	42	69 0	7 43.29	17.63	7342	175
40	12.55	.90	1087	41	70 0	7 25.08	18.21	7174	168
50	11.62	.93	1046	41	71 0	7 6.33	18.75	7013	161
2 0	11 10.67	.95	9.999 1005	41	72 0	6 47.06	19.27	9.998 6859	154
10	9.70	.97	0963	42	73 0	6 27.28	19.78	6713	146
20	8.71	.99	0922	41	74 0	6 7.03	20.25	6573	140
30	7.69	1.02	0881	41	75 0	5 46.33	20.70	6441	132
40	6.66	1.03	0840	41	76 0	5 25.20	21.13	6317	124
50	5.60	1.06	0800	40	77 0	5 3.67	21.53	6201	116
3 0	11 4.51	1.09	9.999 0759	41	78 0	4 41.77	21.90	9.998 6093	108
10	3.40	1.11	0718	41	79 0	4 19.53	22.24	5993	100
20	2.27	1.13	0677	41	80 0	3 56.96	22.57	5901	92
30	1.12	1.15	0637	40	81 0	3 34.10	22.86	5818	83
40	10 59.94	1.18	0596	41	82 0	3 10.98	23.12	5743	75
50	58.74	1.20	0556	40	83 0	2 47.63	23.35	5676	67
4 0	10 57.52	1.22	9.999 0515	41	84 0	2 24.07	23.56	9.998 5619	57
10	56.28	1.24	0475	40	85 0	2 0.33	23.74	5570	49
20	55.02	1.26	0435	40	86 0	1 36.44	23.89	5530	40
30	53.73	1.29	0395	40	87 0	1 12.43	24.01	5498	32
40	52.42	1.31	0355	40	88 0	0 48.34	24.09	5476	22
50	51.09	1.33	0315	40	89 0	0 24.18	24.16	5463	13
5 0	10 49.74	1.35	9.999 0275	40	90 0	0 0.00	24.18	9.998 5458	5

TABLE IV. Log A and Log B.

For Computing the Equation of Equal Altitudes.

For Noon, A − } For Midnight, A + }   ARGUMENT = ELAPSED TIME.   { For Noon or { Midnight, B

Elapsed Time.	12h		13h		14h		15h		16h		17h
	Log A	Log B	Log A	Log B	Log A	Log B	Log A	Log B	Log A	Log B	Log A
m											
0	9.6021	Inf.	9.6406	8.7563	9.6841	9.0971	9.7333	9.3162	9.7895	9.4884	9.8539
1	.6027	6.9603	.6412	.7641	.6848	.1014	.7342	.3194	.7905	.4911	.8550
2	.6033	7.2431	.6419	.7718	.6856	.1057	.7351	.3225	.7915	.4937	.8562
3	.6039	.4198	.6426	.7794	.6864	.1099	.7360	.3256	.7925	.4963	.8573
4	.6045	.5453	.6433	.7868	.6872	.1141	.7369	.3287	.7935	.4990	.8585
5	9.6051	7.6428	9.6440	8.7942	9.6879	9.1183	9.7378	9.3319	9.7945	9.5016	9.8597
6	.6057	.7226	.6447	.8015	.6887	.1224	.7386	.3350	.7955	.5042	.8608
7	.6063	.7902	.6454	.8087	.6895	.1265	.7395	.3380	.7965	.5068	.8620
8	.6069	.8488	.6461	.8158	.6903	.1306	.7404	.3411	.7975	.5094	.8632
9	.6075	.9005	.6467	.8227	.6911	.1347	.7413	.3442	.7986	.5120	.8644
10	9.6082	7.9469	9.6474	8.8296	9.6919	9.1387	9.7422	9.3472	9.7996	9.5146	9.8655
11	.6088	.9889	.6481	.8364	.6926	.1428	.7431	.3503	.8006	.5171	.8667
12	.6094	8.0273	.6488	.8432	.6934	.1468	.7440	.3533	.8016	.5197	.8679
13	.6100	.0627	.6495	.8498	.6942	.1507	.7449	.3563	.8027	.5223	.8691
14	.6106	.0955	.6502	.8564	.6950	.1547	.7458	.3593	.8037	.5248	.8703
15	9.6112	8.1260	9.6509	8.8628	9.6958	9.1586	9.7467	9.3623	9.8047	9.5274	9.8715
16	.6119	.1547	.6516	.8692	.6966	.1625	.7476	.3653	.8058	.5300	.8727
17	.6125	.1816	.6523	.8756	.6974	.1664	.7485	.3683	.8068	.5325	.8739
18	.6131	.2071	.6530	.8818	.6982	.1703	.7494	.3713	.8078	.5351	.8751
19	.6137	.2312	.6538	.8880	.6990	.1741	.7503	.3742	.8089	.5376	.8763
20	9.6144	8.2541	9.6545	8.8941	9.6998	9.1779	9.7512	9.3772	9.8099	9.5401	9.8775
21	.6150	.2759	.6552	.9002	.7006	.1817	.7522	.3801	.8110	.5427	.8787
22	.6156	.2967	.6559	.9062	.7014	.1855	.7531	.3831	.8120	.5452	.8799
23	.6163	.3166	.6566	.9121	.7022	.1893	.7540	.3860	.8131	.5477	.8812
24	.6169	.3357	.6573	.9180	.7030	.1930	.7549	.3889	.8141	.5502	.8824
25	9.6175	8.3540	9.6580	8.9238	9.7038	9.1967	9.7558	9.3918	9.8152	9.5528	9.8836
26	.6182	.3717	.6588	.9295	.7047	.2004	.7568	.3947	.8162	.5553	.8848
27	.6188	.3887	.6595	.9352	.7055	.2041	.7577	.3976	.8173	.5578	.8861
28	.6195	.4051	.6602	.9408	.7063	.2078	.7586	.4005	.8184	.5603	.8873
29	.6201	.4210	.6609	.9464	.7071	.2114	.7595	.4033	.8194	.5628	.8885
30	9.6207	8.4363	9.6616	8.9519	9.7079	9.2150	9.7605	9.4062	9.8205	9.5653	9.8898
31	.6214	.4512	.6624	.9573	.7088	.2186	.7614	.4090	.8216	.5677	.8910
32	.6220	.4657	.6631	.9627	.7096	.2222	.7624	.4119	.8227	.5702	.8923
33	.6226	.4796	.6638	.9681	.7104	.2258	.7633	.4147	.8237	.5727	.8935
34	.6233	.4932	.6645	.9734	.7112	.2293	.7642	.4175	.8248	.5752	.8948
35	9.6239	8.5064	9.6653	8.9787	9.7121	9.2329	9.7652	9.4204	9.8259	9.5777	9.8961
36	.6246	.5192	.6660	.9839	.7129	.2364	.7661	.4232	.8270	.5801	.8973
37	.6252	.5318	.6667	.9891	.7137	.2399	.7671	.4260	.8281	.5826	.8986
38	.6259	.5440	.6675	.9942	.7146	.2434	.7680	.4288	.8292	.5850	.8999
39	.6265	.5559	.6682	.9993	.7154	.2468	.7690	.4316	.8303	.5875	.9011
40	9.6272	8.5675	9.6690	9.0043	9.7162	9.2503	9.7699	9.4343	9.8314	9.5900	9.9024
41	.6279	.5788	.6697	.0093	.7171	.2537	.7709	.4371	.8325	.5924	.9037
42	.6285	.5899	.6704	.0142	.7179	.2571	.7718	.4399	.8336	.5948	.9050
43	.6292	.6008	.6712	.0191	.7187	.2605	.7728	.4426	.8347	.5973	.9063
44	.6298	.6114	.6719	.0240	.7196	.2639	.7738	.4454	.8358	.5997	.9075
45	9.6305	8.6218	9.6727	9.0288	9.7204	9.2673	9.7747	9.4481	9.8369	9.6022	9.9088
46	.6311	.6320	.6734	.0336	.7213	.2706	.7757	.4509	.8380	.6046	.9101
47	.6318	.6419	.6742	.0384	.7221	.2740	.7767	.4536	.8391	.6070	.9114
48	.6325	.6517	.6749	.0431	.7230	.2773	.7776	.4563	.8402	.6094	.9127
49	.6331	.6613	.6757	.0478	.7238	.2806	.7786	.4590	.8414	.6119	.9140
50	9.6338	8.6707	9.6764	9.0524	9.7247	9.2839	9.7796	9.4617	9.8425	9.6143	9.9154
51	.6345	.6799	.6772	.0570	.7256	.2872	.7806	.4644	.8436	.6167	.9167
52	.6351	.6890	.6779	.0616	.7264	.2905	.7815	.4671	.8447	.6191	.9180
53	.6358	.6979	.6787	.0662	.7273	.2937	.7825	.4698	.8459	.6215	.9193
54	.6365	.7067	.6795	.0707	.7281	.2970	.7835	.4725	.8470	.6239	.9206
55	9.6372	8.7153	9.6802	9.0752	9.7290	9.3002	9.7845	9.4752	9.8481	9.6263	9.9220
56	.6378	.7237	.6810	.0796	.7299	.3034	.7855	.4778	.8493	.6287	.9233
57	.6385	.7321	.6818	.0840	.7307	.3066	.7865	.4805	.8504	.6311	.9246
58	.6392	.7402	.6825	.0884	.7316	.3098	.7875	.4831	.8516	.6335	.9260
59	.6399	.7483	.6833	.0928	.7324	.3130	.7885	.4858	.8527	.6359	.9273
60	9.6406	8.7563	9.6841	9.0971	9.7333	9.3162	9.7895	9.4884	9.8539	9.6383	9.9287

TABLE IV. Log A and Log B.

For Computing the Equation of Equal Altitudes.

For Noon, *A* —
For Midnight, *A* +

ARGUMENT = ELAPSED TIME.

For Noon or Midnight, *B* —

Elapsed Time	18^h		19^h		20^h		21^h		22^h		23^h	
	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*	Log *A*	Log *B*
0^m	9.9287	9.7782	0.0172	9.9167	0.1249	0.0625	0.2623	0.2279	0.4523	0.4372	0.7689	0.7652
1	.9300	.7804	.0188	.9190	.1269	.0650	.2649	.2309	.4562	.4414	.7765	.7729
2	.9314	.7827	.0204	.9213	.1290	.0676	.2676	.2339	.4601	.4455	.7842	.7807
3	.9327	.7850	.0221	.9237	.1310	.0701	.2702	.2370	.4640	.4497	.7920	.7886
4	.9341	.7873	.0237	.9260	.1330	.0727	.2729	.2401	.4680	.4540	.8000	.7967
5	9.9355	9.7896	0.0253	9.9284	0.1351	0.0753	0.2756	0.2431	0.4720	0.4582	0.8081	0.8049
6	.9368	.7919	.0270	.9307	.1371	.0779	.2783	.2462	.4761	.4625	.8163	.8133
7	.9382	.7942	.0286	.9331	.1392	.0805	.2810	.2493	.4801	.4668	.8247	.8218
8	.9396	.7965	.0303	.9355	.1412	.0830	.2838	.2524	.4842	.4711	.8333	.8305
9	.9410	.7988	.0319	.9378	.1433	.0856	.2865	.2556	.4884	.4755	.8420	.8393
10	9.9424	9.8011	0.0336	9.9402	0.1454	0.0882	0.2893	0.2587	0.4926	0.4799	0.8508	0.8483
11	.9437	.8034	.0353	.9426	.1475	.0909	.2921	.2619	.4968	.4844	.8599	.8574
12	.9451	.8057	.0370	.9449	.1496	.0935	.2949	.2650	.5010	.4889	.8691	.8667
13	.9465	.8080	.0386	.9473	.1517	.0961	.2977	.2682	.5053	.4934	.8786	.8763
14	.9479	.8103	.0403	.9497	.1538	.0987	.3005	.2714	.5097	.4980	.8882	.8860
15	9.9493	9.8126	0.0420	9.9520	0.1559	0.1013	0.3034	0.2746	0.5140	0.5026	0.8980	0.8959
16	.9508	.8149	.0437	.9544	.1581	.1040	.3063	.2778	.5184	.5072	.9080	.9060
17	.9522	.8172	.0454	.9568	.1602	.1066	.3091	.2811	.5229	.5118	.9183	.9164
18	.9536	.8195	.0472	.9592	.1623	.1093	.3120	.2843	.5274	.5165	.9288	.9270
19	.9550	.8218	.0489	.9616	.1645	.1119	.3150	.2876	.5319	.5213	.9396	.9378
20	9.9564	9.8241	0.0506	9.9640	0.1667	0.1146	0.3179	0.2909	0.5365	0.5261	0.9506	0.9489
21	.9579	.8264	.0523	.9664	.1689	.1173	.3208	.2942	.5411	.5309	.9618	.9603
22	.9593	.8287	.0541	.9687	.1711	.1200	.3238	.2975	.5458	.5358	.9734	.9719
23	.9607	.8310	.0558	.9711	.1733	.1226	.3268	.3008	.5505	.5407	.9853	.9839
24	.9622	.8333	.0576	.9735	.1755	.1253	.3298	.3041	.5553	.5457	.9975	.9961
25	9.9636	9.8356	0.0593	9.9760	0.1777	0.1280	0.3328	0.3075	0.5601	0.5507	1.0100	1.0087
26	.9651	.8379	.0611	.9784	.1799	.1308	.3359	.3109	.5649	.5557	.0228	.0216
27	.9665	.8402	.0628	.9808	.1821	.1335	.3389	.3143	.5698	.5608	.0361	.0350
28	.9680	.8425	.0646	.9832	.1844	.1362	.3420	.3177	.5748	.5660	.0497	.0487
29	.9695	.8448	.0664	.9856	.1867	.1389	.3451	.3211	.5798	.5712	.0638	.0628
30	9.9709	9.8471	0.0682	9.9880	0.1889	0.1417	0.3482	0.3245	0.5848	0.5764	1.0783	1.0774
31	.9724	.8494	.0700	.9904	.1912	.1444	.3514	.3280	.5899	.5817	.0934	.0925
32	.9739	.8517	.0718	.9929	.1935	.1472	.3545	.3315	.5951	.5871	.1089	.1081
33	.9754	.8540	.0736	.9953	.1958	.1499	.3577	.3350	.6003	.5925	.1250	.1242
34	.9769	.8563	.0754	.9977	.1981	.1527	.3609	.3385	.6056	.5979	.1416	.1409
35	9.9784	9.8586	0.0772	0.0002	0.2004	0.1555	0.3641	0.3420	0.6110	0.6034	1.1590	1.1583
36	.9798	.8609	.0790	.0026	.2028	.1582	.3674	.3456	.6164	.6090	.1770	.1764
37	.9813	.8632	.0809	.0051	.2051	.1610	.3706	.3491	.6218	.6147	.1958	.1952
38	.9829	.8655	.0827	.0075	.2075	.1638	.3739	.3527	.6273	.6204	.2154	.2149
39	.9844	.8678	.0845	.0100	.2098	.1667	.3772	.3563	.6329	.6261	.2359	.2354
40	9.9859	9.8701	0.0864	0.0124	0.2122	0.1695	0.3805	0.3599	0.6386	0.6319	1.2573	1.2569
41	.9874	.8724	.0883	.0149	.2146	.1723	.3839	.3636	.6443	.6378	.2799	.2795
42	.9889	.8748	.0901	.0173	.2170	.1751	.3873	.3673	.6501	.6438	.3037	.3033
43	.9904	.8771	.0920	.0198	.2194	.1780	.3907	.3710	.6560	.6498	.3288	.3285
44	.9920	.8794	.0939	.0223	.2218	.1808	.3941	.3747	.6619	.6559	.3554	.3552
45	9.9935	9.8817	0.0958	0.0248	0.2243	0.1837	0.3975	0.3784	0.6679	0.6621	1.3837	1.3835
46	.9951	.8840	.0976	.0272	.2267	.1866	.4010	.3822	.6740	.6684	.4140	.4138
47	.9966	.8863	.0995	.0297	.2292	.1895	.4045	.3859	.6802	.6747	.4465	.4463
48	.9982	.8887	.1015	.0322	.2316	.1924	.4080	.3897	.6865	.6811	.4815	.4814
49	.9998	.8910	.1034	.0347	.2341	.1953	.4115	.3936	.6928	.6876	.5196	.5195
50	0.0013	9.8933	0.1053	0.0372	0.2366	0.1982	0.4151	0.3974	0.6993	0.6942	1.5613	1.5612
51	.0029	.8956	.1072	.0397	.2391	.2011	.4187	.4013	.7058	.7008	.6074	.6073
52	.0044	.8980	.1092	.0422	.2416	.2040	.4223	.4052	.7124	.7076	.6588	.6587
53	.0060	.9003	.1111	.0447	.2442	.2070	.4260	.4091	.7191	.7144	.7171	.7171
	.0076	.9026	.1131	.0473	.2467	.2099	.4297	.4130	.7259	.7214	.7844	.7843
	.0092	9.9050	0.1150	0.0498	0.2493	0.2129	0.4334	0.4170	0.7328	0.7284	1.8638	1.8638
	.0108	.9073	.1170	.0523	.2518	.2159	.4371	.4210	.7398	.7355	.9610	.9610
	.0124	.9096	.1190	.0548	.2544	.2189	.4408	.4250	.7469	.7428	2.0863	2.0863
	.0140	.9120	.1209	.0574	.2570	.2219	.4446	.4291	.7541	.7501	.2627	.2627
	.0156	.9143	.1229	.0599	.2596	.2249	.4485	.4331	.7615	.7576	2.5640	2.5640
	0.0172	9.9167	0.1249	0.0625	0.2623	0.2279	0.4523	0.4372	0.7689	0.7652	*Inf.*	*Inf.*

Table V. Reduction to the Meridian.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

t	0^m	1^m	2^m	3^m	4^m	5^m	6^m	7^m
′	″	″	″	″	″	″	″	″
0	0.00	1.96	7.85	17.67	31.42	49.09	70.68	96.20
1	0.00	2.03	7.98	17.87	31.68	49.41	71.07	96.66
2	0.00	2.10	8.12	18.07	31.94	49.74	71.47	97.12
3	0.00	2.16	8.25	18.27	32.20	50.07	71.86	97.58
4	0.01	2.23	8.39	18.47	32.47	50.40	72.26	98.04
5	0.01	2.31	8.52	18.67	32.74	50.73	72.66	98.50
6	0.02	2.38	8.66	18.87	33.01	51.07	73.06	98.97
7	0.02	2.45	8.80	19.07	33.27	51.40	73.46	99.43
8	0.03	2.52	8.94	19.28	33.54	51.74	73.86	99.90
9	0.04	2.60	9.08	19.48	33.81	52.07	74.26	100.37
10	0.05	2.67	9.22	19.69	34.09	52.41	74.66	100.84
11	0.06	2.75	9.36	19.90	34.36	52.75	75.06	101.31
12	0.08	2.83	9.50	20.11	34.64	53.09	75.47	101.78
13	0.09	2.91	9.64	20.32	34.91	53.43	75.88	102.25
14	0.11	2.99	9.79	20.53	35.19	53.77	76.29	102.72
15	0.12	3.07	9.94	20.74	35.46	54.11	76.69	103.20
16	0.14	3.15	10.09	20.95	35.74	54.46	77.10	103.67
17	0.16	3.23	10.24	21.16	36.02	54.80	77.51	104.15
18	0.18	3.32	10.39	21.38	36.30	55.15	77.93	104.63
19	0.20	3.40	10.54	21.60	36.58	55.50	78.34	105.10
20	0.22	3.49	10.69	21.82	36.87	55.84	78.75	105.58
21	0.24	3.58	10.84	22.03	37.15	56.19	79.16	106.06
22	0.26	3.67	11.00	22.25	37.44	56.55	79.58	106.55
23	0.28	3.76	11.15	22.47	37.72	56.90	80.00	107.03
24	0.31	3.85	11.31	22.70	38.01	57.25	80.42	107.51
25	0.34	3.94	11.47	22.92	38.30	57.60	80.84	107.99
26	0.37	4.03	11.63	23.14	38.59	57.96	81.26	108.48
27	0.40	4.12	11.79	23.37	38.88	58.32	81.68	108.97
28	0.43	4.22	11.95	23.60	39.17	58.68	82.10	109.46
29	0.46	4.32	12.11	23.82	39.46	59.03	82.52	109.95
30	0.49	4.42	12.27	24.05	39.76	59.40	82.95	110.44
31	0.52	4.52	12.43	24.28	40.05	59.75	83.38	110.93
32	0.56	4.62	12.60	24.51	40.35	60.11	83.81	111.43
33	0.59	4.72	12.76	24.74	40.65	60.47	84.23	111.92
34	0.63	4.82	12.93	24.98	40.95	60.84	84.66	112.41
35	0.67	4.92	13.10	25.21	41.25	61.20	85.09	112.90
36	0.71	5.03	13.27	25.45	41.55	61.57	85.52	113.40
37	0.75	5.13	13.44	25.68	41.85	61.94	85.95	113.90
38	0.79	5.24	13.62	25.92	42.15	62.31	86.39	114.40
39	0.83	5.34	13.79	26.16	42.45	62.68	86.82	114.90
40	0.87	5.45	13.96	26.40	42.76	63.05	87.26	115.40
41	0.91	5.56	14.13	26.64	43.06	63.42	87.70	115.90
42	0.96	5.67	14.31	26.88	43.37	63.79	88.14	116.40
43	1.01	5.78	14.49	27.12	43.68	64.16	88.57	116.90
44	1.06	5.90	14.67	27.37	43.99	64.54	89.01	117.41
45	1.10	6.01	14.85	27.61	44.30	64.91	89.45	117.92
46	1.15	6.13	15.03	27.86	44.61	65.29	89.89	118.43
47	1.20	6.24	15.21	28.10	44.92	65.67	90.33	118.94
48	1.26	6.36	15.39	28.35	45.24	66.05	90.78	119.45
49	1.31	6.48	15.57	28.60	45.55	66.43	91.23	119.96
50	1.36	6.60	15.76	28.85	45.87	66.81	91.68	120.47
51	1.42	6.72	15.95	29.10	46.18	67.19	92.12	120.98
52	1.48	6.84	16.14	29.36	46.50	67.58	92.57	121.49
53	1.53	6.96	16.32	29.61	46.82	67.96	93.02	122.01
54	1.59	7.09	16.51	29.86	47.14	68.35	93.47	122.53
55	1.65	7.21	16.70	30.12	47.46	68.73	93.92	123.05
56	1.71	7.34	16.89	30.38	47.79	69.12	94.38	123.57
57	1.77	7.46	17.08	30.64	48.11	69.51	94.83	124.09
58	1.83	7.60	17.28	30.90	48.43	69.90	95.29	124.61
59	1.89	7.72	17.47	31.16	48.76	70.29	95.74	125.13

Table V. Reduction to the Meridian.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

t	9ᵐ	10ᵐ	11ᵐ	12ᵐ	13ᵐ	14ᵐ	15ᵐ	16ᵐ
s	″	″	″	″	″	″	″	″
0	159.02	196.32	237.54	282.68	331.74	384.74	441.63	502.46
1	159.61	196.97	238.26	283.47	332.59	385.65	442.62	503.50
2	160.20	197.63	238.98	284.26	333.44	386.56	443.60	504.55
3	160.80	198.28	239.70	285.04	334.29	387.48	444.58	505.60
4	161.39	198.94	240.42	285.83	335.15	388.40	445.56	506.65
5	161.98	199.60	241.14	286.62	336.00	389.32	446.55	507.70
6	162.58	200.26	241.87	287.41	336.86	390.24	447.54	508.76
7	163.17	200.92	242.60	288.20	337.72	391.16	448.53	509.81
8	163.77	201.59	243.33	289.00	338.58	392.09	449.51	510.86
9	164.37	202.25	244.06	289.79	339.44	393.01	450.50	511.92
10	164.97	202.92	244.79	290.58	340.30	393.94	451.50	512.98
11	165.57	203.58	245.52	291.38	341.16	394.86	452.49	514.03
12	166.17	204.25	246.25	292.18	342.02	395.79	453.48	515.09
13	166.77	204.92	246.98	292.98	342.88	396.72	454.48	516.15
14	167.37	205.59	247.72	293.78	343.75	397.65	455.47	517.21
15	167.97	206.26	248.45	294.58	344.62	398.58	456.47	518.27
16	168.58	206.93	249.19	295.38	345.49	399.52	457.47	519.34
17	169.19	207.60	249.93	296.18	346.36	400.45	458.47	520.40
18	169.80	208.27	250.67	296.99	347.23	401.38	459.47	521.47
19	170.41	208.94	251.41	297.79	348.10	402.32	460.47	522.53
20	171.02	209.62	252.15	298.60	348.97	403.26	461.47	523.60
21	171.63	210.30	252.89	299.40	349.84	404.20	462.48	524.67
22	172.24	210.98	253.63	300.21	350.71	405.14	463.48	525.74
23	172.85	211.66	254.37	301.02	351.58	406.08	464.48	526.81
24	173.47	212.34	255.12	301.83	352.46	407.02	465.49	527.89
25	174.08	213.02	255.87	302.64	353.34	407.96	466.50	528.96
26	174.70	213.70	256.62	303.46	354.22	408.90	467.51	530.03
27	175.32	214.38	257.37	304.27	355.10	409.84	468.52	531.11
28	175.94	215.07	258.12	305.09	355.98	410.79	469.53	532.18
29	176.56	215.75	258.87	305.90	356.86	411.73	470.54	533.26
30	177.18	216.44	259.62	306.72	357.74	412.68	471.55	534.33
31	177.80	217.12	260.37	307.54	358.62	413.63	472.57	535.41
32	178.43	217.81	261.12	308.36	359.51	414.59	473.58	536.50
33	179.05	218.50	261.88	309.18	360.39	415.54	474.60	537.58
34	179.68	219.19	262.64	310.00	361.28	416.49	475.62	538.67
35	180.30	219.88	263.39	310.82	362.17	417.44	476.64	539.75
36	180.93	220.58	264.15	311.65	363.07	418.40	477.65	540.83
37	181.56	221.27	264.91	312.47	363.96	419.35	478.67	541.91
38	182.19	221.97	265.68	313.30	364.85	420.31	479.70	543.00
39	182.82	222.66	266.44	314.12	365.75	421.27	480.72	544.09
40	183.46	223.36	267.20	314.95	366.64	422.23	481.74	545.18
41	184.09	224.06	267.96	315.78	367.53	423.19	482.77	546.27
42	184.72	224.76	268.73	316.61	368.42	424.15	483.79	547.36
43	185.35	225.46	269.49	317.44	369.31	425.11	484.82	548.45
44	185.99	226.16	270.26	318.27	370.21	426.07	485.85	549.55
45	186.63	226.86	271.02	319.10	371.11	427.04	486.88	550.64
46	187.27	227.57	271.79	319.94	372.01	428.01	487.91	551.73
47	187.91	228.27	272.56	320.78	372.91	428.97	488.94	552.83
48	188.55	228.98	273.34	321.62	373.82	429.93	489.97	553.93
49	189.19	229.68	274.11	322.45	374.72	430.90	491.01	555.03
50	189.83	230.39	274.88	323.29	375.62	431.87	492.05	556.13
51	190.47	231.10	275.65	324.13	376.52	432.84	493.08	557.24
52	191.12	231.81	276.43	324.97	377.43	433.82	494.12	558.34
53	191.76	232.52	277.20	325.81	378.34	434.79	495.15	559.44
54	192.41	233.24	277.98	326.66	379.26	435.76	496.19	560.55
55	193.06	233.95	278.76	327.50	380.17	436.73	497.23	561.65
56	193.71	234.67	279.55	328.35	381.08	437.71	498.28	562.76
57	194.36	235.38	280.33	329.19	381.99	438.69	499.32	563.87
58	195.01	236.10	281.12	330.04	382.90	439.67	500.37	564.98
59	195.66	236.82	281.90	330.89	383.82	440.65	501.41	566.08

Table V. Reduction to the Meridian.

$$m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''}$$

t	17^m	18^m	19^m	20^m	21^m	22^m	23^m	24^m	25
s	″	″	″	″	″	″	″	″	
0	567.2	635.9	708.4	784.9	865.3	949.6	1037.8	1129.9	122
1	568.3	637.0	709.7	786.2	866.6	951.0	1039.3	1131.4	122
2	569.4	638.2	710.9	787.5	868.0	952.4	1040.8	1133.0	122
3	570.5	639.4	712.1	788.8	869.4	953.8	1042.3	1134.6	123
4	571.6	640.6	713.4	790.1	870.8	955.3	1043.8	1136.2	123
5	572.8	641.7	714.6	791.4	872.1	956.7	1045.3	1137.8	123
6	573.9	642.9	715.9	792.7	873.5	958.2	1046.8	1139.3	123
7	575.0	644.1	717.1	794.0	874.9	959.6	1048.3	1140.9	123
8	576.1	645.3	718.4	795.4	876.3	961.1	1049.8	1142.5	123
9	577.2	646.5	719.6	796.7	877.6	962.5	1051.3	1144.0	124
10	578.4	647.7	720.9	798.0	879.0	963.9	1052.8	1145.6	124
11	579.5	648.9	722.1	799.3	880.4	965.4	1054.3	1147.2	124
12	580.6	650.0	723.4	800.7	881.8	966.9	1055.9	1148.8	124
13	581.7	651.2	724.6	802.0	883.2	968.3	1057.4	1150.4	124
14	582.9	652.4	725.9	803.3	884.6	969.8	1058.9	1152.0	124
15	584.0	653.6	727.2	804.6	886.0	971.2	1060.4	1153.6	125
16	585.1	654.8	728.4	806.0	887.4	972.7	1062.0	1155.2	125
17	586.2	656.0	729.7	807.3	888.8	974.1	1063.5	1156.8	125
18	587.4	657.2	730.9	808.6	890.2	975.5	1065.0	1158.3	125
19	588.5	658.4	732.2	809.9	891.6	977.0	1066.5	1159.9	125
20	589.6	659.6	733.5	811.3	893.0	978.5	1068.1	1161.5	125
21	590.8	660.8	734.7	812.6	894.4	979.9	1069.6	1163.1	126
22	591.9	662.0	736.0	813.9	895.8	981.4	1071.1	1164.7	126
23	593.0	663.2	737.3	815.2	897.2	982.9	1072.6	1166.3	126
24	594.2	664.4	738.5	816.6	898.6	984.4	1074.2	1167.9	126
25	595.3	665.6	739.8	817.9	900.0	985.8	1075.7	1169.5	126
26	596.5	666.8	741.1	819.2	901.4	987.3	1077.2	1171.1	126
27	597.6	668.0	742.3	820.5	902.8	988.8	1078.7	1172.7	127
28	598.7	669.2	743.6	821.9	904.2	990.3	1080.3	1174.3	127
29	599.9	670.4	744.9	823.2	905.6	991.8	1081.8	1175.9	127
30	601.0	671.6	746.2	824.6	907.0	993.2	1083.3	1177.5	127
31	602.2	672.8	747.4	825.9	908.4	994.7	1084.8	1179.1	127
32	603.3	674.1	748.7	827.3	909.8	996.2	1086.4	1180.7	127
33	604.5	675.3	750.0	828.6	911.2	997.6	1087.9	1182.3	128
34	605.6	676.5	751.3	829.9	912.6	999.1	1089.5	1183.9	128
35	606.8	677.7	752.6	831.2	914.0	1000.6	1091.0	1185.5	128
36	607.9	678.9	753.8	832.6	915.5	1002.1	1092.6	1187.1	128
37	609.1	680.1	755.1	833.9	916.9	1003.5	1094.1	1188.7	128
38	610.2	681.3	756.4	835.3	918.3	1005.0	1095.7	1190.3	128
39	611.4	682.6	757.7	836.6	919.7	1006.5	1097.2	1191.9	129
40	612.5	683.8	759.0	838.0	921.1	1008.0	1098.8	1193.5	129
41	613.7	685.0	760.2	839.3	922.5	1009.4	1100.3	1195.1	129
42	614.8	686.2	761.5	840.7	923.9	1010.9	1101.9	1196.7	129
43	616.0	687.4	762.8	842.0	925.3	1012.4	1103.4	1198.3	129
44	617.2	688.7	764.1	843.4	926.8	1013.9	1105.0	1199.9	129
45	618.3	689.9	765.4	844.7	928.2	1015.4	1106.5	1201.5	130
46	619.5	691.1	766.7	846.1	929.6	1016.9	1108.1	1203.1	130
47	620.6	692.4	768.0	847.5	931.0	1018.4	1109.6	1204.7	130
48	621.8	693.6	769.3	848.9	932.4	1019.9	1111.2	1206.4	130
49	623.0	694.8	770.6	850.2	933.8	1021.4	1112.7	1208.0	130
50	624.1	696.0	771.9	851.6	935.2	1022.8	1114.3	1209.6	130
51	625.3	697.3	773.1	852.9	936.6	1024.3	1115.8	1211.2	131
52	626.5	698.5	774.5	854.3	938.1	1025.8	1117.4	1212.9	131
53	627.6	699.7	775.7	855.7	939.5	1027.3	1118.9	1214.5	131
54	628.8	701.0	777.1	857.1	940.9	1028.8	1120.5	1216.1	131
55	630.0	702.2	778.4	858.4	942.3	1030.3	1122.0	1217.7	131
56	631.2	703.5	779.7	859.8	943.8	1031.8	1123.6	1219.4	131
57	632.3	704.7	781.0	861.1	945.2	1033.3	1125.1	1221.0	132
58	633.5	705.9	782.3	862.5	946.6	1034.8	1126.7	1222.6	132
59	634.7	707.1	783.6	863.9	948.1	1036.3	1128.3	1224.2	132

Table V. Reduction to the Meridian.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

t	26m	27m	28m	29m
s	″	″	″	″
0	1325.9	1429.7	1537.5	1649.0
1	1327.6	1431.4	1539.3	1650.9
2	1329.3	1433.2	1541.1	1652.8
3	1331.0	1434.9	1542.9	1654.7
4	1332.7	1436.7	1544.8	1656.6
5	1334.4	1438.5	1546.6	1658.5
6	1336.1	1440.3	1548.4	1660.4
7	1337.8	1442.1	1550.2	1662.3
8	1339.5	1443.9	1552.1	1664.2
9	1341.2	1445.6	1553.9	1666.1
10	1342.9	1447.4	1555.8	1668.0
11	1344.6	1449.2	1557.6	1669.9
12	1346.3	1451.0	1559.5	1671.9
13	1348.0	1452.8	1561.3	1673.8
14	1349.7	1454.5	1563.2	1675.7
15	1351.4	1456.3	1565.0	1677.6
16	1353.2	1458.1	1566.9	1679.5
17	1354.9	1459.9	1568.7	1681.4
18	1356.6	1461.6	1570.5	1683.3
19	1358.3	1463.4	1572.4	1685.2
20	1360.1	1465.2	1574.3	1687.2
21	1361.8	1466.9	1576.1	1689.1
22	1363.5	1468.7	1578.0	1691.0
23	1365.2	1470.5	1579.8	1692.9
24	1367.0	1472.3	1581.7	1694.8
25	1368.7	1474.1	1583.5	1696.7
26	1370.4	1475.9	1585.3	1698.6
27	1372.1	1477.7	1587.2	1700.5
28	1373.9	1479.5	1589.1	1702.5
29	1375.6	1481.3	1590.9	1704.4
30	1377.3	1483.1	1592.7	1706.3
31	1379.0	1484.9	1594.6	1708.2
32	1380.8	1486.7	1596.5	1710.2
33	1382.5	1488.5	1598.3	1712.1
34	1384.2	1490.3	1600.2	1714.0
35	1385.9	1492.1	1602.1	1715.9
36	1387.7	1493.9	1604.0	1717.9
37	1389.4	1495.7	1605.9	1719.8
38	1391.2	1497.5	1607.7	1721.7
39	1392.9	1499.3	1609.6	1723.6
40	1394.7	1501.1	1611.5	1725.6
41	1396.4	1502.9	1613.3	1727.5
42	1398.2	1504.7	1615.2	1729.5
43	1399.9	1506.5	1617.1	1731.5
44	1401.7	1508.4	1619.0	1733.4
45	1403.4	1510.2	1620.8	1735.3
46	1405.2	1512.0	1622.7	1737.2
47	1406.9	1513.8	1624.6	1739.2
48	1408.7	1515.6	1626.5	1741.2
49	1410.4	1517.4	1628.3	1743.1
50	1412.2	1519.2	1630.2	1745.1
51	1413.9	1521.0	1632.1	1747.0
52	1415.7	1522.9	1634.0	1749.0
53	1417.4	1524.7	1635.9	1750.9
54	1419.2	1526.5	1637.7	1752.8
55	1420.9	1528.3	1639.6	1754.8
56	1422.7	1530.2	1641.5	1756.8
57	1424.4	1532.0	1643.3	1758.7
58	1426.2	1533.8	1645.2	1760.7
59	1427.9	1535.6	1647.1	1762.6

$$n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}$$

t	n	t	n
m s	″	m s	″
0 0	0.00	20 0	1.49
1 0	0.00	10	1.54
2 0	0.00	20	1.60
3 0	0.00	30	1.65
4 0	0.00	40	1.70
5 0	0.01	50	1.76
6 0	0.01	21 0	1.82
7 0	0.02	10	1.87
8 0	0.04	20	1.93
9 0	0.06	30	1.99
10 0	0.09	40	2.06
11 0	0.14	50	2.12
12 0	0.19	22 0	2.19
10	0.20	10	2.25
20	0.22	20	2.32
30	0.23	30	2.39
40	0.24	40	2.46
50	0.25	50	2.54
13 0	0.26	23 0	2.61
10	0.28	10	2.69
20	0.30	20	2.77
30	0.31	30	2.85
40	0.33	40	2.93
50	0.34	50	3.01
14 0	0.36	24 0	3.10
10	0.38	10	3.18
20	0.39	20	3.27
30	0.41	30	3.36
40	0.43	40	3.45
50	0.45	50	3.55
15 0	0.47	25 0	3.64
10	0.49	10	3.74
20	0.52	20	3.84
30	0.54	30	3.94
40	0.56	40	4.05
50	0.59	50	4.15
16 0	0.61	26 0	4.26
10	0.64	10	4.37
20	0.67	20	4.48
30	0.69	30	4.60
40	0.72	40	4.72
50	0.75	50	4.83
17 0	0.78	27 0	4.96
10	0.81	10	5.08
20	0.84	20	5.20
30	0.88	30	5.33
40	0.91	40	5.46
50	0.95	50	5.60
18 0	0.98	28 0	5.73
10	1.02	10	5.87
20	1.06	20	6.01
30	1.09	30	6.15
40	1.13	40	6.30
50	1.18	50	6.44
19 0	1.22	29 0	6.59
10	1.26	10	6.75
20	1.30	20	6.90
30	1.35	30	7.06
40	1.40	40	7.22
50	1.44	50	7.38
20 0	1.49	30 0	7.55

For rate.

Rate.	Log *k*
s	
− 30	9.999 6985
29	7085
28	7186
27	7286
26	7387
25	7487
24	7588
23	7688
22	7789
21	7889
20	7990
19	8090
18	8191
17	8291
16	8392
15	8492
14	8593
13	8693
12	8794
11	8894
10	8995
9	9095
8	9196
7	9296
6	9397
5	9497
4	9598
3	9698
2	9799
− 1	9.999 9899
0	0.000 0000
+ 1	0101
2	0201
3	0302
4	0402
5	0503
6	0603
7	0704
8	0804
9	0905
10	1005
11	1106
12	1206
13	1307
14	1407
15	1508
16	1608
17	1709
18	1809
19	1910
20	2010
21	2111
22	2212
23	2312
24	2412
25	2513
26	2613
27	2714
28	2814
29	2915
+ 30	0.000 3015

Table VI. Logarithms of *m* and *n*.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

log *m*

t	0^m	1^m	2^m	3^m	4^m	5^m	6^m
s							
0	— ∞	0.29303	0.89509	1.24727	1.49714	1.69096	1.84931
1	6.73673	.30739	.90230	.25208	.50076	.69385	.85172
2	7.33879	.32151	.90945	.25687	.50435	.69673	.85412
3	7.69097	.33541	.91654	.26163	.50793	.69960	.85651
4	7.94085	.34909	.92357	.26636	.51150	.70246	.85890
5	8.13467	.36255	.93055	.27107	.51505	.70531	.86129
6	8.29303	.37581	.93747	.27575	.51859	.70815	.86366
7	8.42692	.38888	.94434	.28041	.52211	.71099	.86603
8	8.54291	.40174	.95115	.28504	.52562	.71382	.86840
9	8.64521	.41442	.95791	.28965	.52912	.71663	.87075
10	8.73673	.42692	.96462	.29423	.53260	.71944	.87310
11	8.81951	.43925	.97127	.29879	.53606	.72223	.87545
12	8.89509	.45140	.97788	.30332	.53952	.72502	.87779
13	8.96461	.46338	.98443	.30783	.54296	.72780	.88012
14	9.02898	.47519	.99094	.31232	.54639	.73057	.88244
15	9.08891	.48685	.99740	.31679	.54980	.73333	.88476
16	9.14497	.49836	1.00381	.32123	.55320	.73608	.88708
17	9.19763	.50971	.01017	.32566	.55659	.73883	.88938
18	9.24727	.52092	.01649	.33006	.55996	.74157	.89168
19	9.29423	.53198	.02276	.33443	.56332	.74429	.89398
20	9.33879	.54291	.02898	.33878	.56667	.74701	.89627
21	9.38117	.55370	.03517	.34311	.57000	.74972	.89855
22	9.42157	.56436	.04131	.34743	.57332	.75242	.90083
23	9.46018	.57489	.04740	.35172	.57663	.75511	.90310
24	9.49715	.58529	.05345	.35598	.57993	.75780	.90536
25	9.53261	.59557	.05946	.36022	.58321	.76048	.90762
26	9.56667	.60573	.06543	.36445	.58648	.76314	.90987
27	9.59945	.61577	.07136	.36866	.58974	.76580	.91212
28	9.63104	.62570	.07725	.37285	.59299	.76846	.91436
29	9.66152	.63551	.08310	.37702	.59622	.77110	.91660
30	9.69097	.64521	.08891	.38116	.59945	.77373	.91883
31	9.71945	.65481	.09468	.38529	.60266	.77636	.92105
32	9.74703	.66431	.10042	.38940	.60586	.77898	.92327
33	9.77376	.67370	.10611	.39348	.60904	.78160	.92548
34	9.79968	.68299	.11177	.39755	.61222	.78420	.92769
35	9.82486	.69218	.11739	.40160	.61538	.78680	.92990
36	9.84933	.70127	.12298	.40563	.61854	.78938	.93209
37	9.87313	.71027	.12853	.40964	.62168	.79197	.93428
38	9.89629	.71918	.13404	.41364	.62481	.79454	.93646
39	9.91886	.72800	.13952	.41761	.62793	.79710	.93864
40	9.94085	.73673	.14497	.42157	.63103	.79967	.94082
41	9.96229	.74537	.15038	.42551	.63413	.80221	.94299
42	9.98323	.75393	.15576	.42943	.63722	.80476	.94515
43	0.00366	.76240	.16110	.43333	.64029	.80729	.94731
44	0.02363	.77080	.16641	.43722	.64335	.80982	.94946
45	0.04315	.77911	.17169	.44109	.64641	.81234	.95161
46	0.06224	.78734	.17694	.44494	.64945	.81486	.95375
47	0.08092	.79550	.18216	.44877	.65248	.81736	.95589
48	0.09921	.80358	.18735	.45259	.65550	.81986	.95802
49	0.11712	.81158	.19250	.45639	.65851	.82236	.96014
50	0.13467	.81952	.19762	.46018	.66151	.82484	.96226
51	0.15187	.82738	.20271	.46395	.66450	.82732	.96438
52	0.16875	.83517	.20778	.46770	.66748	.82979	.96649
53	0.18528	.84288	.21281	.47143	.67045	.83225	.96860
54	0.20151	.85053	.21782	.47515	.67341	.83471	.97070
55	0.21745	.85813	.22280	.47886	.67636	.83716	.97279
56	0.23310	.86564	.22775	.48255	.67930	.83960	.97488
57	0.24848	.87310	.23267	.48622	.68223	.84204	.97697
58	0.26358	.88049	.23756	.48988	.68515	.84447	.97905
59	0.27843	.88782	.24243	.49352	.68806	.84690	.98112
60	0.29303	0.89509	1.24727	1.49714	1.69096	1.84931	1.98320

Table VI. Logarithms of m and n.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

log m							
8^m	9^m	10^m	11^m	12^m	13^m	14^m	15^m
2.09917	2.20146	2.29296	2.37574	2.45130	2.52081	2.58516	2.64506
.10098	.20307	.29441	.37705	.45250	.52192	.58619	.64603
.10278	.20467	.29586	.37836	.45371	.52303	.58722	.64699
.10458	.20627	.29730	.37967	.45491	.52414	.58825	.64795
.10637	.20787	.29874	.38098	.45611	.52525	.58928	.64891
.10817	.20946	.30017	.38229	.45731	.52635	.59031	.64987
.10995	.21106	.30161	.38360	.45850	.52746	.59134	.65083
.11174	.21264	.30304	.38490	.45970	.52856	.59236	.65179
.11352	.21423	.30447	.38619	.46089	.52967	.59339	.65274
.11530	.21581	.30590	.38749	.46209	.53077	.59441	.65370
.11707	.21739	.30732	.38879	.46328	.53187	.59543	.65466
.11884	.21897	.30874	.39009	.46446	.53297	.59645	.65561
.12061	.22055	.31016	.39138	.46565	.53406	.59747	.65656
.12237	.22212	.31158	.39267	.46684	.53516	.59849	.65751
.12413	.22369	.31300	.39396	.46802	.53625	.59951	.65846
.12589	.22525	.31441	.39525	.46920	.53735	.60052	.65941
.12764	.22682	.31582	.39654	.47038	.53844	.60154	.66036
.12939	.22838	.31723	.39782	.47156	.53953	.60255	.66131
.13114	.22994	.31864	.39910	.47274	.54062	.60357	.66225
.13288	.23150	.32004	.40038	.47392	.54170	.60458	.66320
.13462	.23304	.32144	.40166	.47509	.54279	.60559	.66414
.13635	.23459	.32284	.40294	.47626	.54387	.60660	.66509
.13809	.23614	.32424	.40421	.47743	.54496	.60760	.66603
.13982	.23768	.32563	.40548	.47860	.54604	.60861	.66697
.14154	.23922	.32703	.40675	.47977	.54712	.60961	.66791
.14326	.24076	.32842	.40802	.48094	.54820	.61062	.66885
.14498	.24230	.32980	.40929	.48210	.54928	.61162	.66979
.14670	.24383	.33019	.41055	.48327	.55035	.61263	.67073
.14841	.24536	.33258	.41181	.48443	.55143	.61363	.67166
.15011	.24689	.33396	.41307	.48559	.55250	.61463	.67260
.15182	.24842	.33534	.41434	.48675	.55358	.61563	.67353
.15352	.24994	.33671	.41560	.48790	.55465	.61662	.67446
.15522	.25146	.33809	.41685	.48906	.55572	.61762	.67539
.15691	.25297	.33946	.41811	.49021	.55679	.61861	.67633
.15860	.25449	.34083	.41936	.49136	.55785	.61961	.67726
.16029	.25600	.34220	.42061	.49251	.55892	.62060	.67818
.16198	.25751	.34357	.42186	.49366	.55999	.62159	.67911
.16366	.25902	.34493	.42310	.49481	.56105	.62258	.68004
.16534	.26052	.34630	.42435	.49596	.56211	.62357	.68097
.16701	.26202	.34766	.42559	.49711	.56317	.62456	.68189
.16868	.26352	.34901	.42683	.49825	.56423	.62555	.68281
.17035	.26501	.35037	.42807	.49939	.56529	.62654	.68374
.17202	.26651	.35172	.42931	.50053	.56635	.62752	.68466
.17368	.26800	.35307	.43055	.50167	.56740	.62850	.68558
.17534	.26949	.35442	.43178	.50281	.56846	.62949	.68650
.17700	.27097	.35577	.43302	.50394	.56951	.63047	.68742
.17865	.27246	.35712	.43425	.50508	.57056	.63145	.68834
.18030	.27394	.35846	.43548	.50621	.57161	.63243	.68926
.18194	.27542	.35980	.43670	.50734	.57266	.63341	.69017
.18359	.27689	.36114	.43793	.50847	.57371	.63438	.69109
.18523	.27836	.36248	.43915	.50960	.57476	.63536	.69201
.18687	.27984	.36381	.44037	.51073	.57580	.63634	.69292
.18850	.28130	.36515	.44159	.51185	.57685	.63731	.69383
.19013	.28277	.36648	.44281	.51298	.57789	.63828	.69474
.19176	.28423	.36781	.44403	.51410	.57893	.63925	.69565
.19338	.28569	.36913	.44525	.51522	.57997	.64022	.69656
.19500	.28715	.37046	.44646	.51634	.58101	.64119	.69747
.19662	.28861	.37178	.44767	.51746	.58205	.64216	.69838
.19824	.29006	.37310	.44888	.51858	.58309	.64313	.69929
.19985	.29151	.37442	.45009	.51969	.58412	.64410	.70019
2.20146	2.29296	2.37574	2.45130	2.52081	2.58516	2.64506	2.70109

Table VI. Logarithms of *m* and *n*.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

log *m*

t	0^m	1^m	2^m	3^m	4^m	5^m	6^m	7^m
s								
0	— ∞	0.29303	0.89509	1.24727	1.49714	1.69096	1.84931	1.98320
1	6.73673	.30739	.90230	.25208	.50076	.69385	.85172	.98528
2	7.33879	.32151	.90945	.25687	.50435	.69673	.85412	.98732
3	7.69097	.33541	.91654	.26163	.50793	.69960	.85651	.98937
4	7.94085	.34909	.92357	.26636	.51150	.70246	.85890	.99142
5	8.13467	.36255	.93055	.27107	.51505	.70531	.86129	.99347
6	8.29303	.37581	.93747	.27575	.51859	.70815	.86366	.99551
7	8.42692	.38888	.94434	.28041	.52211	.71099	.86603	.99755
8	8.54291	.40174	.95115	.28504	.52562	.71382	.86840	.99958
9	8.64521	.41442	.95791	.28965	.52912	.71663	.87075	2.00161
10	8.73673	.42692	.96462	.29423	.53260	.71944	.87310	.00363
11	8.81951	.43925	.97127	.29879	.53606	.72223	.87545	.00565
12	8.89509	.45140	.97788	.30332	.53952	.72502	.87779	.00766
13	8.96461	.46338	.98443	.30783	.54296	.72780	.88012	.00967
14	9.02898	.47519	.99094	.31232	.54639	.73057	.88244	.01167
15	9.08891	.48685	.99740	.31679	.54980	.73333	.88476	.01367
16	9.14497	.49836	1.00381	.32123	.55320	.73608	.88708	.01566
17	9.19763	.50971	.01017	.32566	.55659	.73883	.88938	.01765
18	9.24727	.52092	.01649	.33006	.55996	.74157	.89168	.01964
19	9.29423	.53198	.02276	.33443	.56332	.74429	.89398	.02162
20	9.33879	.54291	.02898	.33878	.56667	.74701	.89627	.02360
21	9.38117	.55370	.03517	.34311	.57000	.74972	.89855	.02557
22	9.42157	.56436	.04131	.34743	.57332	.75242	.90083	.02753
23	9.46018	.57489	.04740	.35172	.57663	.75511	.90310	.02954
24	9.49715	.58529	.05345	.35598	.57993	.75780	.90536	.03148
25	9.53261	.59557	.05946	.36022	.58321	.76048	.90762	.03342
26	9.56667	.60573	.06543	.36445	.58648	.76314	.90987	.03536
27	9.59945	.61577	.07136	.36866	.58974	.76580	.91212	.03730
28	9.63104	.62570	.07725	.37285	.59299	.76846	.91436	.03924
29	9.66152	.63551	.08310	.37702	.59622	.77110	.91660	.04118
30	9.69097	.64521	.08891	.38116	.59945	.77373	.91883	.04311
31	9.71945	.65481	.09468	.38529	.60266	.77636	.92105	.04504
32	9.74703	.66431	.10042	.38940	.60586	.77898	.92327	.04697
33	9.77376	.67370	.10611	.39348	.60904	.78160	.92548	.04888
34	9.79968	.68299	.11177	.39755	.61222	.78420	.92769	.05080
35	9.82486	.69218	.11739	.40160	.61538	.78680	.92990	.05271
36	9.84933	.70127	.12298	.40563	.61854	.78938	.93209	.05462
37	9.87313	.71027	.12853	.40964	.62168	.79197	.93428	.05652
38	9.89629	.71918	.13404	.41364	.62481	.79454	.93646	.05842
39	9.91886	.72800	.13952	.41761	.62793	.79710	.93864	.06031
40	9.94085	.73673	.14497	.42157	.63103	.79967	.94082	.06220
41	9.96229	.74537	.15038	.42551	.63413	.80221	.94299	.06409
42	9.98323	.75393	.15576	.42943	.63722	.80476	.94515	.06597
43	0.00366	.76240	.16110	.43333	.64029	.80729	.94731	.06785
44	0.02363	.77080	.16641	.43722	.64335	.80982	.94946	.06972
45	0.04315	.77911	.17169	.44109	.64641	.81234	.95161	.07159
46	0.06224	.78734	.17694	.44494	.64945	.81486	.95375	.07346
47	0.08092	.79550	.18216	.44877	.65248	.81736	.95589	.07532
48	0.09921	.80358	.18735	.45259	.65550	.81986	.95802	.07718
49	0.11712	.81158	.19250	.45639	.65851	.82236	.96014	.07903
50	0.13467	.81952	.19762	.46018	.66151	.82484	.96226	.08088
51	0.15187	.82738	.20271	.46395	.66450	.82732	.96438	.08273
52	0.16875	.83517	.20778	.46770	.66748	.82979	.96649	.08457
53	0.18528	.84288	.21281	.47143	.67045	.83225	.96860	.08641
54	0.20151	.85053	.21782	.47515	.67341	.83471	.97070	.08824
55	0.21745	.85813	.22280	.47886	.67636	.83716	.97279	.09007
56	0.23310	.86564	.22775	.48255	.67930	.83960	.97488	.09190
57	0.24848	.87310	.23267	.48622	.68223	.84204	.97697	.09372
58	0.26358	.88049	.23756	.48988	.68515	.84447	.97905	.09554
59	0.27843	.88782	.24243	.49352	.68806	.84690	.98112	.09735
60	0.29303	0.89509	1.24727	1.49714	1.69096	1.84931	1.98320	2.09917

Table VI. Logarithms of *m* and *n*.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''} \qquad n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}$$

log m									
24^m	**25^m**	**26^m**	**27^m**	**28^m**	**29^m**	*t*	log *n*	*t*	log *n*
						m s		m s	
3.05306	3.08848	3.12252	3.15526	3.18681	3.21725	**0** 0	−∞	**20** 0	0.1742
.05366	.08906	.12307	.15580	.18733	.21775	**1** 0	4.9706	10	0.1886
.05426	.08964	.12363	.15633	.18784	.21825	**2** 0	6.1747	20	0.2029
.05487	.09022	.12418	.15666	.18836	.21875	**3** 0	6.8791	30	0.2170
.05547	.09079	.12474	.15740	.18887	.21924	**4** 0	7.3788	40	0.2311
.05607	.09137	.12529	.15793	.18939	.21974	**5** 0	7.7665	50	0.2450
.05667	.09195	.12585	.15847	.18990	.22024	**6** 0	8.0832	**21** 0	0.2589
.05727	.09252	.12640	.15900	.19042	.22073	**7** 0	8.3509	10	0.2726
.05787	.09310	.12695	.15953	.19093	.22123	**8** 0	8.5829	20	0.2862
.05847	.09367	.12751	.16007	.19145	.22172	**9** 0	8.7875	30	0.2997
.05907	.09425	.12806	.16060	.19196	.22222	**10** 0	8.9705	40	0.3131
.05966	.09482	.12861	.16113	.19247	.22272	**11** 0	9.1360	50	0.3264
.06026	.09540	.12916	.16166	.19299	.22321	**12** 0	9.2871	**22** 0	0.3396
.06086	.09597	.12971	.16210	.19350	.22371	10	9.3111	10	0.3527
.06146	.09655	.13026	.16373	.19401	.22420	20	9.3347	20	0.3657
.06205	.09712	.13081	.16326	.19452	.22470	30	9.3580	30	0.3786
.06265	.09769	.13136	.16379	.19503	.22519	40	9.3810	40	0.3915
.06324	.09826	.13191	.16432	.19554	.22568	50	9.4037	50	0.4042
.06384	.09883	.13246	.16485	.19606	.22618	**13** 0	9.4262	**23** 0	0.4168
.06444	.09941	.13301	.16538	.19657	.22667	10	9.4483	10	0.4293
.06503	.09998	.13356	.16591	.19708	.22716	20	9.4701	20	0.4418
.06562	.10055	.13411	.16643	.19759	.22766	30	9.4917	30	0.4541
.06622	.10112	.13466	.16696	.19810	.22815	40	9.5130	40	0.4664
.06681	.10169	.13521	.16749	.19861	.22864	50	9.5341	50	0.4786
.06740	.10226	.13576	.16802	.19912	.22913	**14** 0	9.5549	**24** 0	0.4907
.06800	.10283	.13631	.16855	.19962	.22963	10	9.5754	10	0.5027
.06859	.10340	.13686	.16907	.20013	.23012	20	9.5957	20	0.5146
.06918	.10396	.13740	.16960	.20064	.23061	30	9.6158	30	0.5264
.06977	.10453	.13795	.17013	.20115	.23110	40	9.6356	40	0.5382
.07036	.10510	.13850	.17066	.20166	.23159	50	9.6553	50	0.5499
.07095	.10567	.13904	.17118	.20216	.23208	**15** 0	9.6747	**25** 0	0.5615
.07154	.10623	.13959	.17170	.20267	.23257	10	9.6939	10	0.5730
.07213	.10680	.14013	.17223	.20318	.23306	20	9.7128	20	0.5845
.07272	.10737	.14068	.17275	.20369	.23355	30	9.7316	30	0.5959
.07331	.10793	.14122	.17327	.20419	.23404	40	9.7502	40	0.6072
.07389	.10850	.14177	.17380	.20470	.23453	50	9.7686	50	0.6184
.07448	.10906	.14231	.17433	.20520	.23501	**16** 0	9.7867	**26** 0	0.6296
.07507	.10963	.14285	.17485	.20571	.23550	10	9.8047	10	0.6407
.07566	.11019	.14340	.17538	.20621	.23599	20	9.8225	20	0.6517
.07625	.11076	.14394	.17590	.20672	.23648	30	9.8402	30	0.6626
.07683	.11132	.14448	.17642	.20722	.23697	40	9.8576	40	0.6735
.07742	.11188	.14502	.17694	.20772	.23745	50	9.8749	50	0.6843
.07801	.11245	.14557	.17746	.20822	.23794	**17** 0	9.8920	**27** 0	0.6951
.07859	.11301	.14611	.17799	.20873	.23843	10	9.9089	10	0.7057
.07918	.11357	.14665	.17851	.20924	.23891	20	9.9257	20	0.7164
.07976	.11413	.14719	.17903	.20974	.23940	30	9.9423	30	0.7269
.08035	.11469	.14773	.17955	.21024	.23988	40	9.9588	40	0.7374
.08093	.11525	.14827	.18007	.21075	.24037	50	9.9751	50	0.7478
.08151	.11582	.14881	.18059	.21125	.24086	**18** 0	9.9913	**28** 0	0.7582
.08210	.11638	.14935	.18111	.21175	.24134	10	0.0072	10	0.7685
.08268	.11694	.14989	.18163	.21225	.24182	20	0.0231	20	0.7787
.08326	.11750	.15043	.18215	.21275	.24231	30	0.038	30	0.7889
.08384	.11805	.15096	.18267	.21325	.24279	40	0.0544	40	0.7990
.08442	.11861	.15150	.18319	.21375	.24328	50	0.0698	50	0.8090
.08501	.11917	.15204	.18371	.21425	.24376	**19** 0	0.0851	**29** 0	0.8190
.08559	.11973	.15258	.18422	.21475	.24424	10	0.1003	10	0.8290
.08617	.12029	.15312	.18474	.21525	.24473	20	0.1153	20	0.8389
.08675	.12085	.15365	.18526	.21575	.24521	30	0.1302	30	0.8487
.08733	.12140	.15419	.18578	.21625	.24569	40	0.1450	40	0.8585
.08791	.12196	.15472	.18629	.21675	.24617	50	0.1597	50	0.8682
3.08848	3.12252	3.15526	3.18681	3.21725	3.24665	**20** 0	0.1742	**30** 0	0.8789

Table VII. Limits of Circum-meridian Altitudes.

A. *Limiting hour angle at which the second reduction amounts to one second.*

Lat.	Declination same sign as latitude.								Declination different sign from latitude.								
	80°	70°	60°	50°	40°	30°	20°	10°	0°	10°	20°	30°	40°	50°	60°	70°	80
°	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	
0	67	39	27	21	16	12	8	5	0	5	8	12	16	21	27	39	6
10	54	33	24	17	13	9	5	0	5	8	12	15	19	24	32	48	
20	48	29	20	14	10	5	0	5	8	12	15	18	23	29	40		
30	43	26	17	11	6	0	5	9	12	15	18	22	28	37			
40	38	22	13	7	0	6	10	13	16	19	23	28	36				
50	33	18	9	0	7	11	14	17	21	24	29	37					
60	28	12	0	9	13	17	20	24	27	32	40						
70	20	0	12	18	22	26	29	33	39	48							

B. *Limiting hour angle at which the third reduction amounts to one second.*

Lat.	Declination same sign as latitude.								Declination different sign from latitude.								
	80°	70°	60°	50°	40°	30°	20°	10°	0°	10°	20°	30°	40°	50°	60°	70°	8
°	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	
0	135	90	67	51	40	29	20	11	0	11	20	29	40	51	67	90	13
10	128	82	59	43	32	21	11	0	11	20	28	37	47	59	75	96	
20	118	73	51	35	23	12	0	11	20	28	37	46	56	67	82		
30	107	64	42	26	14	0	12	21	29	37	46	55	64	75			
40	95	54	32	16	0	14	23	32	40	47	56	64	73				
50	82	42	19	0	16	26	35	43	51	59	67	75					
60	67	27	0	19	32	42	51	59	67	75	82						
70	45	0	27	42	54	64	73	82	90	96							

The following approximate rules are sufficiently exact for practi purposes :

A. The limit at which the second reduction amounts to $0''.1$ is ½ the h angle of Table VII. **A.**

The limit at which the second reduction amounts to $0''.01$ is ⅓ the h angle of Table VII. **A.**

B. The limit at which the third reduction amounts to $0''.1$ is ⅔ the h angle of Table VII. **B.**

The limit at which the third reduction amounts to $0''.01$ is ½ the h angle of Table VII. **B.**

Table VI. Logarithms of *m* and *n*.

$$m = \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

$$n = \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}$$

log *m*

24^m	25^m	26^m	27^m	28^m	29^m
3.05306	3.08848	3.12252	3.15526	3.18681	3.21725
.05366	.08906	.12307	.15580	.18733	.21775
.05426	.08964	.12363	.15633	.18784	.21825
.05487	.09022	.12418	.15666	.18836	.21875
.05547	.09079	.12474	.15740	.18887	.21924
.05607	.09137	.12529	.15793	.18939	.21974
.05667	.09195	.12585	.15847	.18990	.22024
.05727	.09252	.12640	.15900	.19042	.22073
.05787	.09310	.12695	.15953	.19093	.22123
.05847	.09367	.12751	.16007	.19145	.22172
.05907	.09425	.12806	.16060	.19196	.22222
.05966	.09482	.12861	.16113	.19247	.22272
.06026	.09540	.12916	.16166	.19299	.22321
.06086	.09597	.12971	.16210	.19350	.22371
.06146	.09655	.13026	.16373	.19401	.22420
.06205	.09712	.13081	.16326	.19452	.22470
.06265	.09769	.13136	.16379	.19503	.22519
.06324	.09826	.13191	.16432	.19554	.22568
.06384	.09883	.13246	.16485	.19606	.22618
.06444	.09941	.13301	.16538	.19657	.22667
.06503	.09998	.13356	.16591	.19708	.22716
.06562	.10055	.13411	.16643	.19759	.22766
.06622	.10112	.13466	.16696	.19810	.22815
.06681	.10169	.13521	.16749	.19861	.22864
.06740	.10226	.13576	.16802	.19912	.22913
.06800	.10283	.13631	.16855	.19962	.22963
.06859	.10340	.13686	.16907	.20013	.23012
.06918	.10396	.13740	.16960	.20064	.23061
.06977	.10453	.13795	.17013	.20115	.23110
.07036	.10510	.13850	.17066	.20166	.23159
.07095	.10567	.13904	.17118	.20216	.23208
.07154	.10623	.13959	.17170	.20267	.23257
.07213	.10680	.14013	.17223	.20318	.23306
.07272	.10737	.14068	.17275	.20369	.23355
.07331	.10793	.14122	.17327	.20419	.23404
.07389	.10850	.14177	.17380	.20470	.23453
.07448	.10906	.14231	.17433	.20520	.23501
.07507	.10963	.14285	.17485	.20571	.23550
.07566	.11019	.14340	.17538	.20621	.23599
.07625	.11076	.14394	.17590	.20672	.23648
.07683	.11132	.14448	.17642	.20722	.23697
.07742	.11188	.14502	.17694	.20772	.23745
.07801	.11245	.14557	.17746	.20822	.23794
.07859	.11301	.14611	.17799	.20873	.23843
.07918	.11357	.14665	.17851	.20924	.23891
.07976	.11413	.14719	.17903	.20974	.23940
.08035	.11469	.14773	.17955	.21024	.23988
.08093	.11525	.14827	.18007	.21075	.24037
.08151	.11582	.14881	.18059	.21125	.24086
.08210	.11638	.14935	.18111	.21175	.24134
.08268	.11694	.14989	.18163	.21225	.24182
.08326	.11750	.15043	.18215	.21275	.24231
.08384	.11805	.15096	.18267	.21325	.24279
.08442	.11861	.15150	.18319	.21375	.24328
.08501	.11917	.15204	.18371	.21425	.24376
.08559	.11973	.15258	.18422	.21475	.24424
.08617	.12029	.15312	.18474	.21525	.24473
.08675	.12085	.15365	.18526	.21575	.24521
.08733	.12140	.15419	.18578	.21625	.24569
.08791	.12196	.15472	.18629	.21675	.24617
3.08848	3.12252	3.15526	3.18681	3.21725	3.24665

t		log *n*	*t*		log *n*
m	*s*		*m*	*s*	
0	0	— ∞	**20**	0	0.1742
1	0	4.9706		10	0.1886
2	0	6.1747		20	0.2029
3	0	6.8791		30	0.2170
4	0	7.3788		40	0.2311
5	0	7.7665		50	0.2450
6	0	8.0832	**21**	0	0.2589
7	0	8.3509		10	0.2726
8	0	8.5829		20	0.2862
9	0	8.7875		30	0.2997
10	0	8.9705		40	0.3131
11	0	9.1360		50	0.3264
12	0	9.2871	**22**	0	0.3396
	10	9.3111		10	0.3527
	20	9.3347		20	0.3657
	30	9.3580		30	0.3786
	40	9.3810		40	0.3915
	50	9.4037		50	0.4042
13	0	9.4262	**23**	0	0.4168
	10	9.4483		10	0.4293
	20	9.4701		20	0.4418
	30	9.4917		30	0.4541
	40	9.5130		40	0.4664
	50	9.5341		50	0.4786
14	0	9.5549	**24**	0	0.4907
	10	9.5754		10	0.5027
	20	9.5957		20	0.5146
	30	9.6158		30	0.5264
	40	9.6356		40	0.5382
	50	9.6553		50	0.5499
15	0	9.6747	**25**	0	0.5615
	10	9.6939		10	0.5730
	20	9.7128		20	0.5845
	30	9.7316		30	0.5959
	40	9.7502		40	0.6072
	50	9.7686		50	0.6184
16	0	9.7867	**26**	0	0.6296
	10	9.8047		10	0.6407
	20	9.8225		20	0.6517
	30	9.8402		30	0.6626
	40	9.8576		40	0.6735
	50	9.8749		50	0.6843
17	0	9.8920	**27**	0	0.6951
	10	9.9089		10	0.7057
	20	9.9257		20	0.7164
	30	9.9423		30	0.7269
	40	9.9588		40	0.7374
	50	9.9751		50	0.7478
18	0	9.9913	**28**	0	0.7582
	10	0.0072		10	0.7685
	20	0.0231		20	0.7787
	30	0.038		30	0.7889
	40	0.0544		40	0.7990
	50	0.0698		50	0.8090
19	0	0.0851	**29**	0	0.8190
	10	0.1003		10	0.8290
	20	0.1153		20	0.8389
	30	0.1302		30	0.8487
	40	0.1450		40	0.8585
	50	0.1597		50	0.8682
20	0	0.1742	**30**	0	0.8789

TABLE VIII.

For reducing transits over several threads to a common instant.

	Arg. Sidereal time.		Proportional parts of log *k*.									Arg. Mean tim
I	*x*	Log *k*	1ˢ	2ˢ	3ˢ	4ˢ	5ˢ	6ˢ	7ˢ	8ˢ	9ˢ	Log *k*
m s	s											
8 0	1.46	0.000 2646	11	22	33	44	55	66	77	88	100	0.000 2660
10	1.55	2757	11	23	34	45	56	68	79	90	102	2772
20	1.65	2871	11	23	34	46	57	69	81	92	104	2886
30	1.75	2987	12	23	35	47	58	70	82	94	106	3003
40	1.86	3105	12	24	36	48	60	72	84	96	108	3122
50	1.97	3225	12	25	37	49	62	74	86	98	110	3243
9 0	2.08	3348	13	25	37	50	63	75	88	100	112	3367
10	2.20	3473	13	26	38	51	64	76	89	102	114	3492
20	2.32	3601	13	26	38	51	64	77	90	103	116	3621
30	2.45	3731	13	26	39	52	65	78	92	105	118	3751
40	2.58	3863	13	26	40	53	66	79	93	107	120	3884
50	2.72	3997	14	27	41	54	68	82	95	109	122	4019
10 0	2.86	4134	14	27	41	55	69	83	96	110	124	4156
10	3.00	4272	14	28	42	56	70	84	98	112	127	4296
20	3.15	4414	14	28	43	57	71	85	100	114	129	4438
30	3.30	4557	15	29	44	58	73	87	102	117	131	4582
40	3.46	4703	15	29	44	59	74	88	103	118	133	4729
50	3.63	4851	15	30	45	60	75	90	105	120	135	4878
11 0	3.80	5001	15	31	46	61	76	92	107	122	137	5029
10	3.98	5154	15	31	46	62	77	93	108	124	139	5182
20	4.16	5309	16	31	47	63	78	94	110	126	141	5338
30	4.34	5466	16	32	48	64	80	96	112	128	143	5496
40	4.53	5626	16	32	49	64	81	97	113	129	145	5657
50	4.73	5788	16	32	49	65	82	98	114	131	147	5819
12 0	4.93	5952	16	33	50	66	83	99	116	133	150	5985
10	5.14	6118	17	34	51	67	84	101	118	135	152	6152
20	5.36	6287	17	34	51	68	85	102	120	137	154	6322
30	5.58	6458	17	34	52	69	86	104	121	138	156	6493
40	5.80	6631	18	35	53	70	88	105	123	141	158	6668
50	6.04	6807	18	35	53	71	89	106	124	142	160	6844

TABLE VIII. A.

For correcting the mean log *k found from the preceding table.*

Mean log *k*	Correction.
0.0001	0.0000000.1
2	0.3
3	0.8
4	1.4
5	2.2
6	3.3
7	4.3

TABLE IX. Probability of Errors.

(Method of Least Squares.)

$$\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-tt}\,dt$$

Θ (t)	Diff.	t	Θ (t)	Diff.	t	Θ (t)	Diff.	t	Θ (t)	Diff.
0.00000	1128	0.50	0.52050	874	1.00	0.84270	411	1.50	0.96611	117
.01128	1128	0.51	.52924	866	1.01	.84681	403	1.51	.96728	113
.02256	1128	0.52	.53790	856	1.02	.85084	394	1.52	.96841	111
.03384	1127	0.53	.54646	848	1.03	.85478	387	1.53	.96952	107
.04511	1126	0.54	.55494	838	1.04	.85865	379	1.54	.97059	103
0.05637	1125	0.55	0.56332	830	1.05	0.86244	370	1.55	0.97162	101
.06762	1124	0.56	.57162	820	1.06	.86614	363	1.56	.97263	97
.07886	1122	0.57	.57982	810	1.07	.86977	356	1.57	.97360	95
.09008	1120	0.58	.58792	802	1.08	.87333	347	1.58	.97455	91
.10128	1118	0.59	.59594	792	1.09	.87680	341	1.59	.97546	89
0.11246	1116	0.60	0.60386	782	1.10	0.88021	332	1.60	0.97635	86
.12362	1114	0.61	.61168	773	1.11	.88353	326	1.61	.97721	83
.13476	1111	0.62	.61941	764	1.12	.88679	318	1.62	.97804	80
.14587	1108	0.63	.62705	754	1.13	.88997	311	1.63	.97884	78
.15695	1105	0.64	.63459	744	1.14	.89308	304	1.64	.97962	76
0.16800	1101	0.65	0.64203	735	1.15	0.89612	298	1.65	0.98038	72
.17901	1098	0.66	.64938	725	1.16	.89910	290	1.66	.98110	71
.18999	1095	0.67	.65663	715	1.17	.90200	284	1.67	.98181	68
.20094	1090	0.68	.66378	706	1.18	.90484	277	1.68	.98249	66
.21184	1086	0.69	.67084	696	1.19	.90761	270	1.69	.98315	64
0.22270	1082	0.70	0.67780	687	1.20	0.91031	265	1.70	0.98379	62
.23352	1078	0.71	.68467	676	1.21	.91296	257	1.71	.98441	59
.24430	1072	0.72	.69143	667	1.22	.91553	252	1.72	.98500	58
.25502	1068	0.73	.69810	658	1.23	.91805	246	1.73	.98558	55
.26570	1063	0.74	.70468	648	1.24	.92051	239	1.74	.98613	54
0.27633	1057	0.75	0.71116	638	1.25	0.92290	234	1.75	0.98667	52
.28690	1052	0.76	.71754	628	1.26	.92524	227	1.76	.98719	50
.29742	1046	0.77	.72382	619	1.27	.92751	222	1.77	.98769	48
.30788	1040	0.78	.73001	609	1.28	.92973	217	1.78	.98817	47
.31828	1035	0.79	.73610	600	1.29	.93190	211	1.79	.98864	45
0.32863	1028	0.80	0.74210	590	1.30	0.93401	205	1.80	0.98909	43
.33891	1022	0.81	.74800	581	1.31	.93606	201	1.81	.98952	42
.34913	1015	0.82	.75381	571	1.32	.93807	195	1.82	.98994	41
.35928	1008	0.83	.75952	562	1.33	.94002	189	1.83	.99035	39
.36936	1002	0.84	.76514	553	1.34	.94191	185	1.84	.99074	37
0.37938	995	0.85	0.77067	543	1.35	0.94376	180	1.85	0.99111	36
.38933	988	0.86	.77610	534	1.36	.94556	175	1.86	.99147	35
.39921	980	0.87	.78144	525	1.37	.94731	171	1.87	.99182	34
.40901	973	0.88	.78669	515	1.38	.94902	165	1.88	.99216	32
.41874	965	0.89	.79184	507	1.39	.95067	162	1.89	.99248	31
0.42839	958	0.90	0.79691	497	1.40	0.95229	156	1.90	0.99279	30
.43797	950	0.91	.80188	489	1.41	.95385	153	1.91	.99309	29
.44747	942	0.92	.80677	479	1.42	.95538	148	1.92	.99338	28
.45689	934	0.93	.81156	471	1.43	.95686	144	1.93	.99366	26
.46623	925	0.94	.81627	462	1.44	.95830	140	1.94	.99392	26
0.47548	918	0.95	0.82089	453	1.45	0.95970	135	1.95	0.99418	25
.48466	909	0.96	.82542	445	1.46	.96105	132	1.96	.99443	23
.49375	900	0.97	.82987	436	1.47	.96237	128	1.97	.99466	23
.50275	892	0.98	.83423	428	1.48	.96365	125	1.98	.99489	22
.51167	883	0.99	.83851	419	1.49	.96490	121	1.99	.99511	21
0.52050		1.00	0.84270		1.50	0.96611		2.00	0.99532	

TABLE VIII.

For reducing transits over several threads to a common instant.

	Arg. Sidereal time.		Proportional parts of log *k*.									Arg. Mea
I	*x*	Log *k*	1^s	2^s	3^s	4^s	5^s	6^s	7^s	8^s	9^s	Log k
m s	s											
8 0	1.46	0.000 2646	11	22	33	44	55	66	77	88	100	0.000 266
10	1.55	2757	11	23	34	45	56	68	79	90	102	277
20	1.65	2871	11	23	34	46	57	69	81	92	104	288
30	1.75	2987	12	23	35	47	58	70	82	94	106	300
40	1.86	3105	12	24	36	48	60	72	84	96	108	312
50	1.97	3225	12	25	37	49	62	74	86	98	110	324
9 0	2.08	3348	13	25	37	50	63	75	88	100	112	3367
10	2.20	3473	13	26	38	51	64	76	89	102	114	349
20	2.32	3601	13	26	38	51	64	77	90	103	116	3621
30	2.45	3731	13	26	39	52	65	78	92	105	118	3751
40	2.58	3863	13	26	40	53	66	79	93	107	120	3884
50	2.72	3997	14	27	41	54	68	82	95	109	122	4019
10 0	2.86	4134	14	27	41	55	69	83	96	110	124	4156
10	3.00	4272	14	28	42	56	70	84	98	112	127	4296
20	3.15	4414	14	28	43	57	71	85	100	114	129	4438
30	3.30	4557	15	29	44	58	73	87	102	117	131	4582
40	3.46	4703	15	29	44	59	74	88	103	118	133	4729
50	3.63	4851	15	30	45	60	75	90	105	120	135	4878
11 0	3.80	5001	15	31	46	61	76	92	107	122	137	5029
10	3.98	5154	15	31	46	62	77	93	108	124	139	5182
20	4.16	5309	16	31	47	63	78	94	110	126	141	5338
30	4.34	5466	16	32	48	64	80	96	112	128	143	5496
40	4.53	5626	16	32	49	64	81	97	113	129	145	5657
50	4.73	5788	16	32	49	65	82	98	114	131	147	5819
12 0	4.93	5952	16	33	50	66	83	99	116	133	150	598
10	5.14	6118	17	34	51	67	84	101	118	135	152	615
20	5.36	6287	17	34	51	68	85	102	120	137	154	632
30	5.58	6458	17	34	52	69	86	104	121	138	156	649
40	5.80	6631	18	35	53	70	88	105	123	141	158	6668
50	6.04	6807	18	35	53	71	89	106	124	142	160	6844

TABLE VIII. A.

For correcting the mean log *k found from the preceding table.*

Mean log *k*	Correction.
0.0001	0.0000000.1
2	0.3
3	0.8
4	1.4
5	2.2
6	3.3
7	4.3

TABLE IX. Probability of Errors.

(Method of Least Squares.)

$$\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-tt}\, dt$$

$\Theta(t)$	Diff.	t	$\Theta(t)$	Diff.	t	$\Theta(t)$	Diff.	t	$\Theta(t)$	Diff.
0.00000	1128	0.50	0.52050	874	1.00	0.84270	411	1.50	0.96611	117
.01128	1128	0.51	.52924	866	1.01	.84681	403	1.51	.96728	113
.02256	1128	0.52	.53790	856	1.02	.85084	394	1.52	.96841	111
.03384	1127	0.53	.54646	848	1.03	.85478	387	1.53	.96952	107
.04511	1126	0.54	.55494	838	1.04	.85865	379	1.54	.97059	103
0.05637	1125	0.55	0.56332	830	1.05	0.86244	370	1.55	0.97162	101
.06762	1124	0.56	.57162	820	1.06	.86614	363	1.56	.97263	97
.07886	1122	0.57	.57982	810	1.07	.86977	356	1.57	.97360	95
.09008	1120	0.58	.58792	802	1.08	.87333	347	1.58	.97455	91
.10128	1118	0.59	.59594	792	1.09	.87680	341	1.59	.97546	89
0.11246	1116	0.60	0.60386	782	1.10	0.88021	332	1.60	0.97635	86
.12362	1114	0.61	.61168	773	1.11	.88353	326	1.61	.97721	83
.13476	1111	0.62	.61941	764	1.12	.88679	318	1.62	.97804	80
.14587	1108	0.63	.62705	754	1.13	.88997	311	1.63	.97884	78
.15695	1105	0.64	.63459	744	1.14	.89308	304	1.64	.97962	76
0.16800	1101	0.65	0.64203	735	1.15	0.89612	298	1.65	0.98038	72
.17901	1098	0.66	.64938	725	1.16	.89910	290	1.66	.98110	71
.18999	1095	0.67	.65663	715	1.17	.90200	284	1.67	.98181	68
.20094	1090	0.68	.66378	706	1.18	.90484	277	1.68	.98249	66
.21184	1086	0.69	.67084	696	1.19	.90761	270	1.69	.98315	64
0.22270	1082	0.70	0.67780	687	1.20	0.91031	265	1.70	0.98379	62
.23352	1078	0.71	.68467	676	1.21	.91296	257	1.71	.98441	59
.24430	1072	0.72	.69143	667	1.22	.91553	252	1.72	.98500	58
.25502	1068	0.73	.69810	658	1.23	.91805	246	1.73	.98558	55
.26570	1063	0.74	.70468	648	1.24	.92051	239	1.74	.98613	54
0.27633	1057	0.75	0.71116	638	1.25	0.92290	234	1.75	0.98667	52
.28690	1052	0.76	.71754	628	1.26	.92524	227	1.76	.98719	50
.29742	1046	0.77	.72382	619	1.27	.92751	222	1.77	.98769	48
.30788	1040	0.78	.73001	609	1.28	.92973	217	1.78	.98817	47
.31828	1035	0.79	.73610	600	1.29	.93190	211	1.79	.98864	45
0.32863	1028	0.80	0.74210	590	1.30	0.93401	205	1.80	0.98909	43
.33891	1022	0.81	.74800	581	1.31	.93606	201	1.81	.98952	42
.34913	1015	0.82	.75381	571	1.32	.93807	195	1.82	.98994	41
.35928	1008	0.83	.75952	562	1.33	.94002	189	1.83	.99035	39
.36936	1002	0.84	.76514	553	1.34	.94191	185	1.84	.99074	37
0.37938	995	0.85	0.77067	543	1.35	0.94376	180	1.85	0.99111	36
.38933	988	0.86	.77610	534	1.36	.94556	175	1.86	.99147	35
.39921	980	0.87	.78144	525	1.37	.94731	171	1.87	.99182	34
.40901	973	0.88	.78669	515	1.38	.94902	165	1.88	.99216	32
.41874	965	0.89	.79184	507	1.39	.95067	162	1.89	.99248	31
0.42839	958	0.90	0.79691	497	1.40	0.95229	156	1.90	0.99279	30
.43797	950	0.91	.80188	489	1.41	.95385	153	1.91	.99309	29
.44747	942	0.92	.80677	479	1.42	.95538	148	1.92	.99338	28
.45689	934	0.93	.81156	471	1.43	.95686	144	1.93	.99366	26
.46623	925	0.94	.81627	462	1.44	.95830	140	1.94	.99392	26
0.47548	918	0.95	0.82089	453	1.45	0.95970	135	1.95	0.99418	25
.48466	909	0.96	.82542	445	1.46	.96105	132	1.96	.99443	23
.49375	900	0.97	.82987	436	1.47	.96237	128	1.97	.99466	23
.50275	892	0.98	.83423	428	1.48	.96365	125	1.98	.99489	22
.51167	883	0.99	.83851	419	1.49	.96490	121	1.99	.99511	21
0.52050		1.00	0.84270		1.50	0.96611		2.00	0.99532	

TABLE X. Peirce's Criterion.

VALUES OF x^2 FOR $\mu = 1$.

m	n							
	1	2	3	4	5	6	7	8
3	1.480	......	......	......	......	......	......	......
4	1.912	1.163	......	......	......	......	......	......
5	2.278	1.439	......	......	......	......	......	......
6	2.592	1.687	1.208	......	......	......	......	......
7	2.866	1.910	1.409	1.045	......	......	......	......
8	3.109	2.112	1.589	1.229	......	......	......	......
9	3.327	2.295	1.753	1.388	1.091	......	......	.....
10	3.526	2.464	1.904	1.531	1.242	......	......	......
11	3.707	2.621	2.045	1.662	1.373	1.122	......	......
12	3.875	2.766	2.176	1.785	1.492	1.249	1.018	......
13	4.029	2.902	2.299	1.901	1.604	1.362	1.145	......
14	4.173	3.030	2.416	2.009	1.709	1.465	1.255	1.053
15	4.309	3.151	2.526	2.111	1.807	1.561	1.354	1.163
16	4.436	3.264	2.630	2.207	1.898	1.651	1.445	1.259
17	4.555	3.371	2.729	2.300	1.985	1.736	1.529	1.347
18	4.668	3.475	2.824	2.389	2.069	1.817	1.609	1.428
19	4.776	3.571	2.914	2.474	2.150	1.895	1.685	1.504
20	4.878	3.664	3.001	2.556	2.227	1.970	1.757	1.576
21	4.975	3.755	3.084	2.634	2.301	2.041	1.827	1.644
22	5.068	3.840	3.164	2.709	2.373	2.109	1.893	1.710
23	5.157	3.923	3.240	2.782	2.442	2.176	1.957	1.773
24	5.242	4.002	3.315	2.852	2.509	2.240	2.019	1.833
25	5.324	4.078	3.387	2.920	2.573	2.302	2.079	1.892
26	5.403	4.151	3.456	2.986	2.636	2.362	2.137	1.948
27	5.479	4.222	3.523	3.049	2.697	2.420	2.194	2.003
28	5.552	4.291	3.588	3.111	2.756	2.477	2.249	2.056
29	5.622	4.358	3.651	3.171	2.813	2.532	2.302	2.108
30	5.690	4.422	3.712	3.229	2.869	2.586	2.354	2.158
31	5.756	4.484	3.772	3.285	2.923	2.638	2.404	2.207
32	5.820	4.545	3.829	3.340	2.976	2.689	2.454	2.255
33	5.882	4.604	3.884	3.394	3.028	2.738	2.502	2.302
34	5.942	4.661	3.939	3.446	3.078	2.787	2.549	2.347
35	6.001	4.717	3.992	3.497	3.127	2.834	2.594	2.392
36	6.058	4.771	4.044	3.547	3.174	2.880	2.639	2.436
37	6.113	4.823	4.095	3.595	3.221	2.926	2.683	2.478
38	6.167	4.874	4.144	3.643	3.267	2.970	2.726	2.520
39	6.219	4.925	4.192	3.689	3.312	3.013	2.768	2.561
40	6.270	4.974	4.239	3.734	3.356	3.055	2.809	2.601
41	6.320	5.022	4.285	3.779	3.398	3.097	2.849	2.640
42	6.369	5.069	4.331	3.822	3.440	3.138	2.888	2.678
43	6.416	5.114	4.375	3.865	3.481	3.178	2.927	2.716
44	6.463	5.159	4.418	3.906	3.521	3.217	2.965	2.753
45	6.508	5.202	4.460	3.947	3.561	3.255	3.002	2.789
46	6.552	5.245	4.501	3.987	3.600	3.293	3.039	2.825
47	6.596	5.287	4.542	4.026	3.638	3.330	3.075	2.860
48	6.639	5.328	4.581	4.065	3.675	3.366	3.110	2.894
49	6.681	5.368	4.620	4.103	3.712	3.401	3.145	2.928
50	6.720	5.408	4.657	4.140	3.748	3.436	3.179	2.962
51	6.761	5.447	4.695	4.176	3.784	3.471	3.213	2.994
52	6.800	5.484	4.732	4.212	3.819	3.505	3.246	3.027
53	6.838	5.522	4.768	4.247	3.853	3.538	3.279	3.059
54	6.876	5.559	4.804	4.282	3.887	3.571	3.311	3.090
55	6.913	5.595	4.839	4.316	3.920	3.603	3.342	3.121
56	6.950	5.630	4.873	4.349	3.952	3.635	3.373	3.151
57	6.986	5.665	4.907	4.382	3.984	3.666	3.404	3.181
58	7.021	5.699	4.941	4.415	4.016	3.697	3.434	3.210
59	7.056	5.733	4.974	4.447	4.047	3.728	3.463	3.239
60	7.090	5.766	5.006	4.478	4.078	3.758	3.492	3.268

TABLE IX.A. Probability of Errors.

(Method of Least Squares.)

$$\Theta(\rho t') = \frac{2}{\sqrt{\pi}}\int_0^{\rho t'} e^{-t^2}\,dt \qquad t' = \frac{a}{r}$$

Θ (ρt')	Diff.	t'	Θ (ρt')	Diff.	t'	Θ (ρt')	Diff.	t'	Θ (ρt')	Diff.
0.82266	215	2.50	0.90825	129	3.00	0.95698	69	3.50	0.98176	306
.82481	214	2.51	.90954	128	3.01	.95767	68	3.60	.98482	261
.82695	212	2.52	.91082	126	3.02	.95835	67	3.70	.98743	219
.82907	210	2.53	.91208	124	3.03	.95902	66	3.80	.98962	185
.83117	207	2.54	.91332	124	3.04	.95968	65	3.90	.99147	155
0.83324	206	2.55	0.91456	122	3.05	0.96033	65	4.00	0.99302	129
.83530	204	2.56	.91578	120	3.06	.96098	63	4.10	.99431	108
.83734	202	2.57	.91698	119	3.07	.96161	63	4.20	.99539	88
.83936	201	2.58	.91817	118	3.08	.96224	62	4.30	.99627	73
.84137	198	2.59	.91935	116	3.09	.96286	60	4.40	.99700	60
0.84335	196	2.60	0.92051	115	3.10	0.96346	60	4.50	0.99760	48
.84531	195	2.61	.92166	114	3.11	.96406	60	4.60	.99808	40
.84726	193	2.62	.92280	112	3.12	.96466	58	4.70	.99848	31
.84919	190	2.63	.92392	111	3.13	.96524	58	4.80	.99879	26
.85109	189	2.64	.92503	110	3.14	.96582	56	4.90	.99905	21
0.85298	188	2.65	0.92613	108	3.15	0.96638	56	5.00	0.99926	
.85486	185	2.66	.92721	107	3.16	.96694	55	∞	1.00000	
.85671	183	2.67	.92828	106	3.17	.96749	55			
.85854	182	2.68	.92934	104	3.18	.96804	53			
.86036	180	2.69	.93038	103	3.19	.96857	53			
0.86216	178	2.70	0.93141	102	3.20	0.96910	52			
.86394	176	2.71	.93243	101	3.21	.96962	51			
.86570	175	2.72	.93344	99	3.22	.97013	51			
.86745	172	2.73	.93443	98	3.23	.97064	50			
.86917	171	2.74	.93541	97	3.24	.97114	49			
0.87088	170	2.75	0.93638	96	3.25	0.97163	48			
.87258	167	2.76	.93734	94	3.26	.97211	48			
.87425	166	2.77	.93828	94	3.27	.97259	47			
.87591	164	2.78	.93922	92	3.28	.97306	46			
.87755	163	2.79	.94014	91	3.29	.97352	45			
0.87918	160	2.80	0.94105	90	3.30	0.97397	45			
.88078	159	2.81	.94195	89	3.31	.97442	44			
88237	158	2.82	.94284	87	3.32	.97486	44			
.88395	155	2.83	.94371	87	3.33	.97530	43			
.88550	155	2.84	.94458	85	3.34	.97573	42			
0.88705	152	2.85	0.94543	84	3.35	0.97615	42			
.88857	151	2.86	.94627	84	3.36	.97657	41			
.89008	149	2.87	.94711	82	3.37	.97698	40			
.89157	147	2.88	.94793	81	3.38	.97738	40			
.89304	146	2.89	.94874	80	3.39	.97778	39			
0.89450	145	2.90	0.94954	79	3.40	0.97817	38			
.89595	143	2.91	.95033	78	3.41	.97855	38			
.89738	141	2.92	.95111	76	3.42	.97893	37			
.89879	140	2.93	.95187	76	3.43	.97930	37			
.90019	138	2.94	.95263	75	3.44	.97967	36			
0.90157	136	2.95	0.95338	74	3.45	0.98003	36			
.90293	135	2.96	.95412	73	3.46	.98039	35			
.90428	134	2.97	.95485	72	3.47	.98074	35			
.90562	132	2.98	.95557	71	3.48	.98109	34			
.90694	131	2.99	.95628	70	3.49	.98143	33			
0.90825		3.00	0.95698		3.50	0.98176				

TABLE X. A. Peirce's Criterion.

Log *T*.

m	*n*							
	1	2	3	4	5	6	7	8
2	9.3979	....	....	....	....	....	....	....
3	9.1707	9.5853	....	....	....	....	....	....
4	9.0231	9.3979	9.6744	....	....	....	....	....
5	8.9134	9.2693	9.5129	9.7283	....	....	....	....
6	8.8259	9.1707	9.3979	9.5853	9.7652	....	....	....
7	8.7532	9.0906	9.3080	9.4810	9.6362	9.7922	....	....
8	8.6910	9.0231	9.2338	9.3979	9.5403	9.6744	9.8130	....
9	8.6365	8.9648	9.1707	9.3287	9.4630	9.5853	9.7042	9.8296
10	8.5882	8.9134	9.1157	9.2693	9.3979	9.5129	9.6210	9.7253
11	8.5447	8.8675	9.0669	9.2172	9.3417	9.4514	9.5527	9.6501
12	8.5051	8.8259	9.0231	9.1707	9.2921	9.3979	9.4943	9.5853
13	8.4689	8.7881	8.9834	9.1288	9.2477	9.3506	9.4433	9.5298
14	8.4355	8.7532	8.9470	9.0906	9.2074	9.3080	9.3979	9.4810
15	8.4044	8.7210	8.9134	9.0555	9.1707	9.2693	9.3570	9.4374
16	8.3754	8.6910	8.8822	9.0231	9.1368	9.2338	9.3197	9.3979
17	8.3483	8.6629	8.8532	8.9930	9.1055	9.2011	9.2854	9.3619
18	8.3227	8.6365	8.8259	8.9648	9.0762	9.1707	9.2537	9.3287
19	8.2986	8.6117	8.8003	8.9383	9.0489	9.1423	9.2242	9.2980
20	8.2757	8.5882	8.7761	8.9134	9.0231	9.1157	9.1966	9.2693
21	8.2540	8.5659	8.7532	8.8898	8.9988	9.0906	9.1707	9.2424
22	8.2333	8.5447	8.7315	8.8675	8.9758	9.0669	9.1463	9.2172
23	8.2136	8.5245	8.7107	8.8462	8.9540	9.0445	9.1231	9.1933
24	8.1947	8.5051	8.6910	8.8259	8.9332	9.0231	9.1012	9.1707
25	8.1766	8.4867	8.6721	8.8066	8.9134	9.0028	9.0803	9.1492
26	8.1592	8.4689	8.6539	8.7881	8.8944	8.9834	9.0604	9.1288
27	8.1425	8.4519	8.6365	8.7703	8.8763	8.9648	9.0414	9.1093
28	8.1264	8.4354	8.6198	8.7532	8.8588	8.9470	9.0231	9.0906
29	8.1109	8.4197	8.6037	8.7368	8.8421	8.9299	9.0056	9.0727
30	8.0959	8.4044	8.5882	8.7210	8.8259	8.9134	8.9888	9.0555
31	8.0814	8.3897	8.5732	8.7057	8.8104	8.8975	8.9726	9.0390
32	8.0674	8.3754	8.5587	8.6910	8.7954	8.8822	8.9571	9.0231
33	8.0538	8.3617	8.5447	8.6767	8.7809	8.8675	8.9420	9.0078
34	8.0407	8.3483	8.5311	8.6629	8.7668	8.8532	8.9275	8.9930
35	8.0279	8.3353	8.5179	8.6495	8.7532	8.8393	8.9134	8.9786
36	8.0155	8.3227	8.5051	8.6365	8.7400	8.8259	8.8998	8.9648
37	8.0034	8.3105	8.4927	8.6239	8.7272	8.8129	8.8865	8.9513
38	7.9917	8.2986	8.4807	8.6117	8.7148	8.8003	8.8737	8.9383
39	7.9803	8.2870	8.4689	8.5998	8.7027	8.7881	8.8613	8.9257
40	7.9691	8.2757	8.4575	8.5882	8.6910	8.7761	8.8492	8.9134
41	7.9583	8.2647	8.4463	8.5769	8.6795	8.7645	8.8374	8.9014
42	7.9477	8.2540	8.4355	8.5659	8.6684	8.7532	8.8259	8.8898
43	7.9373	8.2435	8.4249	8.5552	8.6575	8.7422	8.8148	8.8785
44	7.9272	8.2333	8.4145	8.5447	8.6469	8.7315	8.8039	8.8675
45	7.9174	8.2233	8.4044	8.5345	8,6365	8.7210	8.7933	8.8567
46	7.9077	8.2136	8.3945	8.5245	8.6264	8.7107	8.7829	8.8462
47	7.8983	8.2040	8.3849	8.5147	8.6165	8.7007	8.7728	8.8360
48	7.8890	8.1947	8.3754	8.5051	8.6069	8.6910	8.7629	8.8259
49	7.8800	8.1855	8.3662	8.4958	8.5974	8.6814	8.7532	8.8162
50	7.8711	8.1766	8.3572	8.4867	8.5882	8.6721	8.7438	8.8066
51	7.8624	8.1678	8.3483	8.4777	8.5791	8.6629	8.7345	8.7972
52	7.8539	8.1592	8.3396	8.4689	8.5703	8.6539	8.7254	8.7881
53	7.8456	8.1508	8.3311	8.4603	8.5616	8.6451	8.7166	8.7791
54	7.8374	8.1425	8.3227	8.4519	8.5530	8.6365	8.7079	8.7703
55	7.8293	8.1344	8.3145	8.4436	8.5447	8.6281	8.6993	8.7617
56	7.8214	8.1264	8.3065	8.4355	8.5365	8.6198	8.6910	8.7532
57	7.8137	8.1186	8.2986	8.4275	8.5284	8.6117	8.6828	8.7449
58	7.8060	8.1109	8.2908	8.4197	8.5205	8.6037	8.6747	8.7368
59	7.7986	8.1033	8.2832	8.4120	8.5128	8.5959	8.6668	8.7288
60	7.7912	8.0959	8.2757	8.4044	8.5051	8.5882	8.6590	8.7210

TABLE X. A. Peirce's Criterion.

Log *T*.

m	n								
	1	2	3	4	5	6	7	8	9
61	7.7840	8.0886	8.2684	8.3970	8.4977	8.5806	8.6514	8.7133	8.7684
62	7.7768	8.0814	8.2611	8.3897	8.4903	8.5732	8.6439	8.7057	8.7607
63	7.7698	8.0744	8.2540	8.3825	8.4830	8.5659	8.6365	8.6983	8.7532
64	7.7629	8.0674	8.2470	8.3754	8.4759	8.5587	8.6293	8.6910	8.7458
65	7.7562	8.0606	8.2401	8.3685	8.4689	8.5516	8.6222	8.6838	8.7386
66	7.7495	8.0538	8.2333	8.3617	8.4620	8.5447	8.6152	8.6767	8.7315
67	7.7429	8.0472	8.2266	8.3549	8.4552	8.5378	8.6082	8.6697	8.7244
68	7.7364	8.0407	8.2200	8.3483	8.4485	8.5311	8.6015	8.6629	8.7175
69	7.7300	8.0342	8.2136	8.3418	8.4420	8.5245	8.5948	8.6562	8.7107
70	7.7237	8.0279	8.2072	8.3353	8.4355	8.5179	8.5882	8.6495	8.7040
71	7.7175	8.0217	8.2009	8.3290	8.4291	8.5115	8.5817	8.6430	8.6975
72	7.7114	8.0155	8.1947	8.3227	8.4228	8.5051	8.5753	8.6365	8.6910
73	7.7054	8.0094	8.1886	8.3166	8.4166	8.4989	8.5690	8.6302	8.6846
74	7.6994	8.0034	8.1825	8.3106	8.4105	8.4927	8.5628	8.6239	8.6783
75	7.6936	7.9975	8.1766	8.3045	8.4044	8.4867	8.5567	8.6178	8.6721
76	7.6878	7.9917	8.1707	8.2986	8.3985	8.4807	8.5506	8.6117	8.6659
77	7.6820	7.9859	8.1649	8.2928	8.3926	8.4747	8.5447	8.6057	8.6599
78	7.6764	7.9803	8.1592	8.2870	8.3868	8.4689	8.5388	8.5998	8.6539
79	7.6708	7.9747	8.1536	8.2813	8.3811	8.4632	8.5330	8.5939	8.6481
80	7.6653	7.9691	8.1480	8.2757	8.3754	8.4575	8.5273	8.5882	8.6423
81	7.6599	7.9637	8.1425	8.2702	8.3699	8.4519	8.5216	8.5825	8.6365
82	7.6546	7.9583	8.1371	8.2647	8.3644	8.4463	8.5161	8.5769	8.6309
83	7.6493	7.9529	8.1317	8.2593	8.3589	8.4409	8.5106	8.5714	8.6253
84	7.6440	7.9477	8.1264	8.2540	8.3536	8.4355	8.5051	8.5659	8.6198
85	7.6389	7.9425	8.1212	8.2487	8.3483	8.4301	8.4998	8.5605	8.6144
86	7.6337	7 9373	8.1160	8.2435	8.3431	8.4249	8.4945	8.5552	8.6090
87	7.6287	7.9322	8.1109	8.2384	8.3379	8.4197	8.4892	8.5499	8.6037
88	7.6237	7.9272	8.1058	8.2333	8.3328	8.4145	8.4841	8.5447	8.5985
89	7.6187	7.9223	8.1008	8.2283	8.3277	8.4094	8.4790	8.5395	8.5933
90	7.6139	7.9174	8.0959	8.2233	8.3227	8.4044	8.4739	8.5345	8.5882

Log *R*.

x	0	1	2	3	4	5	6	7	8	9
1.0	9.5015	9.4992	9.4969	9.4947	9.4924	9.4902	9.4880	9.4857	9.4835	9.4813
1.1	9.4791	9.4769	9.4747	9.4725	9.4704	9.4682	9.4661	9.4639	9.4618	9.4597
1.2	9.4575	9.4554	9.4533	9.4512	9.4491	9.4470	9.4450	9.4429	9.4408	9.4388
1.3	9.4367	9.4347	9.4327	9.4306	9.4286	9.4266	9.4246	9.4226	9.4206	9.4186
1.4	9.4167	9.4147	9.4127	9.4108	9.4088	9.4069	9.4050	9.4030	9.4011	9.3992
1.5	9.3973	9.3954	9.3935	9.3916	9.3897	9.3878	9.3860	9.3841	9.3823	9.3804
1.6	9.3786	9.3767	9.3749	9.3731	9.3712	9.3694	9.3676	9.3658	9.3640	9.3622
1.7	9.3604	9.3587	9.3569	9.3551	9.3534	9.3516	9.3498	9.3481	9.3464	9.3446
1.8	9.3429	9.3412	9.3395	9.3377	9.3360	9.3343	9.3326	9.3310	9.3293	9.3276
1.9	9.3259	9.3242	9.3226	9.3209	9.3193	9.3176	9.3160	9.3143	9.3127	9.3111
2.0	9.3095	9.3078	9.3062	9.3046	9.3030	9.3014	9.2998	9.2982	9.2966	9.2951
2.1	9.2935	9.2919	9.2904	9.2888	9.2872	9.2857	9.2841	9.2826	9.2811	9.2795
2.2	9.2780	9.2765	9.2750	9.2734	9.2719	9.2704	9.2689	9.2674	9.2659	9.2644
2.3	9.2630	9.2615	9.2600	9.2585	9.2571	9.2556	9.2541	9.2527	9.2512	9.2498
2.4	9.2483	9.2469	9.2455	9.2440	9.2426	9.2412	9.2398	9.2383	9.2369	9.2355
2.5	9.2341	9.2327	9.2313	9.2299	9.2285	9.2272	9.2258	9.2244	9.2230	9.2217
2.6	9.2203	9.2189	9.2176	9.2162	9.2149	9.2135	9.2122	9.2108	9.2095	9.2082
2.7	9.2068	9.2055	9.2042	9.2029	9.2016	9.2002	9.1989	9.1976	9.1963	9.1950
2.8	9.1937	9.1924	9.1912	9.1899	9.1886	9.1873	9.1860	9.1848	9.1835	9.1823
2.9	9.1810	9.1797	9.1785	9.1773	9.1760	9.1748	9.1735	9.1723	9.1711	9.1698
3.0	9.1686									

TABLES

FOR CORRECTING

LUNAR DISTANCES.

TABLES

FOR CORRECTING

LUNAR DISTANCES.

TABLE XIV. A.

Correction of the Mean Refraction for the Height of the Barometer.

Barom.	MEAN REFRACTION.																					Barom.
Subtract.	0'		1'		2'		3'		4'		5'		6'		7'		8'		9'		10'	Add.
	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	
	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	
27.50	0	2	5	7	10	12	15	17	20	23	25	28	30	33	35	38	40	43	45	48	51	
27.55	0	2	5	7	10	12	15	17	20	22	25	27	30	32	35	37	40	42	45	47	50	
27.60	0	2	5	7	10	12	14	17	19	22	24	27	29	31	34	36	39	41	44	46	49	
27.65	0	2	5	7	9	12	14	16	19	21	24	26	28	31	33	36	38	40	43	45	48	
27.70	0	2	5	7	9	11	14	16	18	21	23	25	28	30	32	35	37	39	42	44	47	
27.75	0	2	4	7	9	11	13	16	18	20	23	25	27	29	32	34	36	39	41	43	46	
27.80	0	2	4	7	9	11	13	15	18	20	22	24	27	29	31	33	35	38	40	42	45	
27.85	0	2	4	6	9	11	13	15	17	19	22	24	26	28	30	32	35	37	39	41	44	
27.90	0	2	4	6	8	10	13	15	17	19	21	23	25	27	30	32	34	36	38	40	43	
27.95	0	2	4	6	8	10	12	14	16	18	21	23	25	27	29	31	33	35	37	39	42	
28.00	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	41	
28.05	0	2	4	6	8	10	12	14	16	18	20	22	24	25	27	29	31	33	35	37	39	
28.10	0	2	4	6	8	9	11	13	15	17	19	21	23	25	27	29	31	33	34	36	38	
28.15	0	2	4	6	7	9	11	13	15	17	19	20	22	24	26	28	30	32	34	36	37	
28.20	0	2	4	5	7	9	11	13	14	16	18	20	22	24	25	27	29	31	33	35	36	
28.25	0	2	3	5	7	9	10	12	14	16	18	19	21	23	25	26	28	30	32	34	35	
28.30	0	2	3	5	7	8	10	12	14	15	17	19	21	22	24	26	27	29	31	33	34	
28.35	0	2	3	5	7	8	10	12	13	15	17	18	20	22	23	25	27	28	30	32	33	
28.40	0	2	3	5	6	8	10	11	13	14	16	18	19	21	23	24	26	27	29	31	32	
28.45	0	2	3	5	6	8	9	11	12	14	16	17	19	20	22	23	25	27	28	30	31	
28.50	0	1	3	4	6	7	9	10	12	14	15	17	18	20	21	23	24	26	27	29	30	31.50
28.55	0	1	3	4	6	7	9	10	12	13	15	16	17	19	20	22	23	25	26	28	29	31.45
28.60	0	1	3	4	6	7	8	10	11	13	14	15	17	18	20	21	23	24	25	27	28	31.40
28.65	0	1	3	4	5	7	8	9	11	12	14	15	16	18	19	20	22	23	25	26	27	31.35
28.70	0	1	3	4	5	6	8	9	10	12	13	14	16	17	18	20	21	22	24	25	26	31.30
28.75	0	1	2	4	5	6	7	9	10	11	13	14	15	16	18	19	20	21	23	24	25	31.25
28.80	0	1	2	4	5	6	7	8	10	11	12	13	14	16	17	18	19	21	22	23	24	31.20
28.85	0	1	2	3	5	6	7	8	9	10	12	13	14	15	16	17	19	20	21	22	23	31.15
28.90	0	1	2	3	4	5	7	8	9	10	11	12	13	14	16	17	18	19	20	21	22	31.10
28.95	0	1	2	3	4	5	6	7	8	9	11	12	13	14	15	16	17	18	19	20	21	31.05
29.00	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	31.00
29.05	0	1	2	3	4	5	6	7	8	9	10	11	11	12	13	14	15	16	17	18	19	30.95
29.10	0	1	2	3	4	4	5	6	7	8	9	10	11	12	13	14	15	15	16	17	18	30.90
29.15	0	1	2	3	3	4	5	6	7	8	9	9	10	11	12	13	14	15	15	16	17	30.85
29.20	0	1	2	2	3	4	5	6	6	7	8	9	10	10	11	12	13	14	15	15	16	30.80
29.25	0	1	1	2	3	4	4	5	6	7	8	8	9	10	11	11	12	13	14	14	15	30.75
29.30	0	1	1	2	3	3	4	5	6	6	7	8	8	9	10	11	11	12	13	13	14	30.70
29.35	0	1	1	2	3	3	4	5	5	6	7	7	8	9	9	10	10	11	12	13	13	30.65
29.40	0	1	1	2	2	3	4	4	5	5	6	7	7	8	8	9	10	10	11	12	12	30.60
29.45	0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	8	9	9	10	11	11	30.55
29.50	0	0	1	1	2	2	3	3	4	5	5	6	6	7	7	8	8	9	9	10	10	30.50
29.55	0	0	1	1	2	2	3	3	4	4	5	5	5	6	6	7	7	8	8	9	9	30.45
29.60	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6	6	6	7	7	8	8	30.40
29.65	0	0	1	1	1	2	2	2	3	3	4	4	4	5	5	5	6	6	6	7	7	30.35
29.70	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	5	6	6	30.30
29.75	0	0	0	1	1	1	1	2	2	2	3	3	3	3	4	4	4	4	5	5	5	30.25
29.80	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	30.20
29.85	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	30.15
29.90	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	30.10
29.95	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	30.05
30.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30.00
	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	30"	0"	
Subtract.	0'		1'		2'		3'		4'		5'		6'		7'		8'		9'		10'	Add.
Barom.	MEAN REFRACTION.																					Barom.

TABLE XIV. B.

Correction of the Mean Refraction for the Height of the Thermometer.

Thermo.	MEAN REFRACTION.																				
Add.	0′		1′		2′		3′		4′		5′		6′		7′		8′		9′		10′
	0″	30″	0″	30″	0″	30″	0″	0″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″
°	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″
— 10	0	4	8	12	16	20	24	28	33	37	41	46	50	55	60	65	70	75	80	85	90
— 8	0	4	8	12	15	19	23	27	31	36	40	44	48	53	58	62	67	72	77	82	87
— 6	0	4	7	11	15	19	22	26	30	34	38	42	47	51	55	60	64	69	74	79	84
— 4	0	4	7	11	14	18	22	25	29	33	37	41	45	49	53	57	62	66	71	76	80
— 2	0	3	7	10	14	17	21	24	28	31	35	39	43	47	51	55	59	64	68	72	77
0	0	3	7	10	13	16	20	23	27	30	34	37	41	45	49	53	57	61	65	69	74
+ 2	0	3	6	9	12	16	19	22	25	29	32	36	39	43	47	50	54	58	62	66	70
4	0	3	6	9	12	15	18	21	24	28	31	34	37	41	44	48	52	55	59	63	67
6	0	3	6	8	11	14	17	20	23	26	29	32	36	39	42	46	49	53	56	60	64
8	0	3	5	8	11	14	16	19	22	25	28	31	34	37	40	43	47	50	54	57	61
10	0	3	5	8	10	13	15	18	21	24	26	29	32	35	38	41	44	48	51	54	58
11	0	2	5	7	10	13	15	18	20	23	26	28	31	34	37	40	43	46	49	53	56
12	0	2	5	7	10	12	15	17	20	22	25	28	30	33	36	39	42	45	48	51	54
13	0	2	5	7	9	12	14	17	19	22	24	27	30	32	35	38	41	44	47	50	53
14	0	2	5	7	9	11	14	16	19	21	24	26	29	31	34	37	40	42	45	48	51
15	0	2	4	7	9	11	13	16	18	20	23	25	28	30	33	36	38	41	44	47	50
16	0	2	4	6	9	11	13	15	18	20	22	25	27	29	32	35	37	40	43	45	48
17	0	2	4	6	8	10	13	15	17	19	21	24	26	29	31	33	36	39	41	44	47
18	0	2	4	6	8	10	12	14	16	19	21	23	25	28	30	32	35	37	40	43	45
19	0	2	4	6	8	10	12	14	16	18	20	22	24	27	29	31	34	36	39	41	44
20	0	2	4	6	8	9	11	13	15	17	19	22	24	26	28	30	33	35	37	40	42
21	0	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29	31	34	36	38	41
22	0	2	3	5	7	9	11	12	14	16	18	20	22	24	26	28	30	32	35	37	39
23	0	2	3	5	7	8	10	12	14	15	17	19	21	23	25	27	29	31	33	36	38
24	0	2	3	5	6	8	10	11	13	15	17	18	20	22	24	26	28	30	32	34	36
25	0	2	3	5	6	8	9	11	13	14	16	18	19	21	23	25	27	29	31	33	35
26	0	1	3	4	6	7	9	11	12	14	15	17	19	20	22	24	26	28	29	31	33
27	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	23	25	26	28	30	32
28	0	1	3	4	5	7	8	10	11	12	14	15	17	19	20	22	23	25	27	29	30
29	0	1	3	4	5	6	8	9	11	12	13	15	16	18	19	21	22	24	26	27	29
30	0	1	2	4	5	6	7	9	10	11	13	14	15	17	18	20	21	23	24	26	28
31	0	1	2	3	5	6	7	8	9	11	12	13	15	16	17	19	20	22	23	25	26
32	0	1	2	3	4	6	7	8	9	10	11	13	14	15	16	18	19	20	22	23	25
33	0	1	2	3	4	5	6	7	8	10	11	12	13	14	15	17	18	19	21	22	23
34	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	17	18	19	21	22
35	0	1	2	3	4	5	6	6	7	8	9	10	11	13	14	15	16	17	18	19	20
36	0	1	2	3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
37	0	1	2	2	3	4	5	6	6	7	8	9	10	11	12	13	14	15	16	17	18
38	0	1	1	2	3	4	4	5	6	7	7	8	9	10	11	12	13	13	14	15	16
39	0	1	1	2	3	3	4	5	5	6	7	8	8	9	10	11	11	12	13	14	15
40	0	1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	10	11	12	13	13
41	0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	9	9	10	11	11	12
42	0	0	1	1	2	2	3	3	4	4	5	5	6	7	7	8	8	9	9	10	11
43	0	0	1	1	2	2	3	3	3	4	4	5	5	6	6	7	7	8	8	9	9
44	0	0	1	1	1	2	2	3	3	3	4	4	4	5	5	6	6	7	7	8	8
45	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	6	6	6	7
46	0	0	0	1	1	1	1	2	2	2	2	2	3	3	4	4	4	4	5	5	5
47	0	0	0	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4
48	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3
49	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30′	0″	30″	0″	30″	0″	30″	0″
Add.	0′		1′		2′		3′		4′		5′		6′		7′		8′		9′		10′
Thermo.	MEAN REFRACTION.																				

TABLE XIV. A.

Correction of the Mean Refraction for the Height of the Barometer.

MEAN REFRACTION.																					Barom.
0′		1′		2′		3′		4′		5′		6′		7′		8′		9′		10′	Add.
0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	
″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	
0	2	5	7	10	12	15	17	20	23	25	28	30	33	35	38	40	43	45	48	51	
0	2	5	7	10	12	15	17	20	22	25	27	30	32	35	37	40	42	45	47	50	
0	2	5	7	10	12	14	17	19	22	24	27	29	31	34	36	39	41	44	46	49	
0	2	5	7	9	12	14	16	19	21	24	26	28	31	33	36	38	40	43	45	48	
0	2	5	7	9	11	14	16	18	21	23	25	28	30	32	35	37	39	42	44	47	
0	2	4	7	9	11	13	16	18	20	23	25	27	29	32	34	36	39	41	43	46	
0	2	4	7	9	11	13	15	18	20	22	24	27	29	31	33	35	38	40	42	45	
0	2	4	6	9	11	13	15	17	19	22	24	26	28	30	32	35	37	39	41	44	
0	2	4	6	8	10	13	15	17	19	21	23	25	27	30	32	34	36	38	40	43	
0	2	4	6	8	10	12	14	16	18	21	23	25	27	29	31	33	35	37	39	42	
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	41	
0	2	4	6	8	10	12	14	16	18	20	22	24	25	27	29	31	33	35	37	39	
0	2	4	6	8	9	11	13	15	17	19	21	23	25	27	29	31	33	34	36	38	
0	2	4	6	7	9	11	13	15	17	19	20	22	24	26	28	30	32	34	36	37	
0	2	4	5	7	9	11	13	14	16	18	20	22	24	25	27	29	31	33	35	36	
0	2	3	5	7	9	10	12	14	16	18	19	21	23	25	26	28	30	32	34	35	
0	2	3	5	7	8	10	12	14	15	17	19	21	22	24	26	27	29	31	33	34	
0	2	3	5	7	8	10	12	13	15	17	18	20	22	23	25	27	28	30	32	33	
0	2	3	5	6	8	10	11	13	14	16	18	19	21	23	24	26	27	29	31	32	
0	2	3	5	6	8	9	11	12	14	16	17	19	20	22	23	25	27	28	30	31	
0	1	3	4	6	7	9	10	12	14	15	17	18	20	21	23	24	26	27	29	30	31.50
0	1	3	4	6	7	9	10	12	13	15	16	17	19	20	22	23	25	26	28	29	31.45
0	1	3	4	6	7	8	10	11	13	14	15	17	18	20	21	23	24	25	27	28	31.40
0	1	3	4	5	7	8	9	11	12	14	15	16	18	19	20	22	23	25	26	27	31.35
0	1	3	4	5	6	8	9	10	12	13	14	16	17	18	20	21	22	24	25	26	31.30
0	1	2	4	5	6	7	9	10	11	13	14	15	16	18	19	20	21	23	24	25	31.25
0	1	2	4	5	6	7	8	10	11	12	13	14	16	17	18	19	21	22	23	24	31.20
0	1	2	3	5	6	7	8	9	10	12	13	14	15	16	17	19	20	21	22	23	31.15
0	1	2	3	4	5	7	8	9	10	11	12	13	14	16	17	18	19	20	21	22	31.10
0	1	2	3	4	5	6	7	8	9	11	12	13	14	15	16	17	18	19	20	21	31.05
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	31.00
0	1	2	3	4	5	6	7	8	9	10	11	11	12	13	14	15	16	17	18	19	30.95
0	1	2	3	4	4	5	6	7	8	9	10	11	12	13	14	15	15	16	17	18	30.90
0	1	2	3	3	4	5	6	7	8	9	9	10	11	12	13	14	15	15	16	17	30.85
0	1	2	2	3	4	5	6	6	7	8	9	10	10	11	12	13	14	15	15	16	30.80
0	1	1	2	3	4	4	5	6	7	8	8	9	10	11	11	12	13	14	14	15	30.75
0	1	1	2	3	3	4	5	6	6	7	8	8	9	10	11	11	12	13	13	14	30.70
0	1	1	2	3	3	4	5	5	6	7	7	8	9	9	10	10	11	12	13	13	30.65
0	1	1	2	2	3	4	4	5	5	6	7	7	8	8	9	10	10	11	12	12	30.60
0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	8	9	9	10	11	11	30.55
0	0	1	1	2	2	3	3	4	5	5	6	6	7	7	8	8	9	9	10	10	30.50
0	0	1	1	2	2	3	3	4	4	5	5	5	6	6	7	7	8	8	9	9	30.45
0	0	1	1	2	2	2	3	3	4	4	4	5	5	6	6	6	7	7	8	8	30.40
0	0	1	1	1	2	2	2	3	3	4	4	4	5	5	5	6	6	6	7	7	30.35
0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	5	6	6	30.30
0	0	0	1	1	1	1	2	2	2	3	3	3	3	4	4	4	4	5	5	5	30.25
0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	30.20
0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	30.15
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	30.10
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	30.05
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30.00
0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	
0′		1′		2′		3′		4′		5′		6′		7′		8′		9′		10′	Add.
MEAN REFRACTION.																					Barom.

TABLE XV. Log A.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.													
	41′	42′	43′	44′	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	5
° ′															
5 0	.0288	0295	0301	0308	0315	0321	0328	0335	0341	0348	0355	0361	0368		
2	.0286	0293	0299	0306	0313	0319	0326	0333	0339	0346	0352	0359	0366		
4	.0284	0291	0297	0304	0311	0317	0324	0330	0337	0344	0350	0357	0363		
6	.0282	0289	0296	0302	0309	0315	0322	0328	0335	0341	0348	0354	0361		
8	.0281	0287	0294	0300	0307	0313	0320	0326	0333	0339	0346	0352	0359		
5 10	.0279	0285	0292	0298	0305	0311	0318	0324	0331	0337	0344	0350	0356		
12	.0277	0284	0290	0296	0303	0309	0316	0322	0329	0335	0341	0348	0354		
14	.0275	0282	0288	0295	0301	0307	0314	0320	0327	0333	0339	0346	0352		
16	.0274	0280	0286	0293	0299	0306	0312	0318	0325	0331	0337	0344	0350		
18	.0272	0278	0285	0291	0297	0304	0310	0316	0323	0329	0335	0341	0348		
5 20	.0270	0277	0283	0289	0296	0302	0308	0314	0321	0327	0333	0339	0346	......	
22	.0269	0275	0281	0288	0294	0300	0306	0313	0319	0325	0331	0337	0344	......	
24	.0267	0273	0280	0286	0292	0298	0304	0311	0317	0323	0329	0335	0341	......	
26	.0265	0272	0278	0284	0290	0296	0303	0309	0315	0321	0327	0333	0339	0346	
28	.0264	0270	0276	0282	0289	0295	0301	0307	0313	0319	0325	0331	0337	0344	
5 30	.0262	0268	0275	0281	0287	0293	0299	0305	0311	0317	0323	0329	0335	0342	
32	.0261	0267	0273	0279	0285	0291	0297	0303	0309	0315	0321	0327	0334	0340	
34	.0259	0265	0271	0277	0283	0290	0296	0302	0308	0314	0320	0326	0332	0338	
36	.0258	0264	0270	0276	0282	0288	0294	0300	0306	0312	0318	0324	0330	0336	
38	......	0262	0268	0274	0280	0286	0292	0298	0304	0310	0316	0322	0328	0334	
5 40		0261	0267	0273	0279	0285	0290	0296	0302	0308	0314	0320	0326	0332	
42		0259	0265	0271	0277	0283	0289	0295	0301	0306	0312	0318	0324	0330	
44		0258	0264	0270	0275	0281	0287	0293	0299	0305	0311	0316	0322	0328	
46		0256	0262	0268	0274	0280	0286	0291	0297	0303	0309	0315	0320	0326	
48		0255	0261	0267	0272	0278	0284	0290	0296	0301	0307	0313	0319	0324	
5 50		0253	0259	0265	0271	0277	0282	0288	0294	0300	0305	0311	0317	0323	
52		0252	0258	0264	0269	0275	0281	0287	0292	0298	0304	0309	0315	0321	
54		0251	0256	0262	0268	0274	0279	0285	0291	0296	0302	0308	0313	0319	
56		0249	0255	0261	0266	0272	0278	0283	0289	0295	0300	0306	0312	0317	
58		0248	0254	0259	0265	0271	0276	0282	0287	0293	0299	0304	0310	0316	
6 0		0247	0252	0258	0263	0269	0275	0280	0286	0291	0297	0303	0308	0314	
2		0245	0251	0256	0262	0268	0273	0279	0284	0290	0295	0301	0307	0312	
4		0244	0249	0255	0261	0266	0272	0277	0283	0288	0294	0299	0305	0310	
6		0243	0248	0254	0259	0265	0270	0276	0281	0287	0292	0298	0303	0309	
8		0241	0247	0252	0258	0263	0269	0274	0280	0285	0291	0296	0302	0307	
6 10		0240	0246	0251	0256	0262	0267	0273	0278	0284	0289	0295	0300	0306	
12		0239	0244	0250	0255	0261	0266	0271	0277	0282	0288	0293	0299	0304	
14		0237	0243	0248	0254	0259	0265	0270	0275	0281	0286	0292	0297	0302	
16		0236	0242	0247	0252	0258	0263	0269	0274	0279	0285	0290	0295	0301	
18		0235	0240	0246	0251	0257	0262	0267	0273	0278	0283	0289	0294	0299	
6 20		0234	0239	0245	0250	0255	0261	0266	0271	0276	0282	0287	0292	0298	....
22		0233	0238	0243	0249	0254	0259	0264	0270	0275	0280	0286	0291	0296	....
24		0231	0237	0242	0247	0253	0258	0263	0268	0274	0279	0284	0289	0295	....
26		......	0236	0241	0246	0251	0257	0262	0267	0272	0277	0283	0288	0293	....
28		......	0234	0240	0245	0250	0255	0260	0266	0271	0276	0281	0286	0292	029
6 30			0233	0238	0244	0249	0254	0259	0264	0270	0275	0280	0285	0290	029
32			0232	0237	0242	0248	0253	0258	0263	0268	0273	0278	0284	0289	029
34			0231	0236	0241	0246	0251	0257	0262	0267	0272	0277	0282	0287	029
36			0230	0235	0240	0245	0250	0255	0260	0266	0271	0276	0281	0286	029
38			0229	0234	0239	0244	0249	0254	0259	0264	0269	0274	0279	0284	029
6 40			0227	0232	0238	0243	0248	0253	0258	0263	0268	0273	0278	0283	028
42			0226	0231	0236	0241	0246	0252	0257	0262	0267	0272	0277	0282	028
44			0225	0230	0235	0240	0245	0250	0255	0260	0265	0270	0275	0280	028
46			0224	0229	0234	0239	0244	0249	0254	0259	0264	0269	0274	0279	028
48			0223	0228	0233	0238	0243	0248	0253	0258	0263	0268	0273	0278	028
6 50			0222	0227	0232	0237	0242	0247	0252	0257	0262	0266	0271	0276	028
52			0221	0226	0231	0236	0241	0246	0250	0255	0260	0265	0270	0275	028
54			0220	0225	0230	0235	0239	0244	0249	0254	0259	0264	0269	0274	027
56			0219	0224	0229	0233	0238	0243	0248	0253	0258	0263	0267	0272	027
58			0218	0223	0227	0232	0237	0242	0247	0252	0257	0261	0266	0271	027
7 0			0217	0222	0226	0231	0236	0241	0246	0251	0255	0260	0265	0270	027

TABLE XV. Log A.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.

4′	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′
22	0226	0231	0236	0241	0246	0251	0255	0260	0265	0270	0275		
20	0225	0230	0234	0239	0244	0249	0254	0258	0263	0268	0273		
18	0223	0228	0233	0238	0242	0247	0252	0257	0261	0266	0271		
17	0222	0226	0231	0236	0241	0245	0250	0255	0260	0264	0269		
15	0220	0225	0230	0234	0239	0244	0248	0253	0258	0262	0267		
14	0219	0223	0228	0233	0237	0242	0247	0251	0256	0261	0265		
13	0217	0222	0226	0231	0236	0240	0245	0250	0254	0259	0263		
11	0216	0220	0225	0230	0234	0239	0243	0248	0253	0257	0262		
10	0214	0219	0223	0228	0233	0237	0242	0246	0251	0255	0260		
08	0213	0217	0222	0227	0231	0236	0240	0245	0249	0254	0258		
07	0211	0216	0220	0225	0230	0234	0239	0243	0248	0252	0257		
06	0210	0215	0219	0224	0228	0232	0237	0241	0246	0250	0255		
04	0209	0213	0218	0222	0227	0231	0235	0240	0244	0249	0253		
03	0207	0212	0216	0221	0225	0229	0234	0238	0243	0247	0252		
02	0206	0210	0215	0219	0224	0228	0232	0237	0241	0246	0250		
00	0205	0209	0213	0218	0222	0227	0231	0235	0240	0244	0248	......	
99	0203	0208	0212	0216	0221	0225	0229	0234	0238	0242	0247	......	
98	0202	0206	0211	0215	0219	0224	0228	0232	0237	0241	0245	0249	
96	0201	0205	0209	0214	0218	0222	0227	0231	0235	0239	0244	0248	
95	0200	0204	0208	0212	0217	0221	0225	0229	0234	0238	0242	0246	
94	0198	0203	0207	0211	0215	0219	0224	0228	0232	0236	0241	0245	
93	0197	0201	0206	0210	0214	0218	0222	0227	0231	0235	0239	0243	
92	0196	0200	0204	0208	0213	0217	0221	0225	0229	0233	0238	0242	
...	0195	0199	0203	0207	0211	0215	0220	0224	0228	0232	0236	0240	
...	0193	0198	0202	0206	0210	0214	0218	0222	0227	0231	0235	0239	
	0192	0196	0201	0205	0209	0213	0217	0221	0225	0229	0233	0237	
	0191	0195	0199	0203	0207	0212	0216	0220	0224	0228	0232	0236	
	0190	0194	0198	0202	0206	0210	0214	0218	0222	0226	0231	0235	
	0189	0193	0197	0201	0205	0209	0213	0217	0221	0225	0229	0233	
	0188	0192	0196	0200	0204	0208	0212	0216	0220	0224	0228	0232	
	0187	0191	0195	0199	0203	0207	0211	0215	0219	0223	0226	0230	
	0186	0190	0193	0197	0201	0205	0209	0213	0217	0221	0225	0229	
	0184	0188	0192	0196	0200	0204	0208	0212	0216	0220	0224	0228	
	0183	0187	0191	0195	0199	0203	0207	0211	0215	0219	0223	0226	
	0182	0186	0190	0194	0198	0202	0206	0210	0214	0217	0221	0225	
	0181	0185	0189	0193	0197	0201	0205	0208	0212	0216	0220	0224	
	0180	0184	0188	0192	0196	0200	0203	0207	0211	0215	0219	0223	
	0179	0183	0187	0191	0195	0198	0202	0206	0210	0214	0218	0221	
	0178	0182	0186	0190	0193	0197	0201	0205	0209	0212	0216	0220	
	0177	0181	0185	0189	0192	0196	0200	0204	0208	0211	0215	0219	
	0176	0180	0184	0188	0191	0195	0199	0203	0206	0210	0214	0218	
	0175	0179	0183	0186	0190	0194	0198	0201	0205	0209	0213	0216	
	0174	0178	0182	0185	0189	0193	0197	0200	0204	0208	0211	0215	
	0173	0177	0181	0184	0188	0192	0196	0199	0203	0207	0210	0214	
	0172	0176	0180	0183	0187	0191	0194	0198	0202	0206	0209	0213	
	0171	0175	0179	0182	0186	0190	0193	0197	0201	0204	0208	0212	
	0170	0174	0178	0181	0185	0189	0192	0196	0200	0203	0207	0211	
	0170	0173	0177	0180	0184	0188	0191	0195	0199	0202	0206	0209	
	......	0172	0176	0179	0183	0187	0190	0194	0198	0201	0205	0208	
	......	0171	0175	0179	0182	0186	0189	0193	0196	0200	0204	0207	
		0170	0174	0178	0181	0185	0188	0192	0195	0199	0203	0206	
		0170	0173	0177	0180	0184	0187	0191	0194	0198	0201	0205	
		0169	0172	0176	0179	0183	0186	0190	0193	0197	0200	0204	
		0168	0171	0175	0178	0182	0185	0189	0192	0196	0199	0203	
		0167	0170	0174	0177	0181	0184	0188	0191	0195	0198	0202	
		0166	0169	0173	0176	0180	0183	0187	0190	0194	0197	0201	......
		0165	0169	0172	0176	0179	0182	0186	0189	0193	0196	0200	0203
		0164	0168	0171	0175	0178	0182	0185	0188	0192	0195	0199	0202
		0163	0167	0170	0174	0177	0181	0184	0187	0191	0194	0198	0201
		0163	0166	0169	0173	0176	0180	0183	0186	0190	0193	0197	0200
		0162	0165	0169	0172	0175	0179	0182	0186	0189	0192	0196	0199

TABLE XV. Log A.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.													
	41′	42′	43′	44′	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	5
° ′															
5 0	.0288	0295	0301	0308	0315	0321	0328	0335	0341	0348	0355	0361	0368		
2	.0286	0293	0299	0306	0313	0319	0326	0333	0339	0346	0352	0359	0366		
4	.0284	0291	0297	0304	0311	0317	0324	0330	0337	0344	0350	0357	0363		
6	.0282	0289	0296	0302	0309	0315	0322	0328	0335	0341	0348	0354	0361		
8	.0281	0287	0294	0300	0307	0313	0320	0326	0333	0339	0346	0352	0359		
5 10	.0279	0285	0292	0298	0305	0311	0318	0324	0331	0337	0344	0350	0356		
12	.0277	0284	0290	0296	0303	0309	0316	0322	0329	0335	0341	0348	0354		
14	.0275	0282	0288	0295	0301	0307	0314	0320	0327	0333	0339	0346	0352		
16	.0274	0280	0286	0293	0299	0306	0312	0318	0325	0331	0337	0344	0350		
18	.0272	0278	0285	0291	0297	0304	0310	0316	0323	0329	0335	0341	0348		
5 20	.0270	0277	0283	0289	0296	0302	0308	0314	0321	0327	0333	0339	0346	......	
22	.0269	0275	0281	0288	0294	0300	0306	0313	0319	0325	0331	0337	0344	......	
24	.0267	0273	0280	0286	0292	0298	0304	0311	0317	0323	0329	0335	0341	......	
26	.0265	0272	0278	0284	0290	0296	0303	0309	0315	0321	0327	0333	0339	0346	
28	.0264	0270	0276	0282	0289	0295	0301	0307	0313	0319	0325	0331	0337	0344	
5 30	.0262	0268	0275	0281	0287	0293	0299	0305	0311	0317	0323	0329	0335	0342	
32	.0261	0267	0273	0279	0285	0291	0297	0303	0309	0315	0321	0327	0334	0340	
34	.0259	0265	0271	0277	0283	0290	0296	0302	0308	0314	0320	0326	0332	0338	
36	.0258	0264	0270	0276	0282	0288	0294	0300	0306	0312	0318	0324	0330	0336	
38	......	0262	0268	0274	0280	0286	0292	0298	0304	0310	0316	0322	0328	0334	
5 40		0261	0267	0273	0279	0285	0290	0296	0302	0308	0314	0320	0326	0332	
42		0259	0265	0271	0277	0283	0289	0295	0301	0306	0312	0318	0324	0330	
44		0258	0264	0270	0275	0281	0287	0293	0299	0305	0311	0316	0322	0328	
46		0256	0262	0268	0274	0280	0286	0291	0297	0303	0309	0315	0320	0326	
48		0255	0261	0267	0272	0278	0284	0290	0296	0301	0307	0313	0319	0324	
5 50		0253	0259	0265	0271	0277	0282	0288	0294	0300	0305	0311	0317	0323	
52		0252	0258	0264	0269	0275	0281	0287	0292	0298	0304	0309	0315	0321	
54		0251	0256	0262	0268	0274	0279	0285	0291	0296	0302	0308	0313	0319	
56		0249	0255	0261	0266	0272	0278	0283	0289	0295	0300	0306	0312	0317	
58		0248	0254	0259	0265	0271	0276	0282	0287	0293	0299	0304	0310	0316	
6 0		0247	0252	0258	0263	0269	0275	0280	0286	0291	0297	0303	0308	0314	
2		0245	0251	0256	0262	0268	0273	0279	0284	0290	0295	0301	0307	0312	
4		0244	0249	0255	0261	0266	0272	0277	0283	0288	0294	0299	0305	0310	
6		0243	0248	0254	0259	0265	0270	0276	0281	0287	0292	0298	0303	0309	
8		0241	0247	0252	0258	0263	0269	0274	0280	0285	0291	0296	0302	0307	
6 10		0240	0246	0251	0256	0262	0267	0273	0278	0284	0289	0295	0300	0306	
12		0239	0244	0250	0255	0261	0266	0271	0277	0282	0288	0293	0299	0304	
14		0237	0243	0248	0254	0259	0265	0270	0275	0281	0286	0292	0297	0302	
16		0236	0242	0247	0252	0258	0263	0269	0274	0279	0285	0290	0295	0301	
18		0235	0240	0246	0251	0257	0262	0267	0273	0278	0283	0289	0294	0299	
6 20		0234	0239	0245	0250	0255	0261	0266	0271	0276	0282	0287	0292	0298	......
22		0233	0238	0243	0249	0254	0259	0264	0270	0275	0280	0286	0291	0296	......
24		0231	0237	0242	0247	0253	0258	0263	0268	0274	0279	0284	0289	0295	......
26		......	0236	0241	0246	0251	0257	0262	0267	0272	0277	0283	0288	0293	......
28		......	0234	0240	0245	0250	0255	0260	0266	0271	0276	0281	0286	0292	029
6 30			0233	0238	0244	0249	0254	0259	0264	0270	0275	0280	0285	0290	029
32			0232	0237	0242	0248	0253	0258	0263	0268	0273	0278	0284	0289	029
34			0231	0236	0241	0246	0251	0257	0262	0267	0272	0277	0282	0287	029
36			0230	0235	0240	0245	0250	0255	0260	0266	0271	0276	0281	0286	029
38			0229	0234	0239	0244	0249	0254	0259	0264	0269	0274	0279	0284	029
6 40			0227	0232	0238	0243	0248	0253	0258	0263	0268	0273	0278	0283	028
42			0226	0231	0236	0241	0246	0252	0257	0262	0267	0272	0277	0282	028
44			0225	0230	0235	0240	0245	0250	0255	0260	0265	0270	0275	0280	028
46			0224	0229	0234	0239	0244	0249	0254	0259	0264	0269	0274	0279	028
48			0223	0228	0233	0238	0243	0248	0253	0258	0263	0268	0273	0278	028
6 50			0222	0227	0232	0237	0242	0247	0252	0257	0262	0266	0271	0276	028
52			0221	0226	0231	0236	0241	0246	0250	0255	0260	0265	0270	0275	028
54			0220	0225	0230	0235	0239	0244	0249	0254	0259	0264	0269	0274	027
56			0219	0224	0229	0233	0238	0243	0248	0253	0258	0263	0267	0272	027
58			0218	0223	0227	0232	0237	0242	0247	0252	0257	0261	0266	0271	027
7 0			0217	0222	0226	0231	0236	0241	0246	0251	0255	0260	0265	0270	027

TABLE XV. Log A.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.													
44′	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′
.0222	0226	0231	0236	0241	0246	0251	0255	0260	0265	0270	0275		
.0220	0225	0230	0234	0239	0244	0249	0254	0258	0263	0268	0273		
.0218	0223	0228	0233	0238	0242	0247	0252	0257	0261	0266	0271		
.0217	0222	0226	0231	0236	0241	0245	0250	0255	0260	0264	0269		
.0215	0220	0225	0230	0234	0239	0244	0248	0253	0258	0262	0267		
.0214	0219	0223	0228	0233	0237	0242	0247	0251	0256	0261	0265		
.0213	0217	0222	0226	0231	0236	0240	0245	0250	0254	0259	0263		
.0211	0216	0220	0225	0230	0234	0239	0243	0248	0253	0257	0262		
.0210	0214	0219	0223	0228	0233	0237	0242	0246	0251	0255	0260		
.0208	0213	0217	0222	0227	0231	0236	0240	0245	0249	0254	0258		
.0207	0211	0216	0220	0225	0230	0234	0239	0243	0248	0252	0257		
.0206	0210	0215	0219	0224	0228	0232	0237	0241	0246	0250	0255		
.0204	0209	0213	0218	0222	0227	0231	0235	0240	0244	0249	0253		
.0203	0207	0212	0216	0221	0225	0229	0234	0238	0243	0247	0252		
.0202	0206	0210	0215	0219	0224	0228	0232	0237	0241	0246	0250		
.0200	0205	0209	0213	0218	0222	0227	0231	0235	0240	0244	0248	......	
.0199	0203	0208	0212	0216	0221	0225	0229	0234	0238	0242	0247	......	
.0198	0202	0206	0211	0215	0219	0224	0228	0232	0237	0241	0245	0249	
.0196	0201	0205	0209	0214	0218	0222	0227	0231	0235	0239	0244	0248	
.0195	0200	0204	0208	0212	0217	0221	0225	0229	0234	0238	0242	0246	
.0194	0198	0203	0207	0211	0215	0219	0224	0228	0232	0236	0241	0245	
.0193	0197	0201	0206	0210	0214	0218	0222	0227	0231	0235	0239	0243	
.0192	0196	0200	0204	0208	0213	0217	0221	0225	0229	0233	0238	0242	
......	0195	0199	0203	0207	0211	0215	0220	0224	0228	0232	0236	0240	
......	0193	0198	0202	0206	0210	0214	0218	0222	0227	0231	0235	0239	
	0192	0196	0201	0205	0209	0213	0217	0221	0225	0229	0233	0237	
	0191	0195	0199	0203	0207	0212	0216	0220	0224	0228	0232	0236	
	0190	0194	0198	0202	0206	0210	0214	0218	0222	0226	0231	0235	
	0189	0193	0197	0201	0205	0209	0213	0217	0221	0225	0229	0233	
	0188	0192	0196	0200	0204	0208	0212	0216	0220	0224	0228	0232	
	0187	0191	0195	0199	0203	0207	0211	0215	0219	0223	0226	0230	
	0186	0190	0193	0197	0201	0205	0209	0213	0217	0221	0225	0229	
	0184	0188	0192	0196	0200	0204	0208	0212	0216	0220	0224	0228	
	0183	0187	0191	0195	0199	0203	0207	0211	0215	0219	0223	0226	
	0182	0186	0190	0194	0198	0202	0206	0210	0214	0217	0221	0225	
	0181	0185	0189	0193	0197	0201	0205	0208	0212	0216	0220	0224	
	0180	0184	0188	0192	0196	0200	0203	0207	0211	0215	0219	0223	
	0179	0183	0187	0191	0195	0198	0202	0206	0210	0214	0218	0221	
	0178	0182	0186	0190	0193	0197	0201	0205	0209	0212	0216	0220	
	0177	0181	0185	0189	0192	0196	0200	0204	0208	0211	0215	0219	
	0176	0180	0184	0188	0191	0195	0199	0203	0206	0210	0214	0218	
	0175	0179	0183	0186	0190	0194	0198	0201	0205	0209	0213	0216	
	0174	0178	0182	0185	0189	0193	0197	0200	0204	0208	0211	0215	
	0173	0177	0181	0184	0188	0192	0196	0199	0203	0207	0210	0214	
	0172	0176	0180	0183	0187	0191	0194	0198	0202	0206	0209	0213	
	0171	0175	0179	0182	0186	0190	0193	0197	0201	0204	0208	0212	
	0170	0174	0178	0181	0185	0189	0192	0196	0200	0203	0207	0211	
	0170	0173	0177	0180	0184	0188	0191	0195	0199	0202	0206	0209	
	......	0172	0176	0179	0183	0187	0190	0194	0198	0201	0205	0208	
	......	0171	0175	0179	0182	0186	0189	0193	0196	0200	0204	0207	
		0170	0174	0178	0181	0185	0188	0192	0195	0199	0203	0206	
		0170	0173	0177	0180	0184	0187	0191	0194	0198	0201	0205	
		0169	0172	0176	0179	0183	0186	0190	0193	0197	0200	0204	
		0168	0171	0175	0178	0182	0185	0189	0192	0196	0199	0203	
		0167	0170	0174	0177	0181	0184	0188	0191	0195	0198	0202	
		0166	0169	0173	0176	0180	0183	0187	0190	0194	0197	0201	......
		0165	0169	0172	0176	0179	0182	0186	0189	0193	0196	0200	0203
		0164	0168	0171	0175	0178	0182	0185	0188	0192	0195	0199	0202
		0163	0167	0170	0174	0177	0181	0184	0187	0191	0194	0198	0201
		0163	0166	0169	0173	0176	0180	0183	0186	0190	0193	0197	0200
		0162	0165	0169	0172	0175	0179	0182	0186	0189	0192	0196	0199

TABLE XV. Log A.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.											
	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′
° ′													
10 0	.0162	0165	0169	0172	0175	0179	0182	0186	0189	0192	0196	0199	
5	.0160	0164	0167	0171	0174	0177	0181	0184	0187	0191	0194	0197	
10	.0159	0162	0166	0169	0172	0176	0179	0182	0186	0189	0192	0196	
15	.0158	0161	0164	0168	0171	0174	0178	0181	0184	0187	0191	0194	
20	.0156	0160	0163	0166	0170	0173	0176	0179	0183	0186	0189	0192	
25	.0155	0158	0162	0165	0168	0171	0175	0178	0181	0184	0188	0191	
10 30	.0154	0157	0160	0164	0167	0170	0173	0177	0180	0183	0186	0189	
35	.0153	0156	0159	0162	0166	0169	0172	0175	0178	0181	0185	0188	
40	.0151	0155	0158	0161	0164	0167	0171	0174	0177	0180	0183	0186	
45	.0150	0153	0157	0160	0163	0166	0169	0172	0175	0179	0182	0185	
50	.0149	0152	0155	0158	0162	0165	0168	0171	0174	0177	0180	0183	
55	.0148	0151	0154	0157	0160	0163	0167	0170	0173	0176	0179	0182	
11 0	.0147	0150	0153	0156	0159	0162	0165	0168	0171	0174	0177	0181	
5	.0146	0149	0152	0155	0158	0161	0164	0167	0170	0173	0176	0179	
10	......	0148	0151	0154	0157	0160	0163	0166	0169	0172	0175	0178	
15	......	0146	0149	0152	0155	0158	0161	0164	0167	0170	0173	0176	
20	......	0145	0148	0151	0154	0157	0160	0163	0166	0169	0172	0175	
25	......	0144	0147	0150	0153	0156	0159	0162	0165	0168	0171	0174	
11 30		0143	0146	0149	0152	0155	0158	0161	0164	0167	0170	0172	
35		0142	0145	0148	0151	0154	0157	0160	0162	0165	0168	0171	
40		0141	0144	0147	0150	0153	0156	0158	0161	0164	0167	0170	
45		0140	0143	0146	0149	0151	0154	0157	0160	0163	0166	0169	
50		0139	0142	0145	0148	0150	0153	0156	0159	0162	0165	0167	
55		0138	0141	0144	0146	0149	0152	0155	0158	0161	0163	0166	
12 0		0137	0140	0143	0145	0148	0151	0154	0157	0159	0162	0165	
5		0136	0139	0142	0144	0147	0150	0153	0156	0158	0161	0164	
10		0135	0138	0141	0143	0146	0149	0152	0154	0157	0160	0163	
15		0134	0137	0140	0142	0145	0148	0151	0153	0156	0159	0162	
20		0133	0136	0139	0141	0144	0147	0150	0152	0155	0158	0160	
25		0132	0135	0138	0140	0143	0146	0148	0151	0154	0157	0159	
12 30		0131	0134	0137	0139	0142	0145	0147	0150	0153	0155	0158	......
35		0130	0133	0136	0138	0141	0144	0146	0149	0152	0154	0157	......
40		0129	0132	0135	0137	0140	0143	0145	0148	0151	0153	0156	......
45		0129	0131	0134	0136	0139	0142	0144	0147	0150	0152	0155	0158
50		0128	0130	0133	0136	0138	0141	0143	0146	0149	0151	0154	0156
55		0127	0129	0132	0135	0137	0140	0142	0145	0148	0150	0153	0155
13 0		0126	0129	0131	0134	0136	0139	0141	0144	0147	0149	0152	0154
5		0125	0128	0130	0133	0135	0138	0141	0143	0146	0148	0151	0153
10		0124	0127	0129	0132	0135	0137	0140	0142	0145	0147	0150	0152
15		0123	0126	0129	0131	0134	0136	0139	0141	0144	0146	0149	0151
20		0123	0125	0128	0130	0133	0135	0138	0140	0143	0145	0148	0150
25		0122	0124	0127	0129	0132	0134	0137	0139	0142	0144	0147	0149
13 30		0121	0124	0126	0129	0131	0133	0136	0138	0141	0143	0146	0148
35		0120	0123	0125	0128	0130	0133	0135	0138	0140	0142	0145	0147
40		0120	0122	0124	0127	0129	0132	0134	0137	0139	0142	0144	0146
45		......	0121	0124	0126	0128	0131	0133	0136	0138	0141	0143	0145
50		......	0120	0123	0125	0128	0130	0132	0135	0137	0140	0142	0145
55		......	0120	0122	0124	0127	0129	0132	0134	0136	0139	0141	0144
14 0			0119	0121	0124	0126	0128	0131	0133	0136	0138	0140	0143
5			0118	0121	0123	0125	0128	0130	0132	0135	0137	0139	0142
10			0117	0120	0122	0124	0127	0129	0132	0134	0136	0139	0141
15			0117	0119	0121	0124	0126	0128	0131	0133	0135	0138	0140
20			0116	0118	0121	0123	0125	0128	0130	0132	0135	0137	0139
25			0115	0118	0120	0122	0124	0127	0129	0131	0134	0136	0138
14 30			0114	0117	0119	0121	0124	0126	0128	0131	0133	0135	0137
35			0114	0116	0118	0121	0123	0125	0128	0130	0132	0134	0137
40			0113	0115	0118	0120	0122	0124	0127	0129	0131	0134	0136
45			0112	0115	0117	0119	0121	0124	0126	0128	0130	0133	0135
50			0112	0114	0116	0118	0121	0123	0125	0127	0130	0132	0134
55			0111	0113	0116	0118	0120	0122	0124	0127	0129	0131	0133
15 0			0110	0113	0115	0117	0119	0121	0124	0126	0128	0130	0133

TABLE XV. Log A.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.											
8′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′	59′
110	0113	0115	0117	0119	0121	0124	0126	0128	0130	0133	
109	0111	0113	0116	0118	0120	0122	0124	0127	0129	0131	
108	0110	0112	0114	0116	0119	0121	0123	0125	0127	0129	
107	0109	0111	0113	0115	0117	0119	0121	0124	0126	0128	
105	0107	0110	0112	0114	0116	0118	0120	0122	0124	0126	
104	0106	0108	0110	0112	0115	0117	0119	0121	0123	0125	
103	0105	0107	0109	0111	0113	0115	0117	0119	0121	0124	
102	0104	0106	0108	0110	0112	0114	0116	0118	0120	0122	
101	0103	0105	0107	0109	0111	0113	0115	0117	0119	0121	
100	0102	0103	0105	0107	0109	0111	0113	0115	0117	0119	
098	0100	0102	0104	0106	0108	0110	0112	0114	0116	0118	
097	0099	0101	0103	0105	0107	0109	0111	0113	0115	0117	
096	0098	0100	0102	0104	0106	0108	0110	0112	0114	0116	
095	0097	0099	0101	0103	0105	0107	0109	0110	0112	0114	
094	0096	0098	0100	0102	0104	0106	0107	0109	0111	0113	
......	0095	0097	0099	0101	0103	0104	0106	0108	0110	0112	
......	0094	0096	0098	0100	0101	0103	0105	0107	0109	0111	
......	0093	0095	0097	0099	0100	0102	0104	0106	0108	0109	
	0092	0094	0096	0098	0099	0101	0103	0105	0107	0108	......
	0091	0093	0095	0097	0098	0100	0102	0104	0105	0107	0109
	0090	0092	0094	0096	0097	0099	0101	0103	0104	0106	0108
	0089	0091	0093	0095	0096	0098	0100	0102	0103	0105	0107
	0088	0090	0092	0094	0095	0097	0099	0101	0102	0104	0106
	0088	0089	0091	0093	0094	0096	0098	0099	0101	0103	0105
	0087	0088	0090	0092	0093	0095	0097	0098	0100	0102	0104
	0086	0087	0089	0091	0092	0094	0096	0098	0099	0101	0103
	0085	0087	0088	0090	0092	0093	0095	0097	0098	0100	0102
	0084	0086	0087	0089	0091	0092	0094	0096	0097	0099	0101
	0083	0085	0087	0088	0090	0091	0093	0095	0096	0098	0100
	0082	0084	0086	0087	0089	0090	0092	0094	0095	0097	0099
	0082	0083	0085	0086	0088	0090	0091	0093	0094	0096	0098
	0081	0082	0084	0086	0087	0089	0090	0092	0093	0095	0097
	0080	0082	0083	0085	0086	0088	0089	0091	0093	0094	0096
	0079	0081	0082	0084	0086	0087	0089	0090	0092	0093	0095
	0079	0080	0082	0083	0085	0086	0088	0089	0091	0092	0094
	0078	0079	0081	0082	0084	0085	0087	0088	0090	0091	0093
	0077	0079	0080	0082	0083	0085	0086	0088	0089	0091	0092
	0076	0078	0079	0081	0082	0084	0085	0087	0088	0090	0091
	0076	0077	0079	0080	0082	0083	0085	0086	0087	0089	0090
	0075	0076	0078	0079	0081	0082	0084	0085	0087	0088	0090
	0074	0076	0077	0079	0080	0082	0083	0084	0086	0087	0089
	0074	0075	0076	0078	0079	0081	0082	0084	0085	0086	0088
	0073	0074	0076	0077	0079	0080	0081	0083	0084	0086	0087
	0072	0074	0075	0076	0078	0079	0081	0082	0083	0085	0086
	0072	0073	0074	0076	0077	0079	0080	0081	0083	0084	0086
	0071	0072	0074	0075	0076	0078	0079	0081	0082	0083	0085
	0070	0072	0073	0074	0076	0077	0079	0080	0081	0083	0084
	0070	0071	0072	0074	0075	0076	0078	0079	0081	0082	0083
	0069	0070	0072	0073	0074	0076	0077	0078	0080	0081	0082
	0068	0070	0071	0072	0074	0075	0076	0078	0079	0080	0082
	0068	0069	0070	0072	0073	0074	0076	0077	0078	0080	0081
	0067	0069	0070	0071	0072	0074	0075	0076	0078	0079	0080
	0067	0068	0069	0071	0072	0073	0074	0076	0077	0078	0080
	0066	0067	0069	0070	0071	0073	0074	0075	0076	0078	0079
		0067	0068	0069	0071	0072	0073	0074	0076	0077	0078
		0066	0067	0069	0070	0071	0073	0074	0075	0076	0078
		0066	0067	0068	0069	0071	0072	0073	0074	0076	0077
		0065	0066	0068	0069	0070	0071	0072	0074	0075	0076
		0065	0066	0067	0068	0069	0071	0072	0073	0074	0076
		0064	0065	0066	0068	0069	0070	0071	0072	0074	0075
		0063	0065	0066	0067	0068	0069	0071	0072	0073	0074

TABLE XV. Log A.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.										
	50'	51'	52'	53'	54'	55'	56'	57'	58'	59'	60'
° '											
25 0	.0063	0065	0066	0067	0068	0069	0071	0072	0073	0074	
20	.0062	0064	0065	0066	0067	0068	0069	0071	0072	0073	
40	.0061	0062	0064	0065	0066	0067	0068	0069	0071	0072	
26 0	.0060	0061	0063	0064	0065	0066	0067	0068	0069	0071	
20	.0059	0060	0062	0063	0064	0065	0066	0067	0068	0069	
40	.0058	0059	0061	0062	0063	0064	0065	0066	0067	0068	
27 0	.0057	0058	0060	0061	0062	0063	0064	0065	0066	0067	
20	.0056	0057	0059	0060	0061	0062	0063	0064	0065	0066	
40	.0055	0057	0058	0059	0060	0061	0062	0063	0064	0065	
28 0	.0055	0056	0057	0058	0059	0060	0061	0062	0063	0064	
20	.0054	0055	0056	0057	0058	0059	0060	0061	0062	0063	
40	.0053	0054	0055	0056	0057	0058	0059	0060	0061	0062	
29 0	.0052	0053	0054	0055	0056	0057	0058	0059	0060	0061	
20	.0051	0052	0053	0054	0055	0056	0057	0058	0059	0060	
40	.0050	0051	0052	0053	0054	0055	0056	0057	0058	0059	
30 0	.0050	0051	0051	0052	0053	0054	0055	0056	0057	0058	
20	.0049	0050	0051	0052	0052	0053	0054	0055	0056	0057	
40	.0048	0049	0050	0051	0052	0053	0053	0054	0055	0056	
31 0	.0047	0048	0049	0050	0051	0052	0053	0053	0054	0055	......
20	.0047	0047	0048	0049	0050	0051	0052	0053	0054	0054	0055
40	.0046	0047	0048	0048	0049	0050	0051	0052	0053	0054	0054
32 0	.0045	0046	0047	0048	0048	0049	0050	0051	0052	0053	0054
20	.0044	0045	0046	0047	0048	0049	0049	0050	0051	0052	0053
40	.0044	0045	0045	0046	0047	0048	0049	0049	0050	0051	0052
33 0	.0043	0044	0045	0045	0046	0047	0048	0049	0049	0050	0051
20	.0042	0043	0044	0045	0046	0046	0047	0048	0049	0050	0050
40	.0042	0043	0043	0044	0045	0046	0046	0047	0048	0049	0050
34 0	.0041	0042	0043	0043	0044	0045	0046	0046	0047	0048	0049
20	.0040	0041	0042	0043	0043	0044	0045	0046	0047	0047	0048
40	.0040	0041	0041	0042	0043	0044	0044	0045	0046	0047	0047
35 0	.0039	0040	0041	0041	0042	0043	0044	0044	0045	0046	0047
20	.0039	0039	0040	0041	0042	0042	0043	0044	0044	0045	0046
40	.0038	0039	0039	0040	0041	0042	0042	0043	0044	0044	0045
36 0	.0037	0038	0039	0040	0040	0041	0042	0042	0043	0044	0044
20	.0037	0038	0038	0039	0040	0040	0041	0042	0042	0043	0044
40	.0036	0037	0038	0038	0039	0040	0040	0041	0042	0042	0043
37 0	.0036	0036	0037	0038	0038	0039	0040	0040	0041	0042	0042
20	.0035	0036	0037	0037	0038	0039	0039	0040	0040	0041	0042
40	.0035	0035	0036	0037	0037	0038	0039	0039	0040	0040	0041
38 0	.0034	0035	0035	0036	0037	0037	0038	0039	0039	0040	0040
20	.0034	0034	0035	0036	0036	0037	0037	0038	0039	0039	0040
40	.0033	0034	0034	0035	0036	0036	0037	0037	0038	0039	0039
39 0		0033	0034	0034	0035	0036	0036	0037	0037	0038	0039
20		0033	0033	0034	0035	0035	0036	0036	0037	0037	0038
40		0032	0033	0033	0034	0035	0035	0036	0036	0037	0037
40 0		0032	0032	0033	0033	0034	0035	0035	0036	0036	0037
20		0031	0032	0032	0033	0034	0034	0035	0035	0036	0036
40		0031	0031	0032	0032	0033	0034	0034	0035	0035	0036
41 0		0030	0031	0031	0032	0033	0033	0034	0034	0035	0035
20		0030	0030	0031	0031	0032	0033	0033	0034	0034	0035
40		0029	0030	0030	0031	0032	0032	0033	0033	0034	0034
42 0		0029	0029	0030	0031	0031	0032	0032	0033	0033	0034
20		0029	0029	0030	0030	0031	0031	0032	0032	0033	0033
40		0028	0029	0029	0030	0030	0031	0031	0032	0032	0033
43 0		0028	0028	0029	0029	0030	0030	0031	0031	0032	0032
20		0027	0028	0028	0029	0029	0030	0030	0031	0031	0032
40		0027	0027	0028	0028	0029	0029	0030	0030	0031	0031
44 0		0026	0027	0027	0028	0028	0029	0029	0030	0030	0031
20		0026	0026	0027	0027	0028	0028	0029	0029	0030	0030
40		0026	0026	0026	0027	0027	0028	0028	0029	0029	0030
45 0		0025	0026	0026	0027	0027	0027	0028	0028	0029	0029

TABLE XV. Log A.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.													
52'	53'	54'	55'	56'	57'	58'	59'	60'					
0026	0026	0027	0027	0027	0028	0028	0029	0029					
0025	0025	0026	0026	0027	0027	0028	0028	0028					
0024	0025	0025	0026	0026	0027	0027	0027	0028					
0024	0024	0025	0025	0026	0026	0026	0027	0027					
0023	0024	0024	0025	0025	0025	0026	0026	0026					
0023	0023	0024	0024	0024	0025	0025	0025	0026					
0022	0023	0023	0023	0024	0024	0024	0025	0025					
0022	0022	0022	0023	0023	0024	0024	0024	0025					
0021	0022	0022	0022	0023	0023	0023	0024	0024					
0021	0021	0021	0022	0022	0022	0023	0023	0023					
0020	0020	0021	0021	0022	0022	0022	0023	0023					
0020	0020	0020	0021	0021	0021	0022	0022	0022					
0019	0020	0020	0020	0020	0021	0021	0021	0022					
0019	0019	0019	0020	0020	0020	0021	0021	0021					
0018	0019	0019	0019	0019	0020	0020	0020	0021					
0018	0018	0018	0019	0019	0019	0020	0020	0020					
0017	0018	0018	0018	0018	0019	0019	0019	0020					
0017	0017	0017	0018	0018	0018	0019	0019	0019					
0016	0017	0017	0017	0018	0018	0018	0018	0019					
0016	0016	0017	0017	0017	0017	0018	0018	0018					
0016	0016	0016	0016	0017	0017	0017	0017	0018					
0015	0015	0016	0016	0016	0016	0017	0017	0017					
0015	0015	0015	0016	0016	0016	0016	0017	0017					
0014	0015	0015	0015	0015	0016	0016	0016	0016					
0014	0014	0015	0015	0015	0015	0015	0016	0016					
0014	0014	0014	0014	0015	0015	0015	0015	0015					
0013	0014	0014	0014	0014	0014	0015	0015	0015					
0013	0013	0013	0014	0014	0014	0014	0014	0015					
0013	0013	0013	0013	0013	0014	0014	0014	0014					
0012	0012	0013	0013	0013	0013	0013	0014	0014					
0012	0012	0012	0013	0013	0013	0013	0013	0013					
0011	0011	0012	0012	0012	0012	0012	0012	0013					
0011	0011	0011	0011	0011	0011	0012	0012	0012					
0010	0010	0010	0011	0011	0011	0011	0011	0011					
0010	0010	0010	0010	0010	0010	0010	0010	0011					
0009	0009	0009	0009	0009	0010	0010	0010	0010					
0008	0009	0009	0009	0009	0009	0009	0009	0009					
0008	0008	0008	0008	0008	0008	0009	0009	0009					
0007	0008	0008	0008	0008	0008	0008	0008	0008					
0007	0007	0007	0007	0007	0007	0008	0008	0008					
0007	0007	0007	0007	0007	0007	0007	0007	0007					
0006	0006	0006	0006	0007	0007	0007	0007	0007					
0006	0006	0006	0006	0006	0006	0006	0006	0006					
0005	0006	0006	0006	0006	0006	0006	0006	0006					
0005	0005	0005	0005	0005	0005	0005	0005	0006					
0005	0005	0005	0005	0005	0005	0005	0005	0005					
0005	0005	0005	0005	0005	0005	0005	0005	0005					
0004	0004	0004	0004	0004	0004	0004	0004	0004					
0004	0004	0004	0004	0004	0004	0004	0004	0004					
0004	0004	0004	0004	0004	0004	0004	0004	0004					
0004	0004	0004	0004	0004	0004	0004	0004	0004					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					
0003	0003	0003	0003	0003	0003	0003	0003	0003					

TABLE XV. Log B.

For Correcting Lunar Distances.

App. Alt. of ⊙ or ✳.	REDUCED REFRACTION AND PARALLAX OF ⊙ OR ✳.										
	0' 0"	0' 30"	1' 0"	1' 30"	2' 0"	2' 30"	3' 0"	3' 30"	4' 0"	4' 30"	5'
5° 0'	......	......	......	......	......	......	......	......	......	......	...
10	......	......	......	......	......	......	......	......	......	......	...
20	......	......	......	......	......	......	......	......	......	......	...
30	......	......	......	......	......	......	......	......	......	......	...
40	......	......	......	......	......	......	......	......	......	......	...
50	......	......	......	......	......	......	......	......	......	......	...
6 0	......	......	......	......	......	......	......	......	......	......	...
20	......	......	......	......	......	......	......	......	......	......	...
40	......	......	......	......	......	......	......	......	......	......	...
7 0	......	......	......	......	......	......	......	......	......	......	9·9
20	......	......	......	......	......	......	......	......	......	......	9·9
40	......	......	......	......	......	......	......	......	......	9·9981	9·9
8 0	......	......	......	......	......	......	......	......	......	9·9982	9·9
20	......	......	......	......	......	......	......	......	......	9·9982	9·9
40	......	......	......	......	......	......	......	......	......	9·9983	9·9
9 0	......	......	......	......	......	......	......	......	9·9986	9·9984	9·9
20	......	......	......	......	......	......	......	......	9·9986	9·9985	9·9
40	......	......	......	......	......	......	......	......	9·9987	9·9985	9·9
10 0	......	......	......	......	......	......	......	9·9989	9·9988	9·9986	9·9
11	......	......	......	......	......	......	9·9992	9·9991	9·9989	9·9987	9·9
12	......	......	......	......	......	......	9·9993	9·9992	9·9990	9·9989	9·9
13	......	......	......	......	......	9·9995	9·9994	9·9992	9·9991	9·9990	9·9
14	......	......	......	......	......	9·9995	9·9994	9·9993	9·9992	9·9991	9·9
15	......	......	......	......	9·9997	9·9996	9·9995	9·9994	9·9993	9·9992	9·9
16	......	......	......	......	9·9997	9·9996	9·9995	9·9994	9·9993	9·9993	...
18	......	......	......	9·9990	9·9998	9·9997	9·9996	9·9995	9·9995	......	...
20	......	......	0.0000	9·9999	9·9998	9·9998	9·9997	9·9996	9·9996	......	...
25	......	......	0.0000	0.0000	9·9999	9·9999	9·9998	9·9998	......	......	...
30	......	0.0001	0.0001	0.0000	0.0000	0.0000	9·9999	......	......	......	...
50	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	......	......	......	......	...
90	0.0001	0.0002	0.0002	0.0002	......	......	......	......	......	... ..	...

Log C.

App. Alt. of ⊙ or ✳.	REDUCED REFRACTION AND PARALLAX OF ⊙ OR ✳.										
	0' 0"	0' 30"	1' 0"	1' 30"	2' 0"	2' 30"	3' 0"	3' 30"	4' 0"	4' 30"	5'
5° 0'	......	......	......	......	......	......	......	......	......	......	...
20	......	......	......	......	......	......	......	......	......	......	...
40	......	......	......	......	......	......	......	......	......	......	...
6 0	......	......	......	......	......	......	......	......	......	......	...
20	......	......	......	......	......	......	......	......	......	......	...
40	......	......	......	......	......	......	......	......	......	......	...
7 0	......	......	......	......	......	......	......	......	......	......	9·9
8	......	......	......	......	......	......	......	......	......	9·9980	9·9
9	......	......	......	......	......	......	......	......	9·9984	9·9982	9·9
10	......	......	......	......	......	......	......	9·9988	9·9986	9·9984	9·9
11	......	.....	......	......	......	......	9·9990	9·9989	9·9987	9·9986	9·9
12	......	......	......	......	......	......	9·9991	9·9990	9·9988	9·9987	9·9
13	......	......	......	......	......	9·9993	9·9992	9·9991	9·9989	9·9988	9·9
14	......	......	......	......	......	9·9994	9·9993	9·9991	9·9990	9·9989	9·9
15	......	......	......	......	9·9995	9·9994	9·9993	9·9992	9·9991	9·9990	9·9
16	......	......	......	......	9·9996	9·9995	9·9994	9·9993	9·9992	9·9990	...
17	......	......	......	......	9·9996	9·9995	9·9994	9·9993	9·9992	9·9991	...
18	......	......	......	9·9997	9·9996	9·9995	9·9994	9·9994	9·9993	......	...
20	......	......	9·9998	9·9998	9·9997	9·9996	9·9995	9·9994	9·9993	......	...
25	......	......	9·9999	9·9998	9·9998	9·9997	9·9996	9·9996	......	......	...
30	......	0.0000	9·9999	9·9999	9·9998	9·9998	9·9997	......	......	......	...
40	......	0.0000	9·9999	9·9999	9·9999	9·9999	......	......	......	......	...
50	0.0000	0.0000	0.0000	9·9999	9·9999	......	......	......	......	......	...
90	0.0000	0.0000	0.0000	0.0000	......	......	......	......	......	......	...

TABLE XV. Log B.

For Correcting Lunar Distances.

REDUCED REFRACTION AND PARALLAX OF ⊙ OR *.											
6' 0"	6' 30'	7' 0"	7' 30"	8' 0"	8' 30"	9' 0"	9' 30"	10' 0"	10' 30"	11' 0"	11' 30"
......	......	9·9951	9·9947	9·9944	9·9940	9·9937	9·9933	9·9929	9·9926	9·9922	9·9919
......	......	9·9953	9·9949	9·9946	9·9942	9·9939	9·9935	9·9932	9·9928	9·9925	9·9921
......	......	9·9954	9·9951	9·9948	9·9944	9·9941	9·9937	9·9934	9·9931	9·9927	9·9924
......	9·9959	9·9956	9·9952	9·9949	9·9946	9·9943	9·9939	9·9936	9·9933	9·9929	......
......	9·9960	9·9957	9·9954	9·9951	9·9948	9·9944	9·9941	9·9938	9·9935	9·9932	......
9·9965	9·9962	9·9958	9·9955	9·9952	9·9949	9·9946	9·9943	9·9940	9·9937	......	......
9·9966	9·9963	9·9960	9·9957	9·9954	9·9951	9·9948	9·9945	9·9942	9·9939	......	......
9·9968	9·9965	9·9962	9·9959	9·9956	9·9954	9·9951	9·9948	9·9945	......	......	......
9·9969	9·9967	9·9964	9·9961	9·9959	9·9956	9·9953	9·9951	9·9948	......	......	......
9·9971	9·9968	9·9966	9·9963	9·9961	9·9958	9·9956	9·9953	......	......	......	......
9·9972	9·9970	9·9968	9·9965	9·9963	9·9960	9·9958	......	......	......	......	......
9·9974	9·9971	9·9969	9·9967	9·9965	9·9962	......	......	......	......	......	......
9·9975	9·9973	9·9971	9·9968	9·9966	9·9964	......	......	......	......	......	......
9·9976	9·9974	9·9972	9·9970	9·9968	......	......	......	......	......	......	......
9·9977	9·9975	9·9973	9·9971	......	......	......	......	......	......	......	......
9·9978	9·9976	9·9974	9·9972	......	......	......	......	......	......	......	......
9·9979	9·9977	9·9975	......	......	......	......	......	......	......	......	......
9·9980	9·9978	9·9976	......	......	......	......	......	......	......	......	......
9·9981	9·9979	9·9977	......	......	......	......	......	......	......	......	......
9·9983	9·9981	......	......	......	......	......	......	......	......	......	......
9·9985	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......

Log C.

REDUCED REFRACTION AND PARALLAX OF ⊙ OR *.											
6' 0"	6' 30"	7' 0"	7' 30"	8' 0"	8' 30"	9' 0"	9' 30"	10' 0"	10' 30"	11' 0"	11' 30"
......	......	9·9949	9·9946	9·9942	9·9938	9·9935	9·9931	9·9927	9·9924	9·9920	9·9916
......	9·9956	9·9953	9·9949	9·9946	9·9942	9·9939	9·9936	9·9932	9·9929	9·9925	9·9922
9·9962	9·9959	9·9955	9·9952	9·9949	9·9946	9·9943	9·9939	9·9936	9·9933	9·9930	......
9·9964	9·9961	9·9958	9·9955	9·9952	9·9949	9·9946	9·9943	9·9940	9·9937	......	......
9·9966	9·9963	9·9960	9·9957	9·9955	9·9952	9·9949	9·9946	9·9943	......	......	......
9·9968	9·9965	9·9962	9·9960	9·9957	9·9954	9·9951	9·9949	9·9946	......	......	......
9·9969	9·9967	9·9964	9·9962	9·9959	9·9956	9·9954	9·9951	......	......	......	......
9·9973	9·9971	9·9969	9·9966	9·9964	9·9962	9·9960	......	......	......	......	......
9·9976	9·9974	9·9972	9·9970	9·9968	......	......	......	......	......	......	......
9·9979	9·9977	9·9975	......	......	......	......	......	......	......	......	......
9·9981	9·9979	......	......	......	......	......	......	......	......	......	......
9·9983	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......

TABLE XV. Log D.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.												
	41′	42′	43′	44′	45′	46′	47′	48′	49′	50′	51′	52′	53′
° ′													
5 0	.0283	0290	0296	0303	0310	0316	0323	0329	0336	0343	0349	0356	0362
3	.0280	0287	0293	0300	0307	0313	0320	0326	0333	0339	0346	0352	0359
6	.0277	0284	0291	0297	0304	0310	0317	0323	0330	0336	0342	0349	0355
9	.0275	0281	0288	0294	0301	0307	0313	0320	0326	0333	0339	0345	0352
12	.0272	0279	0285	0291	0298	0304	0310	0317	0323	0330	0336	0342	0349
5 15	.0270	0276	0282	0289	0295	0301	0308	0314	0320	0326	0333	0339	0345
18	.0267	0273	0280	0286	0292	0298	0305	0311	0317	0323	0330	0336	0342
21	.0264	0271	0277	0283	0289	0296	0302	0308	0314	0320	0327	0333	0339
24	.0262	0268	0274	0281	0287	0293	0299	0305	0311	0317	0324	0330	0336
27	.0260	0266	0272	0278	0284	0290	0296	0302	0308	0314	0321	0327	0333
5 30	.0257	0263	0269	0275	0282	0288	0294	0300	0306	0312	0318	0324	0330
33	.0255	0261	0267	0273	0279	0285	0291	0297	0303	0309	0315	0321	0327
36	.0253	0259	0265	0271	0276	0282	0288	0294	0300	0306	0312	0318	0324
39	......	0256	0262	0268	0274	0280	0286	0292	0298	0303	0309	0315	0321
42	......	0254	0260	0266	0272	0277	0283	0289	0295	0301	0306	0312	0318
5 45		0252	0258	0263	0269	0275	0281	0287	0292	0298	0304	0310	0315
48		0250	0255	0261	0267	0273	0278	0284	0290	0295	0301	0307	0313
51		0247	0253	0259	0265	0270	0276	0282	0287	0293	0299	0304	0310
54		0245	0251	0257	0262	0268	0274	0279	0285	0290	0296	0302	0307
57		0243	0249	0254	0260	0266	0271	0277	0282	0288	0294	0299	0305
6 0		0241	0247	0252	0258	0263	0269	0275	0280	0286	0291	0297	0302
3		0239	0245	0250	0256	0261	0267	0272	0278	0283	0289	0294	0300
6		0237	0243	0248	0254	0259	0265	0270	0275	0281	0286	0292	0297
9		0235	0241	0246	0252	0257	0262	0268	0273	0279	0284	0289	0295
12		0233	0239	0244	0249	0255	0260	0266	0271	0276	0282	0287	0292
6 15		0231	0237	0242	0247	0253	0258	0263	0269	0274	0279	0285	0290
18		0230	0235	0240	0245	0251	0256	0261	0267	0272	0277	0282	0288
21		0228	0233	0238	0243	0249	0254	0259	0264	0270	0275	0280	0285
24		0226	0231	0236	0242	0247	0252	0257	0262	0267	0273	0278	0283
27		......	0229	0234	0240	0245	0250	0255	0260	0265	0271	0276	0281
6 30			0227	0233	0238	0243	0248	0253	0258	0263	0268	0274	0279
33			0226	0231	0236	0241	0246	0251	0256	0261	0266	0271	0276
36			0224	0229	0234	0239	0244	0249	0254	0259	0264	0269	0274
39			0222	0227	0232	0237	0242	0247	0252	0257	0262	0267	0272
42			0220	0225	0230	0235	0240	0245	0250	0255	0260	0265	0270
6 45			0219	0224	0229	0234	0239	0244	0248	0253	0258	0263	0268
48			0217	0222	0227	0232	0237	0242	0247	0251	0256	0261	0266
51			0216	0220	0225	0230	0235	0240	0245	0250	0254	0259	0264
54			0214	0219	0224	0228	0233	0238	0243	0248	0253	0257	0262
57			0212	0217	0222	0227	0232	0236	0241	0246	0251	0255	0260
7 0			0211	0216	0220	0225	0230	0235	0239	0244	0249	0254	0258
3			0209	0214	0219	0223	0228	0233	0238	0242	0247	0252	0256
6			0208	0212	0217	0222	0227	0231	0236	0241	0245	0250	0255
9			......	0211	0216	0220	0225	0230	0234	0239	0243	0248	0253
12			......	0209	0214	0219	0223	0228	0232	0237	0242	0246	0251
7 15				0208	0212	0217	0222	0226	0231	0235	0240	0245	0249
18				0206	0211	0216	0220	0225	0229	0234	0238	0243	0247
21				0205	0209	0214	0219	0223	0228	0232	0237	0241	0246
24				0204	0208	0213	0217	0222	0226	0230	0235	0239	0244
27				0202	0207	0211	0216	0220	0224	0229	0233	0238	0242
7 30				0201	0205	0210	0214	0218	0223	0227	0232	0236	0241
33				0199	0204	0208	0213	0217	0221	0226	0230	0234	0239
36				0198	0202	0207	0211	0215	0220	0224	0229	0233	0237
39				0197	0201	0205	0210	0214	0218	0223	0227	0231	0236
42				0195	0200	0204	0208	0213	0217	0221	0225	0230	0234
7 45				0194	0198	0203	0207	0211	0215	0220	0224	0228	0232
48				0193	0197	0201	0205	0210	0214	0218	0222	0227	0231
51				0191	0196	0200	0204	0208	0213	0217	0221	0225	0229
54				0190	0194	0198	0203	0207	0211	0215	0219	0224	0228
57				0189	0193	0197	0201	0206	0210	0214	0218	0222	0226
8 0				0188	0192	0196	0200	0204	0208	0212	0217	0221	0225

TABLE XV. Log D.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.

5′	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′
192	0196	0200	0204	0208	0212	0217	0221	0225	0229	0233	0237		
190	0194	0198	0202	0206	0210	0214	0218	0222	0227	0231	0235		
188	0192	0196	0200	0204	0208	0212	0216	0220	0224	0228	0232		
186	0190	0194	0198	0202	0206	0210	0214	0218	0222	0226	0230		
184	0188	0192	0196	0200	0204	0207	0211	0215	0219	0223	0227		
182	0186	0190	0194	0197	0201	0205	0209	0213	0217	0221	0225		
180	0184	0188	0192	0195	0199	0203	0207	0211	0215	0219	0223		
178	0182	0186	0190	0193	0197	0201	0205	0209	0213	0216	0220		
176	0180	0184	0188	0191	0195	0199	0203	0207	0210	0214	0218		
174	0178	0182	0186	0189	0193	0197	0201	0205	0208	0212	0216		
173	0176	0180	0184	0188	0191	0195	0199	0202	0206	0210	0214		
171	0175	0178	0182	0186	0189	0193	0197	0200	0204	0208	0212		
169	0173	0177	0180	0184	0188	0191	0195	0198	0202	0206	0209		
167	0171	0175	0178	0182	0186	0189	0193	0197	0200	0204	0207		
166	0169	0173	0177	0180	0184	0187	0191	0195	0198	0202	0205		
164	0168	0171	0175	0179	0182	0186	0189	0193	0196	0200	0203		
163	0166	0170	0173	0177	0180	0184	0187	0191	0194	0198	0201		
161	0165	0168	0172	0175	0179	0182	0186	0189	0193	0196	0199		
	0163	0166	0170	0173	0177	0180	0184	0187	0191	0194	0198	......	
	0161	0165	0168	0172	0175	0179	0182	0185	0189	0192	0196	......	
	0160	0163	0167	0170	0174	0177	0180	0184	0187	0191	0194	......	
	0158	0162	0165	0169	0172	0175	0179	0182	0185	0189	0192	0195	
	0157	0160	0164	0167	0170	0174	0177	0180	0184	0187	0190	0194	
	0156	0159	0162	0165	0169	0172	0175	0179	0182	0185	0189	0192	
	0154	0157	0161	0164	0167	0171	0174	0177	0180	0184	0187	0190	
	0153	0156	0159	0162	0166	0169	0172	0175	0179	0182	0185	0188	
	0151	0155	0158	0161	0164	0167	0171	0174	0177	0180	0183	0187	
	0150	0153	0156	0160	0163	0166	0169	0172	0175	0179	0182	0185	
	0149	0152	0155	0158	0161	0164	0168	0171	0174	0177	0180	0183	
	0147	0150	0154	0157	0160	0163	0166	0169	0172	0175	0179	0182	
	0146	0149	0152	0155	0158	0162	0165	0168	0171	0174	0177	0180	
	0145	0148	0151	0154	0157	0160	0163	0166	0169	0172	0175	0179	
	0143	0147	0150	0153	0156	0159	0162	0165	0168	0171	0174	0177	
	0142	0145	0148	0151	0154	0157	0160	0163	0166	0169	0172	0175	
	0141	0144	0147	0150	0153	0156	0159	0162	0165	0168	0171	0174	
	0140	0143	0146	0149	0152	0155	0158	0161	0164	0167	0170	0172	
	0139	0142	0145	0147	0150	0153	0156	0159	0162	0165	0168	0171	
	0137	0140	0143	0146	0149	0152	0155	0158	0161	0164	0167	0170	
	......	0139	0142	0145	0148	0151	0154	0157	0159	0162	0165	0168	
	......	0138	0141	0144	0147	0150	0152	0155	0158	0161	0164	0167	
	......	0137	0140	0143	0145	0148	0151	0154	0157	0160	0163	0165	
	......	0136	0139	0141	0144	0147	0150	0153	0156	0158	0161	0164	
		0135	0137	0140	0143	0146	0149	0151	0154	0157	0160	0163	
		0133	0136	0139	0142	0145	0147	0150	0153	0156	0159	0161	
		0132	0135	0138	0141	0143	0146	0149	0152	0154	0157	0160	
		0131	0134	0137	0140	0142	0145	0148	0150	0153	0156	0159	
		0130	0133	0136	0138	0141	0144	0147	0149	0152	0155	0157	
		0129	0132	0135	0137	0140	0143	0145	0148	0151	0153	0156	
		0128	0131	0134	0136	0139	0142	0144	0147	0150	0152	0155	
		0127	0130	0132	0135	0138	0140	0143	0146	0148	0151	0154	
		0126	0129	0131	0134	0137	0139	0142	0145	0147	0150	0152	
		0125	0128	0130	0133	0136	0138	0141	0143	0146	0149	0151	
		0124	0127	0129	0132	0135	0137	0140	0142	0145	0147	0150	
		0123	0126	0128	0131	0133	0136	0139	0141	0144	0146	0149	
		0122	0125	0127	0130	0132	0135	0138	0140	0143	0145	0148	......
		0121	0124	0126	0129	0131	0134	0136	0139	0141	0144	0147	......
		0120	0123	0125	0128	0130	0133	0135	0138	0140	0143	0145	......
		0119	0122	0124	0127	0129	0132	0134	0137	0139	0142	0144	0147
		0118	0121	0123	0126	0128	0131	0133	0136	0138	0141	0143	0146
		0118	0120	0123	0125	0127	0130	0132	0135	0137	0140	0142	0145
		0117	0119	0122	0124	0126	0129	0131	0134	0136	0139	0141	0143

TABLE XV. Log D.

For correcting Lunar Distances.

App. Alt. of ☽.	REDUCED PARALLAX AND REFRACTION OF ☽.												
	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′	59′
° ′													
13 0	.0117	0119	0122	0124	0126	0129	0131	0134	0136	0139	0141	0143	
10	.0115	0117	0120	0122	0125	0127	0129	0132	0134	0137	0139	0141	
20	.0113	0116	0118	0120	0123	0125	0127	0130	0132	0134	0137	0139	
30	.0112	0114	0116	0119	0121	0123	0125	0128	0130	0132	0135	0137	
40	......	0112	0114	0117	0119	0121	0124	0126	0128	0131	0133	0135	
50	......	0111	0113	0115	0117	0120	0122	0124	0126	0129	0131	0133	
14 0		0109	0111	0113	0116	0118	0120	0122	0125	0127	0129	0131	
10		0107	0110	0112	0114	0116	0118	0121	0123	0125	0127	0129	
20		0106	0108	0110	0112	0114	0117	0119	0121	0123	0125	0127	
30		0104	0106	0109	0111	0113	0115	0117	0119	0121	0123	0126	
40		0103	0105	0107	0109	0111	0113	0115	0118	0120	0122	0124	
50		0101	0103	0106	0108	0110	0112	0114	0116	0118	0120	0122	
15 0		0100	0102	0104	0106	0108	0110	0112	0114	0116	0118	0120	
10		0099	0101	0103	0105	0107	0109	0111	0113	0115	0117	0119	
20		0097	0099	0101	0103	0105	0107	0109	0111	0113	0115	0117	
30		0096	0098	0100	0102	0104	0106	0108	0110	0112	0113	0115	
40		0094	0096	0098	0100	0102	0104	0106	0108	0110	0112	0114	
50		0093	0095	0097	0099	0101	0103	0105	0107	0108	0110	0112	
16 0		0092	0094	0096	0098	0099	0101	0103	0105	0107	0109	0111	
10		0091	0093	0094	0096	0098	0100	0102	0104	0106	0107	0109	
20		0089	0091	0093	0095	0097	0099	0100	0102	0104	0106	0108	
30		0088	0090	0092	0094	0096	0097	0099	0101	0103	0105	0106	
40		0087	0089	0091	0092	0094	0096	0098	0100	0101	0103	0105	
50		0086	0088	0089	0091	0093	0095	0096	0098	0100	0102	0104	
17 0		0085	0087	0088	0090	0092	0093	0095	0097	0099	0100	0102	
10		0084	0085	0087	0089	0091	0092	0094	0096	0097	0099	0101	
20		0083	0084	0086	0088	0089	0091	0093	0094	0096	0098	0099	
30		......	0083	0085	0086	0088	0090	0091	0093	0095	0096	0098	
40		......	0082	0084	0085	0087	0089	0090	0092	0094	0095	0097	
50		......	0081	0083	0084	0086	0087	0089	0091	0092	0094	0096	
18 0			0080	0082	0083	0085	0086	0088	0090	0091	0093	0094	......
20			0078	0079	0081	0083	0084	0086	0087	0089	0090	0092	0093
40			0076	0077	0079	0080	0082	0083	0085	0087	0088	0090	0091
19 0			0074	0075	0077	0078	0080	0081	0083	0084	0086	0087	0089
20			0072	0073	0075	0076	0078	0079	0081	0082	0084	0085	0086
40			0070	0072	0073	0074	0076	0077	0079	0080	0081	0083	0084
20 0			0068	0070	0071	0073	0074	0075	0077	0078	0079	0081	0082
20			0067	0068	0069	0071	0072	0073	0075	0076	0077	0079	0080
40			0065	0066	0068	0069	0070	0072	0073	0074	0075	0077	0078
21 0			0063	0065	0066	0067	0068	0070	0071	0072	0074	0075	0076
20			0062	0063	0064	0065	0067	0068	0069	0070	0072	0073	0074
40			0060	0061	0063	0064	0065	0066	0067	0069	0070	0071	0072
22 0			0059	0060	0061	0062	0063	0065	0066	0067	0068	0069	0070
20			0057	0058	0059	0061	0062	0063	0064	0065	0066	0068	0069
40			0056	0057	0058	0059	0060	0061	0062	0064	0065	0066	0067
23 0			0054	0055	0057	0058	0059	0060	0061	0062	0063	0064	0065
20			0053	0054	0055	0056	0057	0058	0059	0060	0061	0063	0064
40			0052	0053	0054	0055	0056	0057	0058	0059	0060	0061	0062
24 0			0050	0051	0052	0053	0054	0055	0056	0057	0058	0059	0060
20			......	0050	0051	0052	0053	0054	0055	0056	0057	0058	0059
40			......	0049	0050	0051	0052	0053	0053	0054	0055	0056	0057
25 0			......	0047	0048	0049	0050	0051	0052	0053	0054	0055	0056
20			......	0046	0047	0048	0049	0050	0051	0052	0053	0053	0054
40			......	0045	0046	0047	0048	0049	0049	0050	0051	0052	0053
26 0				0044	0045	0046	0046	0047	0048	0049	0050	0051	0052
20				0043	0043	0044	0045	0046	0047	0048	0048	0049	0050
40				0041	0042	0043	0044	0045	0046	0046	0047	0048	0049
27 0				0040	0041	0042	0043	0044	0044	0045	0046	0047	0047
20				0039	0040	0041	0042	0042	0043	0044	0045	0045	0046
40				0038	0039	0040	0040	0041	0042	0043	0043	0044	0045
28 0				0037	0038	0039	0039	0040	0041	0042	0042	0043	0044

TABLE XV. Log D.

For correcting Lunar Distances.

REDUCED PARALLAX AND REFRACTION OF ☽.

50′	51′	52′	53′	54′	55′	56′	57′	58′	59′	60′
0037	0.0038	0.0039	0.0039	0.0040	0.0041	0.0042	0.0042	0.0043	0.0044	
0036	0.0036	0.0037	0.0038	0.0038	0.0039	0.0040	0.0040	0.0041	0.0042	
0034	0.0035	0.0035	0.0036	0.0037	0.0037	0.0038	0.0039	0.0039	0.0040	
0033	0.0033	0.0034	0.0035	0.0035	0.0036	0.0036	0.0037	0.0038	0.0038	
0031	0.0032	0.0032	0.0033	0.0034	0.0034	0.0035	0.0035	0.0036	0.0037	
0030	0.0030	0.0031	0.0031	0.0032	0.0033	0.0033	0.0034	0.0034	0.0035	
0028	0.0029	0.0029	0.0030	0.0031	0.0031	0.0032	0.0032	0.0033	0.0033	....
0027	0.0028	0.0028	0.0029	0.0029	0.0030	0.0030	0.0031	0.0031	0.0032	0.0032
0026	0.0026	0.0027	0.0027	0.0028	0.0028	0.0029	0.0029	0.0030	0.0030	0.0031
0024	0.0025	0.0025	0.0026	0.0026	0.0027	0.0027	0.0028	0.0028	0.0029	0.0029
0023	0.0024	0.0024	0.0025	0.0025	0.0025	0.0026	0.0026	0.0027	0.0027	0.0028
0022	0.0022	0.0023	0.0023	0.0024	0.0024	0.0025	0.0025	0.0025	0.0026	0.0026
0021	0.0021	0.0022	0.0022	0.0022	0.0023	0.0023	0.0024	0.0024	0.0024	0.0025
0020	0.0020	0.0020	0.0021	0.0021	0.0022	0.0022	0.0022	0.0023	0.0023	0.0023
0018	0.0019	0.0019	0.0020	0.0020	0.0020	0.0021	0.0021	0.0021	0.0022	0.0022
0017	0.0018	0.0018	0.0018	0.0019	0.0019	0.0019	0.0020	0.0020	0.0020	0.0021
0016	0.0017	0.0017	0.0017	0.0018	0.0018	0.0018	0.0019	0.0019	0.0019	0.0019
0015	0.0016	0.0016	0.0016	0.0016	0.0017	0.0017	0.0017	0.0018	0.0018	0.0018
0014	0.0014	0.0015	0.0015	0.0015	0.0016	0.0016	0.0016	0.0016	0.0017	0.0017
0013	0.0013	0.0014	0.0014	0.0014	0.0014	0.0015	0.0015	0.0015	0.0015	0.0016
0012	0.0012	0.0013	0.0013	0.0013	0.0013	0.0014	0.0014	0.0014	0.0014	0.0014
0011	0.0011	0.0012	0.0012	0.0012	0.0012	0.0012	0.0013	0.0013	0.0013	0.0013
0010	0.0010	0.0011	0.0011	0.0011	0.0011	0.0011	0.0012	0.0012	0.0012	0.0012
......	0.0009	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0011	0.0011	0.0011
	0.0008	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009	0.0010	0.0010	0.0010
	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0008	0.0008
	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0006
	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0004
	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002	0.0002	0.0002
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998	9.9998
	9.9997	9.9997	9.9997	9.9997	9.9996	9.9996	9.9996	9.9996	9.9996	9.9996
	9.9995	9.9995	9.9995	9.9995	9.9995	9.9995	9.9995	9.9995	9.9994	9.9994
	9.9994	9.9994	9.9994	9.9993	9.9993	9.9993	9.9993	9.9993	9.9993	9.9993
	9.9992	9.9992	9.9992	9.9992	9.9992	9.9992	9.9992	9.9991	9.9991	9.9991
	9.9991	9.9991	9.9991	9.9991	9.9990	9.9990	9.9990	9.9990	9.9990	9.9990
	9.9990	9.9990	9.9990	9.9989	9.9989	9.9989	9.9989	9.9989	9.9988	9.9988
	9.9989	9.9988	9.9988	9.9988	9.9988	9.9988	9.9987	9.9987	9.9987	9.9987
	9.9988	9.9987	9.9987	9.9987	9.9987	9.9986	9.9986	9.9986	9.9986	9.9985
	9.9986	9.9986	9.9986	9.9986	9.9985	9.9985	9.9985	9.9984	9.9984	9.9984
	9.9985	9.9985	9.9985	9.9984	9.9984	9.9984	9.9984	9.9983	9.9983	9.9983
	9.9984	9.9984	9.9984	9.9983	9.9983	9.9983	9.9982	9.9982	9.9982	9.9981
	9.9983	9.9983	9.9983	9.9982	9.9982	9.9982	9.9981	9.9981	9.9981	9.9980
	9.9982	9.9982	9.9981	9.9981	9.9981	9.9980	9.9980	9.9980	9.9979	9.9979
	9.9981	9.9981	9.9980	9.9980	9.9980	9.9979	9.9979	9.9979	9.9978	9.9978
	9.9980	9.9980	9.9980	9.9979	9.9979	9.9978	9.9978	9.9978	9.9977	9.9977
	9.9979	9.9979	9.9979	9.9978	9.9978	9.9977	9.9977	9.9977	9.9976	9.9976
	9.9979	9.9978	9.9978	9.9977	9.9977	9.9976	9.9976	9.9976	9.9975	9.9975
	9.9978	9.9977	9.9977	9.9976	9.9976	9.9976	9.9975	9.9975	9.9974	9.9974
	9.9977	9.9977	9.9976	9.9976	9.9975	9.9975	9.9974	9.9974	9.9973	9.9972
	9.9976	9.9976	9.9975	9.9975	9.9974	9.9974	9.9973	9.9973	9.9973	9.9972
	9.9976	9.9975	9.9975	9.9974	9.9974	9.9973	9.9973	9.9972	9.9972	9.9971
	9.9975	9.9974	9.9974	9.9973	9.9973	9.9972	9.9972	9.9971	9.9971	9.9970
	9.9974	9.9974	9.9973	9.9973	9.9972	9.9972	9.9971	9.9971	9.9970	9.9970
	9.9974	9.9973	9.9973	9.9972	9.9972	9.9971	9.9970	9.9970	9.9969	9.9969
	9.9972	9.9972	9.9971	9.9971	9.9970	9.9970	9.9969	9.9969	9.9968	9.9968
	9.9971	9.9971	9.9970	9.9970	9.9969	9.9969	9.9968	9.9968	9.9967	9.9966
	9.9971	9.9970	9.9969	9.9969	9.9968	9.9968	9.9967	9.9966	9.9966	9.9965
	9.9970	9.9969	9.9969	9.9968	9.9967	9.9967	9.9966	9.9966	9.9965	9.9964
	9.9969	9.9969	9.9968	9.9967	9.9967	9.9966	9.9965	9.9965	9.9964	9.9964
	9.9968	9.9967	9.9966	9.9966	9.9965	9.9964	9.9964	9.9963	9.9963	9.9962

TABLE XVI.

Second Correction of the Lunar Distance.

Apparent Distance.	FIRST CORRECTION OF DISTANCE.																					
	3'	7'	10'	12'	14'	16'	18'	20	21'	22'	23'	24'	25'	26'	27'	28'	29'	30'	31'	32'	33'	34'
Subtr.	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
15° 0'	0	2	3	5	6	8	11	13	14	16	17	19	20	22	24	26	27	29	31	33	35	38
30	0	2	3	5	6	8	10	13	14	15	17	18	20	21	23	25	26	28	30	32	34	36
16 0	0	1	3	4	6	8	10	12	13	15	16	18	19	21	22	24	26	27	29	31	33	35
30	0	1	3	4	6	8	10	12	13	14	16	17	18	20	21	23	25	27	28	30	32	34
17 0	0	1	3	4	6	7	9	11	13	14	15	16	18	19	21	22	24	26	27	29	31	33
30	0	1	3	4	5	7	9	11	12	13	15	16	17	19	20	22	23	25	27	28	30	32
18 0	0	1	3	4	5	7	9	11	12	13	14	15	17	18	20	21	23	24	26	28	29	31
30	0	1	3	4	5	7	8	10	12	13	14	15	16	18	19	20	22	23	25	27	28	30
19 0	0	1	3	4	5	6	8	10	11	12	13	15	16	17	18	20	21	23	24	26	28	29
30	0	1	2	4	5	6	8	10	11	12	13	14	15	17	18	19	21	22	24	25	27	28
20	0	1	2	3	5	6	8	10	11	12	13	14	15	16	17	19	20	22	23	25	26	28
21	0	1	2	3	4	6	7	9	10	11	12	13	14	15	17	18	19	20	22	23	25	26
22	0	1	2	3	4	6	7	9	10	10	11	12	14	15	16	17	18	19	21	22	24	25
23	0	1	2	3	4	5	7	8	9	10	11	12	13	14	15	16	17	19	20	21	22	24
24	0	1	2	3	4	5	6	8	9	9	10	11	12	13	14	15	16	18	19	20	21	23
25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22
26	0	1	2	3	4	5	6	7	8	9	9	10	11	12	13	14	15	16	17	18	19	21
27	0	1	2	2	3	4	6	7	8	8	9	10	11	12	12	13	14	15	16	18	19	20
28	0	1	2	2	3	4	5	7	7	8	9	9	10	11	12	13	14	15	16	17	18	19
29	0	1	2	2	3	4	5	6	7	8	8	9	10	11	11	12	13	14	15	16	17	18
30	0	1	2	2	3	4	5	6	7	7	8	9	9	10	11	12	13	14	14	15	16	17
31	0	1	1	2	3	4	5	6	6	7	8	8	9	10	11	11	12	13	14	15	16	17
32	0	1	1	2	3	4	5	6	6	7	7	8	9	9	10	11	12	13	13	14	15	16
33	0	1	1	2	3	3	4	5	6	7	7	8	8	9	10	11	11	12	13	14	15	16
34	0	1	1	2	3	3	4	5	6	6	7	7	8	9	9	10	11	12	12	13	14	15
35	0	1	1	2	2	3	4	5	5	6	7	7	8	8	9	10	10	11	12	13	14	14
36	0	1	1	2	2	3	4	5	5	6	6	7	8	8	9	9	10	11	12	12	13	14
37	0	1	1	2	2	3	4	5	5	6	6	7	7	8	8	9	10	10	11	12	13	13
38	0	1	1	2	2	3	4	4	5	5	6	6	7	8	8	9	9	10	11	11	12	13
39	0	1	1	2	2	3	3	4	5	5	6	6	7	7	8	8	9	10	10	11	12	12
40	0	1	1	2	2	3	3	4	5	5	6	6	7	7	8	8	9	9	10	11	11	12
42	0	0	1	1	2	2	3	4	4	5	5	6	6	7	7	8	8	9	9	10	11	11
44	0	0	1	1	2	2	3	4	4	4	5	5	6	6	7	7	8	8	9	9	10	10
46	0	0	1	1	2	2	3	3	4	4	4	5	5	6	6	7	7	8	8	9	9	10
48	0	0	1	1	2	2	3	3	3	4	4	5	5	5	6	6	7	7	8	8	9	9
50	0	0	1	1	1	2	2	3	3	4	4	4	5	5	5	6	6	7	7	8	8	8
52	0	0	1	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	7	7	7	8
54	0	0	1	1	1	2	2	3	3	3	3	4	4	4	5	5	5	6	6	6	7	7
56	0	0	1	1	1	2	2	2	3	3	3	3	4	4	4	5	5	5	6	6	6	7
58	0	0	1	1	1	1	2	2	2	3	3	3	3	4	4	4	5	5	5	6	6	6
60	0	0	0	1	1	1	2	2	2	2	3	3	3	3	4	4	4	5	5	5	5	6
62	0	0	0	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5
64	0	0	0	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5
66	0	0	0	1	1	1	1	2	2	2	2	2	2	3	3	3	3	4	4	4	4	4
68	0	0	0	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	4	4	4
70	0	0	0	0	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	4
74	0	0	0	0	0	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3
78	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
82	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
86	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
90°	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Apparent Distance	3'	7'	10'	12'	14'	16'	18'	20'	21'	22'	23'	24'	25'	26'	27'	28'	29'	30'	31'	32'	33'	34'
	FIRST CORRECTION OF DISTANCE.																					

TABLE XVI.

Second Correction of the Lunar Distance.

FIRST CORRECTION OF DISTANCE.																								Apparent Distance.
37′	38′	39′	40′	41′	42′	43′	44	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′	59′	60′	
″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	Add.
45	47	50	52	55	57	60	63	66	69	72	75	78	81	85	88	91	95	99	102	106	110	113	117	
43	45	48	50	53	56	58	61	64	67	70	72	76	79	82	85	88	92	95	99	102	106	110	113	
42	44	46	49	51	54	56	59	62	64	67	70	73	76	79	82	85	89	92	95	99	102	106	110	
40	43	45	47	50	52	54	57	60	62	65	68	71	74	77	80	83	86	89	92	96	99	103	106	
39	41	43	46	48	50	53	55	58	60	63	66	69	71	74	77	80	83	86	90	93	96	99	103	
38	40	42	44	47	49	51	54	56	59	61	64	66	69	72	75	78	81	84	87	90	93	96	100	
37	39	41	43	45	47	50	52	54	57	59	62	64	67	70	73	75	78	81	84	87	90	94	97	
36	38	40	42	44	46	48	50	53	55	58	60	63	65	68	71	73	76	79	82	85	88	91	94	
35	37	39	41	43	45	47	49	51	54	56	58	61	63	66	69	71	74	77	79	82	85	88	91	
34	36	37	39	41	43	46	48	50	52	54	57	59	62	64	67	69	72	75	77	80	83	86	89	
33	35	36	38	40	42	44	46	49	51	53	55	58	60	62	65	67	70	73	75	78	81	83	86	
31	33	35	36	38	40	42	44	46	48	50	52	55	57	59	61	64	66	69	71	74	76	79	82	
30	31	33	35	36	38	40	42	44	46	48	50	52	54	56	58	61	63	65	68	70	73	75	78	
28	30	31	33	35	36	38	40	42	44	45	47	49	51	53	56	58	60	62	64	67	69	72	74	
27	28	30	31	33	35	36	38	40	41	43	45	47	49	51	53	55	57	59	61	64	66	68	71	
26	27	28	30	31	33	35	36	38	40	41	43	45	47	49	51	53	55	57	59	61	63	65	67	
25	26	27	29	30	32	33	35	36	38	40	41	43	45	47	48	50	52	54	56	58	60	62	64	
23	25	26	27	29	30	32	33	35	36	38	39	41	43	45	46	48	50	52	54	56	58	60	62	
22	24	25	26	28	29	30	32	33	35	36	38	39	41	43	44	46	48	50	51	53	55	57	59	
22	23	24	25	26	28	29	30	32	33	35	36	38	39	41	43	44	46	48	49	51	53	55	57	
21	22	23	24	25	27	28	29	31	32	33	35	36	38	39	41	42	44	46	47	49	51	53	54	
20	21	22	23	24	26	27	28	29	31	32	33	35	36	38	39	41	42	44	46	47	49	51	52	
19	20	21	22	23	25	26	27	28	30	31	32	34	35	36	38	39	41	42	44	45	47	49	50	
18	19	20	22	23	24	25	26	27	28	30	31	32	34	35	36	38	39	41	42	44	45	47	48	
18	19	20	21	22	23	24	25	26	27	29	30	31	32	34	35	36	38	39	41	42	44	45	47	
17	18	19	20	21	22	23	24	25	26	28	29	30	31	32	34	35	36	38	39	40	42	43	45	
16	17	18	19	20	21	22	23	24	25	27	28	29	30	31	32	34	35	36	38	39	40	42	43	
16	17	18	19	19	20	21	22	23	25	26	27	28	29	30	31	33	34	35	36	38	39	40	42	
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	33	34	35	36	37	39	40	
15	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	33	34	35	36	38	39	
14	15	16	17	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	33	34	35	36	37	140°
13	14	15	16	16	17	18	19	20	21	21	22	23	24	25	26	27	28	29	30	31	33	34	35	138
12	13	14	15	15	16	17	17	18	19	20	21	22	23	24	24	25	26	27	28	29	30	31	33	136
12	12	13	14	14	15	16	16	17	18	19	19	20	21	22	23	24	25	26	26	27	28	29	30	134
11	11	12	13	13	14	15	15	16	17	17	18	19	20	20	21	22	23	24	25	26	26	27	28	132
10	11	11	12	12	13	14	14	15	16	16	17	18	18	19	20	21	21	22	23	24	25	25	26	130
9	10	10	11	11	12	13	13	14	14	15	16	16	17	18	18	19	20	21	21	22	23	24	25	128
9	9	10	10	11	11	12	12	13	13	14	15	15	16	16	17	18	18	19	20	21	21	22	23	126
8	9	9	9	10	10	11	11	12	12	13	14	14	15	15	16	17	17	18	18	19	20	20	21	124
7	8	8	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	17	18	18	19	20	122
7	7	8	8	8	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	17	18	18	120
6	7	7	7	8	8	9	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	17	118
6	6	6	7	7	8	8	8	9	9	9	10	10	11	11	12	12	12	13	13	14	14	15	15	116
5	6	6	6	7	7	7	8	8	8	9	9	9	10	10	11	11	11	12	12	13	13	13	14	114
5	5	5	6	6	6	7	7	7	7	8	8	8	9	9	10	10	10	11	11	11	12	12	13	112
4	5	5	5	5	6	6	6	6	7	7	7	8	8	8	9	9	9	10	10	10	11	11	11	110
3	4	4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7	8	8	8	8	9	9	106
3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	6	6	6	6	6	7	102
2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	98
1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	94
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	90°
37′	38′	39′	40′	41′	42′	43′	44′	45′	46′	47′	48′	49′	50′	51′	52′	53′	54′	55′	56′	57′	58′	59′	60′	Apparent Distance.
FIRST CORRECTION OF DISTANCE.																								

TABLE XVII.

For finding the Correction of the Lunar Distance for the Contraction of the Moon's Se

TABLE XVII. A. Giving the Argument for Table XVII. B.

Reduced P. and R. of ☽.	APPARENT ALTITUDE OF ☽.																				
	5°	5½°	6°	6½°	7°	7½°	8°	8½°	9°	9½°	10°	11°	12°	13°	14°	15°	16°	17°	18°	20°	25°
41′	65	56	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
42	63	54	47	41	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
43	62	53	46	40	35	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
44	60	51	45	39	34	30	27	...	...	...	...	...	...	...	...	...	...	...	...	...	...
45	58	50	43	38	33	30	26	24	21	20	...	...	...	...	...	...	...	...	...	...	...
46	57	49	42	37	33	29	26	23	21	19	17	15	...	...	...	...	...	...	...	...	...
47	56	48	41	36	32	28	25	23	20	19	17	14	12	10	...	...	...	...	...	...	...
48	54	46	40	35	31	28	25	22	20	18	17	14	12	10	9	8	7	6	...	...	...
49	53	45	39	35	30	27	24	22	19	18	16	14	12	10	9	8	7	6	6	5	3
50	52	44	38	34	30	26	24	21	19	17	16	13	11	10	9	8	7	6	5	5	3
51	50	43	38	33	29	26	23	21	19	17	15	13	11	10	8	7	7	6	5	5	3
52	49	42	37	32	28	25	23	20	18	17	15	13	11	9	8	7	7	6	5	4	3
53	48	41	36	32	28	25	22	20	18	16	15	12	11	9	8	7	6	6	5	4	3
54	47	41	35	31	27	24	22	19	18	16	15	12	10	9	8	7	6	6	5	4	3
55	...	...	35	30	27	24	21	19	17	16	14	12	10	9	8	7	6	6	5	4	3
56	...	...	...	...	26	23	21	19	17	15	14	12	10	9	8	7	6	5	5	4	3
57	...	...	...	...	...	...	...	18	17	15	14	12	10	9	7	7	6	5	5	4	3
58	...	...	...	...	...	...	...	...	...	...	13	11	10	8	7	7	6	5	5	4	3
59	...	...	...	...	...	...	...	...	...	...	...	...	...	8	7	6	6	5	5	4	3
60	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	4	3

TABLE XVII. B. Contraction of ☽'s Semidiameter.

Whole Correction of ☽.	ARGUMENT = NUMBER FROM TABLE XVII. A.																					
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	44	48
	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″
0′	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
15	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
20	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	4	4
22	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	3	4	4	4	5
24	0	0	1	1	1	1	2	2	2	2	3	3	3	3	3	4	4	4	4	5	5	6
26	0	1	1	1	1	2	2	2	2	3	3	3	4	4	4	4	5	5	5	5	6	6
28	0	1	1	1	2	2	2	3	3	3	3	4	4	4	5	5	5	6	6	6	7	8
30	0	1	1	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	8	9
32	0	1	1	2	2	2	3	3	4	4	5	5	5	6	6	7	7	7	8	8	9	10
34	0	1	1	2	2	3	3	4	4	5	5	6	6	6	7	7	8	8	9	9	10	11
36	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	12
38	1	1	2	2	3	3	4	5	5	6	6	7	8	8	9	9	10	10	11	12	13	14
40	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11	12	12	13	14	15
42	1	1	2	3	4	4	5	6	6	7	8	8	9	10	11	11	12	13	13	14	16	17
44	1	2	2	3	4	5	5	6	7	8	9	9	10	11	12	12	13	14	15	15	17	19
45	1	2	2	3	4	5	6	6	7	8	9	10	11	11	12	13	14	15	15	16	18	19
46	1	2	3	3	4	5	6	7	7	8	9	10	11	12	13	14	14	15	16	17	19	20
47	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	14	15	16	17	18	19	21
48	1	2	3	4	5	6	6	7	8	9	10	11	12	13	14	15	16	17	18	18	20	22
49	1	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	16	17	18	19	21	23
50	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	24
51	1	2	3	4	5	6	7	8	9	10	11	12	14	15	16	17	18	19	20	21	23	25
52	1	2	3	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	21	22	24	26
53	1	2	3	4	6	7	8	9	10	11	12	13	15	16	17	18	19	20	21	22	25	27
54	...	2	3	5	6	7	8	9	10	12	13	14	15	16	17	19	20	21	22	23	26	...
55	...	2	4	5	6	7	8	10	11	12	13	15	16	17	18	19	21	22	...	...	...	...
56	...	3	4	5	6	8	9	10	11	13	14	15	16	...	...	...	...	...	...	...	...	...
57	...	...	4	5	7	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

When the *nearest* limb is observed, subtract this correction; when the *farthest*, *add*.

TABLE XVIII.

finding the Correction of the Lunar Distance for the Contraction of the Sun's Semidiameter.

TABLE XVIII. A. Giving the Argument for Table XVIII. B.

ured d R. ⊙.	APPARENT ALTITUDE OF ⊙.																							
	5°	5½°	6°	6½°	7°	7½°	8°	8½°	9°	9½°	10°	11°	12°	13°	14°	15°	16°	17°	18°	20°	25°	30°	40°	50°
0′	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	22	18
30	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	30	34	24	29
0	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	35	37	42	46	46	...
30	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	40	42	44	47	53	59	...	...
0	...	...	...	...	...	...	...	...	...	...	...	...	...	...	44	46	49	51	53	57	...	...	...	...
30	...	...	...	...	...	...	...	...	...	...	...	...	45	48	51	54	57	60	62	67	...	...	...	...
0	...	...	...	...	...	...	...	...	...	...	45	49	52	55	59	62	65	68	...	...	...	...	...	...
30	...	...	...	...	...	...	...	...	47	49	51	55	59	63	66	70	...	...	...	...	...	...	...	...
0	...	...	...	...	...	...	47	50	52	54	57	61	66	70	74	...	...	...	...	...	...	...	...	...
30	...	...	...	...	47	50	52	55	57	60	62	67	72	...	...	...	...	...	...	...	...	...	...	...
0	...	...	...	49	52	55	57	60	63	66	68	74	...	...	...	...	...	...	...	...	...	...	...	...
30	...	...	50	53	56	59	62	65	68	71	74	...	...	...	...	...	...	...	...	...	...	...	...	...
0	...	51	54	58	61	64	67	70	74	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	...	55	58	62	65	69	72	75	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
0	55	59	62	66	70	73	77	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	59	63	66	70	74	78	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
0	62	66	70	74	79	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	66	70	74	79	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
0	69	74	78	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	73	77	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
0	76	81	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
30	80	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

TABLE XVIII. B. Contraction of ⊙'s Semidiameter.

le ction ⊙.	ARGUMENT = NUMBER FROM TABLE XVIII. A.																							
	20	24	28	32	36	40	44	46	48	50	52	54	56	58	60	62	64	66	68	70	72	74	76	78
	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″
0′	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	...	...	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
30	...	...	...	...	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1
0	...	...	...	...	...	4	4	4	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2
30	...	...	...	...	...	...	5	5	5	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3
0	...	...	...	...	...	...	7	6	6	6	6	5	5	5	5	5	5	4	4	4	4	4	4	4
20	...	...	...	...	...	...	...	7	7	7	7	6	6	6	6	5	5	5	5	5	5	5	4	4
40	...	...	...	...	...	...	...	9	8	8	8	7	7	7	7	6	6	6	6	6	5	5	5	5
0	...	...	...	...	...	...	...	10	9	9	9	8	8	8	8	7	7	7	7	6	6	6	6	6
20	...	...	...	...	...	...	...	...	11	10	10	9	9	9	9	8	8	8	8	7	7	7	7	7
40	...	...	...	...	...	...	...	...	12	12	11	11	10	10	10	9	9	9	9	8	8	8	8	7
0	...	...	...	...	...	...	...	...	...	13	12	12	12	11	11	10	10	10	10	9	9	9	9	8
20	...	...	...	...	...	...	...	...	...	14	14	13	13	12	12	12	11	11	11	10	10	10	10	9
40	...	...	...	...	...	...	...	...	...	16	15	15	14	14	13	13	13	12	12	11	11	11	11	10
0	...	...	...	...	...	...	...	...	...	18	17	16	16	15	15	14	14	13	13	13	12	12	12	11
20	...	...	...	...	...	...	...	...	...	...	19	18	17	17	16	16	15	15	14	14	13	13	13	12
40	...	...	...	...	...	...	...	...	...	...	...	20	19	18	18	17	17	16	16	15	15	14	14	14
0	...	...	...	...	...	...	...	...	...	...	...	21	21	20	19	19	18	17	17	16	16	16	15	15
20	...	...	...	...	...	...	...	...	...	...	...	...	...	22	21	20	20	19	18	18	17	17	16	16
40	...	...	...	...	...	...	...	...	...	...	...	...	...	23	23	22	21	20	20	19	19	18	18	17
0	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	24	23	22	21	21	20	20	19	19
20	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	25	24	23	22	22	21	21	20
40	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	25	25	24	23	23	22	22
0	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	26	26	25	24	24	23
20	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	28	27	26	25	25
40	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	28	28	27	26
0	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	29	28
20	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	30

Subtract **this correction from the distance.**

TABLE XIX.

For finding the value of *N* for correcting lunar distances for the compression of the earth.

TABLE XIX. A. giving 1st Part of *N*.

App. Dist.	Moon's Declination.										
	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°	30°
°	″	″	″	″	″	″	″	″	″	″	″
20	—0	3	6	10	13	16	19	22	25	28	31
22	0	3	6	9	12	14	17	20	23	25	28
24	0	3	5	8	11	13	16	18	21	23	25
26	0	2	5	7	10	12	14	17	19	21	23
28	0	2	4	7	9	11	13	15	17	19	21
30	—0	2	4	6	8	10	12	14	16	18	20
32	0	2	4	6	8	9	11	13	15	16	18
34	0	2	4	5	7	9	10	12	14	15	17
36	0	2	3	5	7	8	10	11	13	14	16
38	0	2	3	5	6	8	9	10	12	13	14
40	—0	1	3	4	6	7	8	10	11	12	13
42	0	1	3	4	5	7	8	9	10	11	13
44	0	1	2	4	5	6	7	8	10	11	12
46	0	1	2	3	5	6	7	8	9	10	11
48	0	1	2	3	4	5	6	7	8	9	10
50	—0	1	2	3	4	5	6	7	8	9	10
52	0	1	2	3	4	5	5	6	7	8	9
54	0	1	2	3	3	4	5	6	7	7	8
56	0	1	2	2	3	4	5	5	6	7	8
58	0	1	1	2	3	4	4	5	6	6	7
60	—0	1	1	2	3	3	4	5	5	6	7
62	0	1	1	2	3	3	4	4	5	5	6
64	0	1	1	2	2	3	3	4	4	5	6
66	0	1	1	2	2	3	3	4	4	5	5
68	0	0	1	1	2	2	3	3	4	4	5
70	—0	0	1	1	2	2	3	3	3	4	4
72	0	0	1	1	2	2	2	3	3	3	4
74	0	0	1	1	1	2	2	2	3	3	3
76	0	0	1	1	1	1	2	2	2	3	3
78	0	0	0	1	1	1	1	2	2	2	2
80	—0	0	0	1	1	1	1	1	2	2	2
82	0	0	0	0	1	1	1	1	1	1	2
84	0	0	0	0	0	1	1	1	1	1	1
86	0	0	0	0	0	0	0	1	1	1	1
88	0	0	0	0	0	0	0	0	0	0	0
90	—0	0	0	0	0	0	0	0	0	0	0
92	+0	0	0	0	0	0	0	0	0	0	0
94	0	0	0	0	0	0	0	1	1	1	1
96	0	0	0	0	0	1	1	1	1	1	1
98	0	0	0	0	1	1	1	1	1	1	2
100	+0	0	0	1	1	1	1	1	2	2	2
102	0	0	0	1	1	1	1	2	2	2	2
104	0	0	1	1	1	1	2	2	2	3	3
106	0	0	1	1	1	2	2	2	3	3	3
108	0	0	1	1	2	2	2	3	3	3	4
110	+0	0	1	1	2	2	3	3	3	4	4
112	0	0	1	1	2	2	3	3	4	4	5
114	0	1	1	2	2	3	3	4	4	5	5
116	0	1	1	2	2	3	3	4	4	5	6
118	0	1	1	2	3	3	4	4	5	5	6
120	+0	1	1	2	3	3	4	5	5	6	7
122	0	1	1	2	3	4	4	5	6	6	7
124	0	1	2	2	3	4	5	5	6	7	8
126	0	1	2	3	3	4	5	6	7	7	8
128	0	1	2	3	4	5	5	6	7	8	9
130	+0	1	2	3	4	5	6	7	8	9	10

TABLE XIX. B. giving 2d Part of *N*.

App. Dist.	Sun's, Planet's, or Star's Declination.									
	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°
°	″	″	″	″	″	″	″	″	″	″
20	+0	3	7	10	14	17	20	24	27	30
22	0	3	6	9	13	16	19	22	25	27
24	0	3	6	9	12	14	17	20	23	25
26	0	3	5	8	11	13	16	18	21	23
28	0	3	5	8	10	12	15	17	20	22
30	+0	2	5	7	9	12	14	16	18	21
32	0	2	4	7	9	11	13	15	17	19
34	0	2	4	6	8	11	13	15	16	18
36	0	2	4	6	8	10	12	14	16	17
38	0	2	4	6	8	10	11	13	15	17
40	+0	2	4	6	7	9	11	13	14	16
42	0	2	4	5	7	9	10	12	14	15
44	0	2	3	5	7	8	10	12	13	15
46	0	2	3	5	6	8	10	11	13	14
48	0	2	3	5	6	8	9	11	12	14
50	+0	2	3	5	6	8	9	11	12	13
52	0	2	3	4	6	7	9	10	12	13
54	0	1	3	4	6	7	9	10	11	13
56	0	1	3	4	6	7	8	10	11	12
58	0	1	3	4	6	7	8	10	11	12
60	+0	1	3	4	5	7	8	9	11	12
62	0	1	3	4	5	7	8	9	10	12
64	0	1	3	4	5	7	8	9	10	11
66	0	1	3	4	5	6	8	9	10	11
68	0	1	3	4	5	6	8	9	10	11
70	+0	1	3	4	5	6	7	9	10	11
72	0	1	2	4	5	6	7	9	10	11
74	0	1	2	4	5	6	7	8	10	11
76	0	1	2	4	5	6	7	8	9	11
78	0	1	2	4	5	6	7	8	9	11
80	+0	1	2	4	5	6	7	8	9	10
82	0	1	2	4	5	6	7	8	9	10
84	0	1	2	4	5	6	7	8	9	10
86	0	1	2	4	5	6	7	8	9	10
88	0	1	2	4	5	6	7	8	9	10
90	+0	1	2	4	5	6	7	8	9	10
92	0	1	2	4	5	6	7	8	9	10
94	0	1	2	4	5	6	7	8	9	10
96	0	1	2	4	5	6	7	8	9	10
98	0	1	2	4	5	6	7	8	9	10
100	+0	1	2	4	5	6	7	8	9	10
102	0	1	2	4	5	6	7	8	9	11
104	0	1	2	4	5	6	7	8	9	11
106	0	1	2	4	5	6	7	8	10	11
108	0	1	2	4	5	6	7	9	10	11
110	+0	1	3	4	5	6	7	9	10	11
112	0	1	3	4	5	6	8	9	10	11
114	0	1	3	4	5	6	8	9	10	11
116	0	1	3	4	5	7	8	9	10	11
118	0	1	3	4	5	7	8	9	10	12
120	+0	1	3	4	5	7	8	9	11	12
122	0	1	3	4	6	7	8	10	11	12
124	0	1	3	4	6	7	8	10	11	12
126	0	1	3	4	6	7	9	10	11	13
128	0	2	3	4	6	7	9	10	12	13
130	+0	2	3	5	6	8	9	11	12	13

The signs in the 0° column apply to all the numbers in the same line, and are to be used when the declination is *No*
When the declination is *South*, change the sign + to — and — to +.

TABLE XX.

TION REQUIRED ON ACCOUNT OF SECOND DIFFERENCES OF THE MOON'S ON, IN FINDING THE GREENWICH TIME CORRESPONDING TO A CORRECTED R DISTANCE.

Difference of the Proportional Logarithms in the Ephemeris.

imate val.		2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52
h	m	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	50	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3
2	40	0	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6	6	6	6
2	30	0	1	1	2	2	2	2	3	3	3	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9
2	20	0	1	1	2	2	3	3	3	4	4	5	5	6	6	6	7	7	8	8	9	9	10	10	10	11	11
2	10	1	1	2	2	3	3	4	4	5	5	5	6	6	7	7	8	8	9	9	10	10	11	12	12	13	13
2	0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	8	9	9	10	10	11	12	12	13	13	14	14
1	50	1	1	2	2	3	4	4	5	5	6	6	7	8	8	9	9	10	11	11	12	12	13	14	14	15	15
1	40	1	1	2	3	3	4	4	5	6	6	7	7	8	9	9	10	10	11	12	12	13	14	14	15	15	16
1	30	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11	11	12	12	13	14	14	15	16	16

Difference of the Proportional Logarithms in the Ephemeris.

		54	56	58	60	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90	92	94	96	98	100	102
h	m	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	50	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	7	7
2	40	7	7	7	7	8	8	8	8	9	9	9	9	10	10	10	10	11	11	11	11	12	12	12	12	13
2	30	9	10	10	10	11	11	12	12	12	13	13	13	14	14	14	14	15	15	16	16	16	17	17	17	18
2	20	12	12	13	13	13	14	14	15	15	16	16	16	17	17	18	18	19	19	19	20	20	21	21	22	22
2	10	14	14	15	15	16	16	16	17	17	18	19	19	20	20	21	21	22	22	22	23	23	24	24	25	26
2	0	15	16	16	17	17	18	18	19	19	20	21	21	22	22	23	23	24	24	25	25	26	27	27	28	28
1	50	16	17	17	18	18	19	19	20	21	21	22	22	23	24	24	25	25	26	27	27	28	28	29	30	30
1	40	17	17	18	19	19	20	20	21	21	22	23	23	24	25	25	26	26	27	28	28	29	29	30	31	31
1	30	17	18	18	19	19	20	21	21	22	23	23	24	24	25	25	26	27	27	28	29	29	30	31	31	32

Difference of the Proportional Logarithms in the Ephemeris.

		104	106	108	110	112	114	116	118	120	122	124	126	128	130	132	134	136	138
h	m	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	50	7	7	7	7	7	7	8	8	8	8	8	8	8	8	9	9	9	9
2	40	13	13	13	14	14	14	14	15	15	15	15	15	16	16	16	16	17	17
2	30	18	18	19	19	19	20	20	20	21	21	21	22	22	22	23	23	24	24
2	20	22	23	23	24	24	25	25	25	26	26	27	27	28	28	28	29	29	30
2	10	26	26	27	27	28	29	29	29	30	30	31	31	32	32	33	33	34	34
2	0	29	29	30	30	31	31	32	33	33	34	34	35	35	36	37	37	38	38
1	50	31	31	32	32	33	34	34	35	35	36	37	37	38	38	39	40	40	41
1	40	32	33	33	34	34	35	35	36	37	38	38	39	39	40	41	41	42	42
1	30	32	33	34	34	35	35	36	36	37	38	39	39	40	40	41	42	42	43

orrection is to be *added* to the approximate Greenwich Time when the Proportional Logarithms in the is are *decreasing*, and *subtracted* when they are *increasing*.

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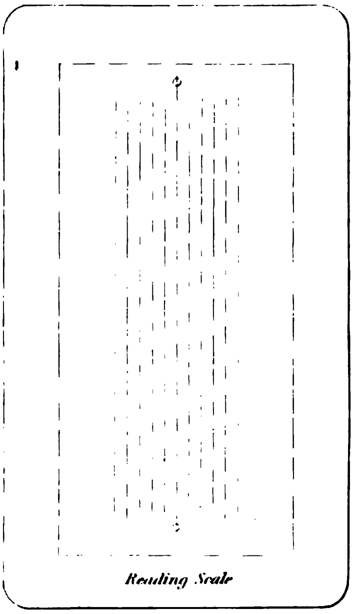
END OF VOL. II.

Fig. 3.



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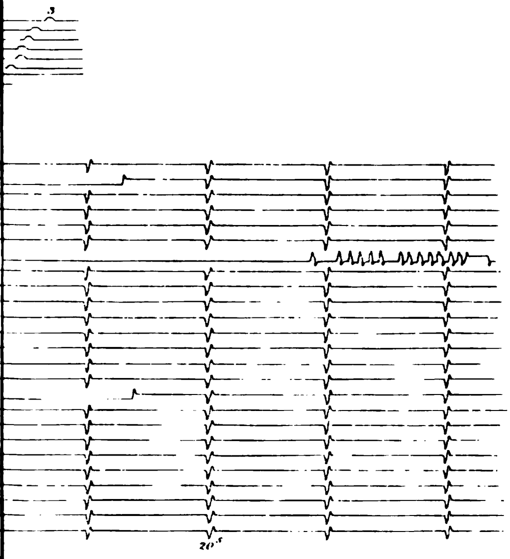


PLATE III.

Fig. 1.

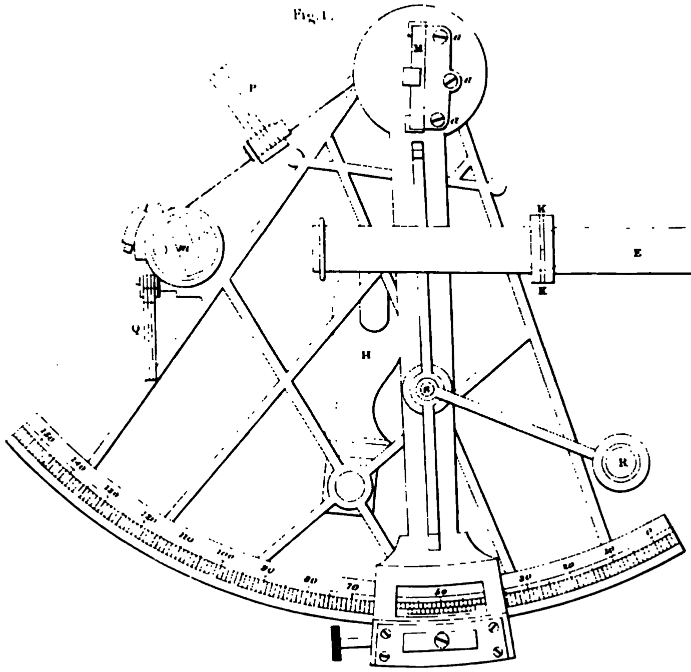
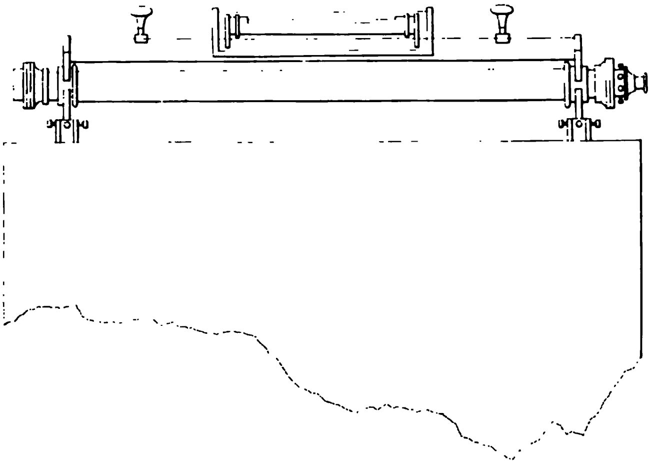


Fig. 2.



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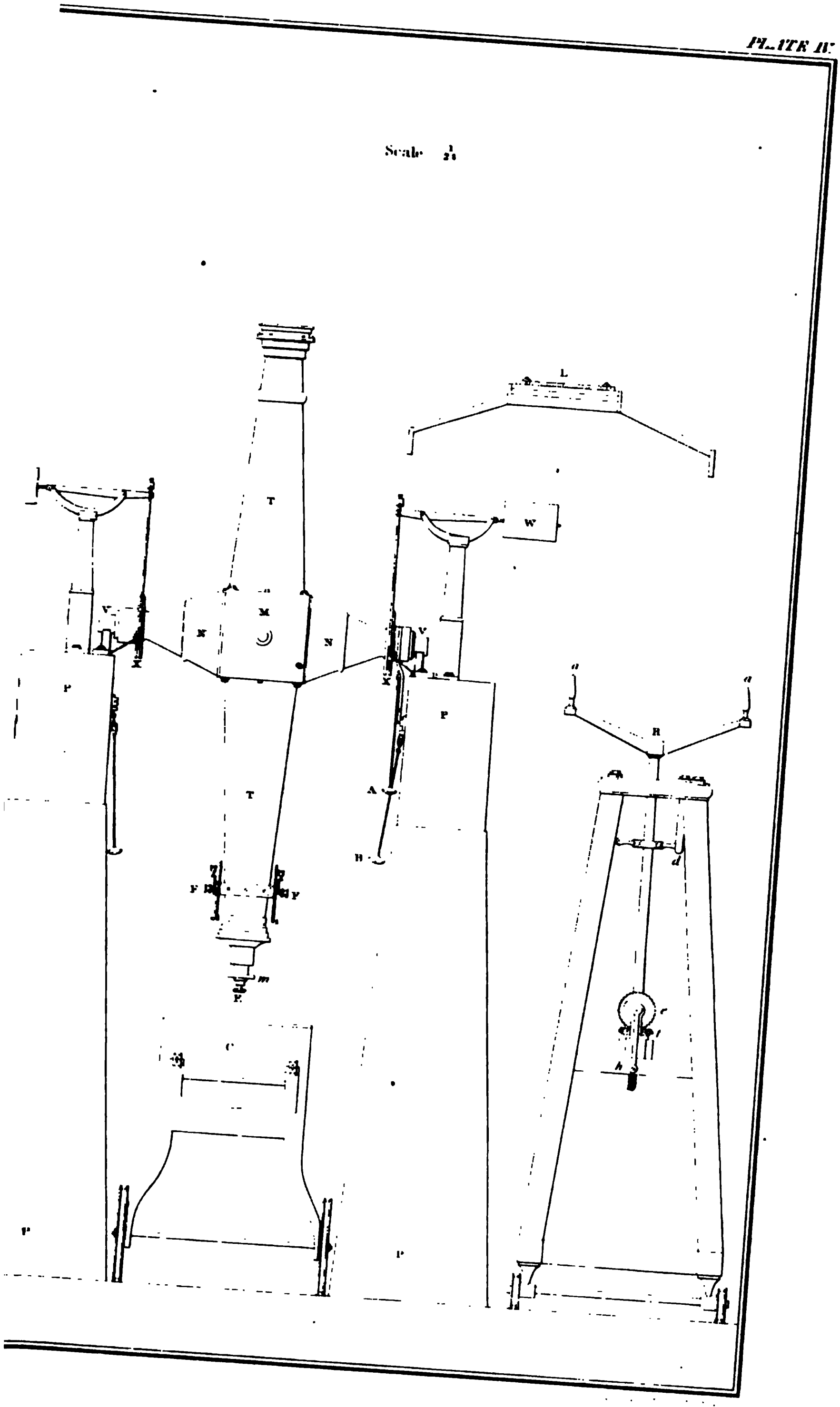

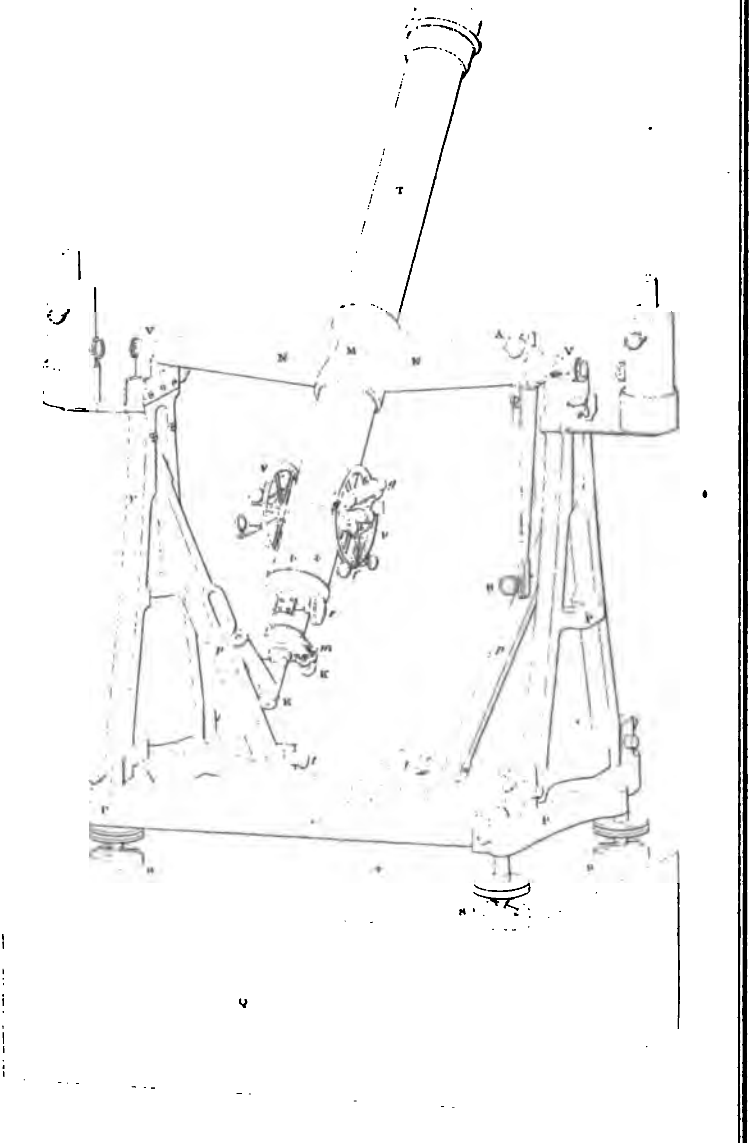
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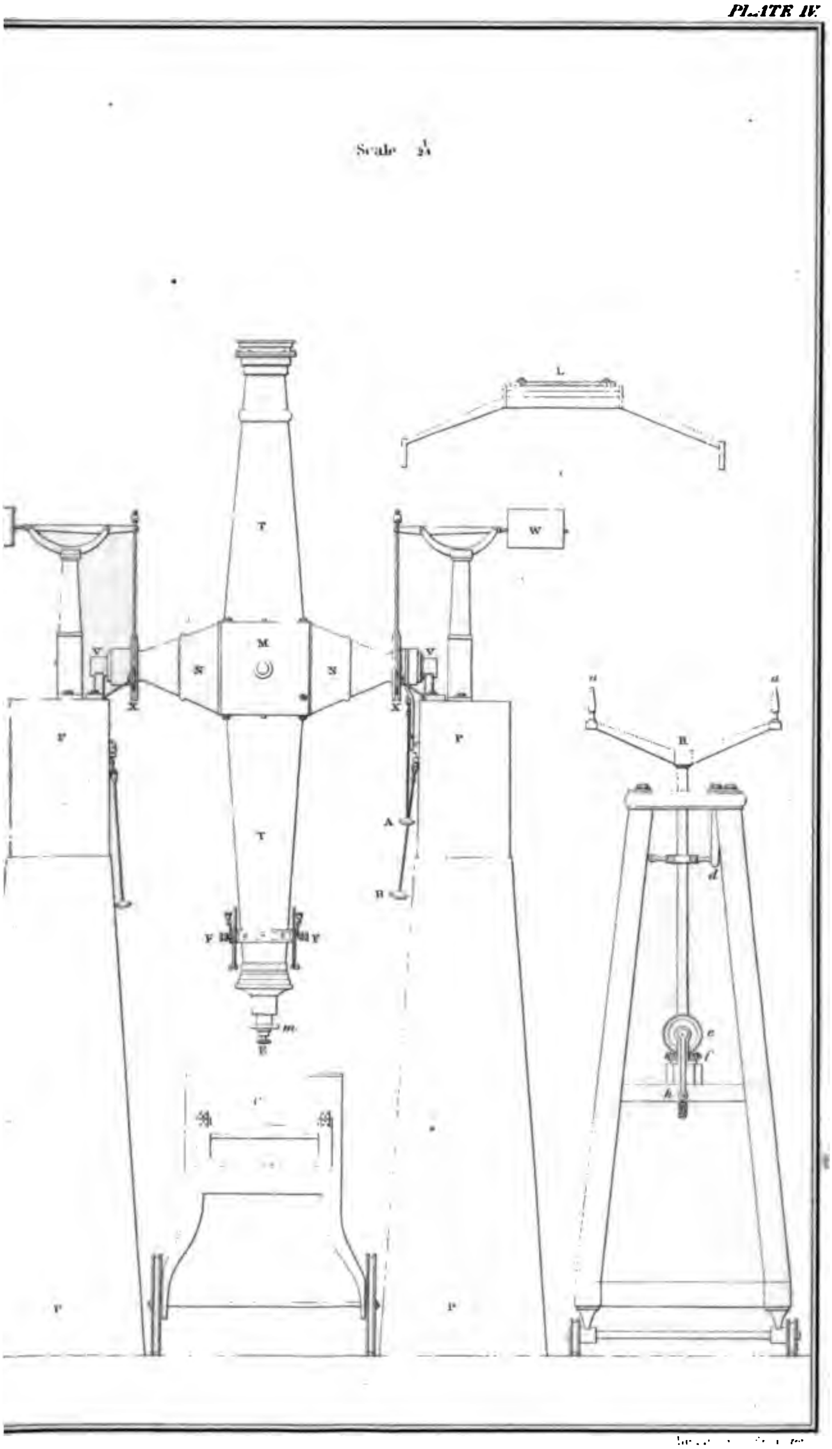
PLATE IV.



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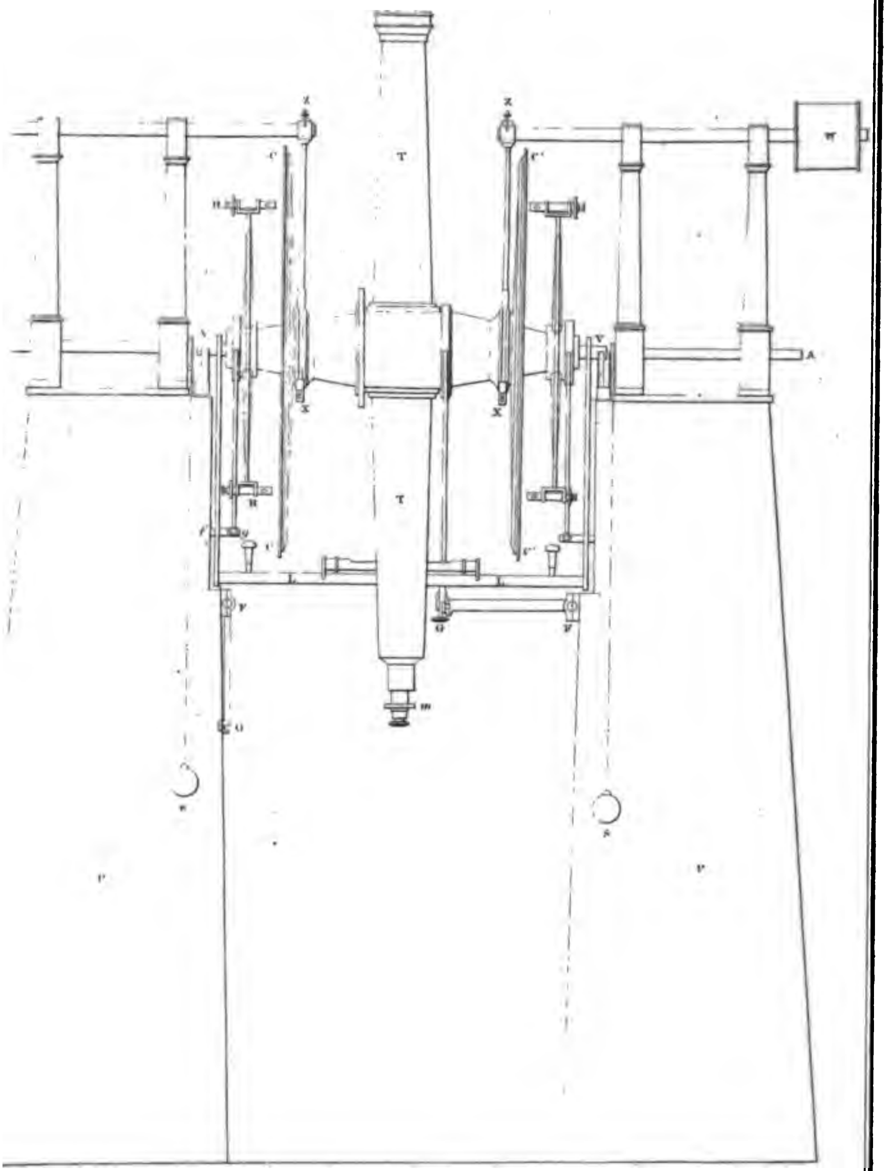
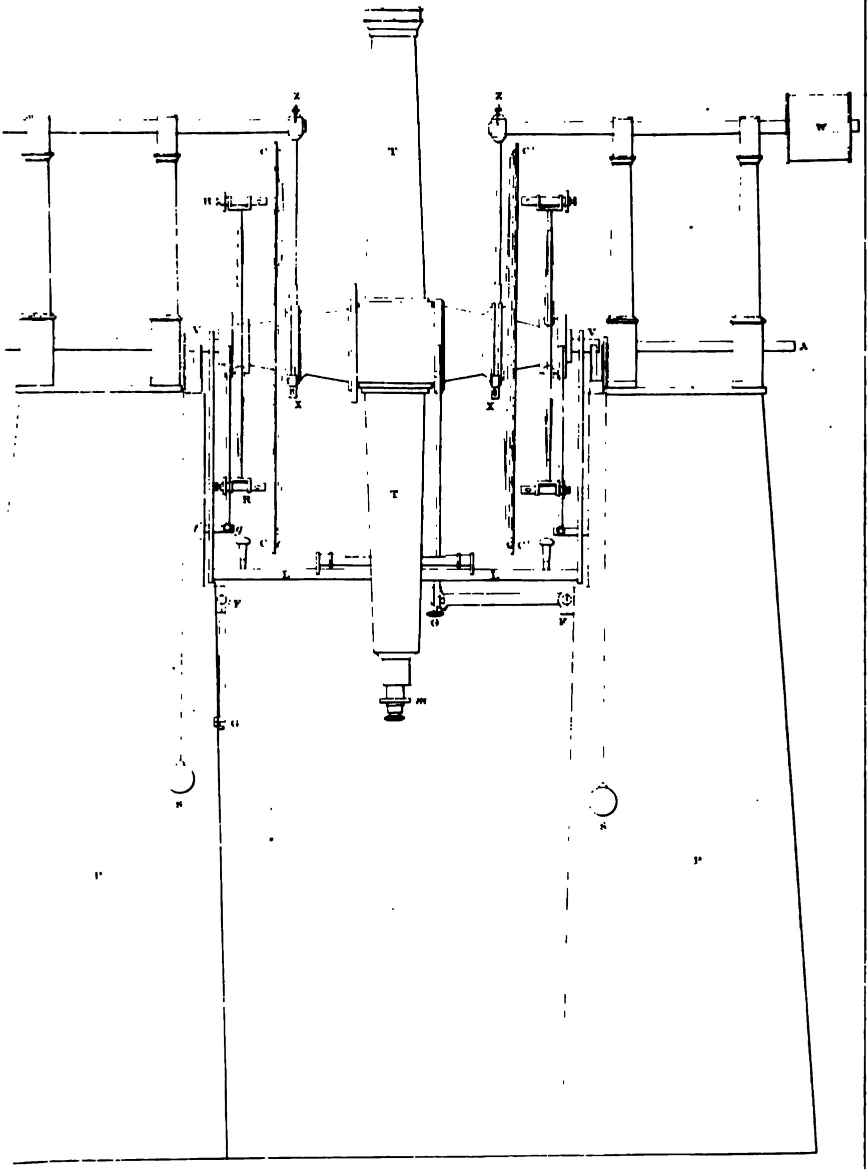
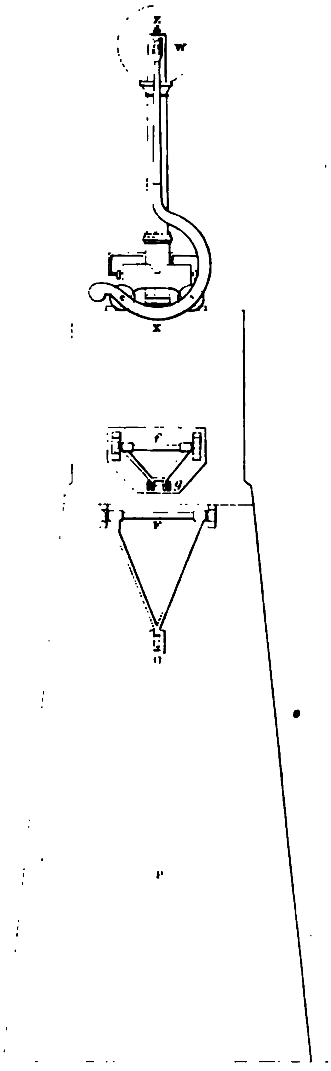


PLATE VII.

Scale $\frac{2}{37}$



Scale - $\frac{1}{24}$



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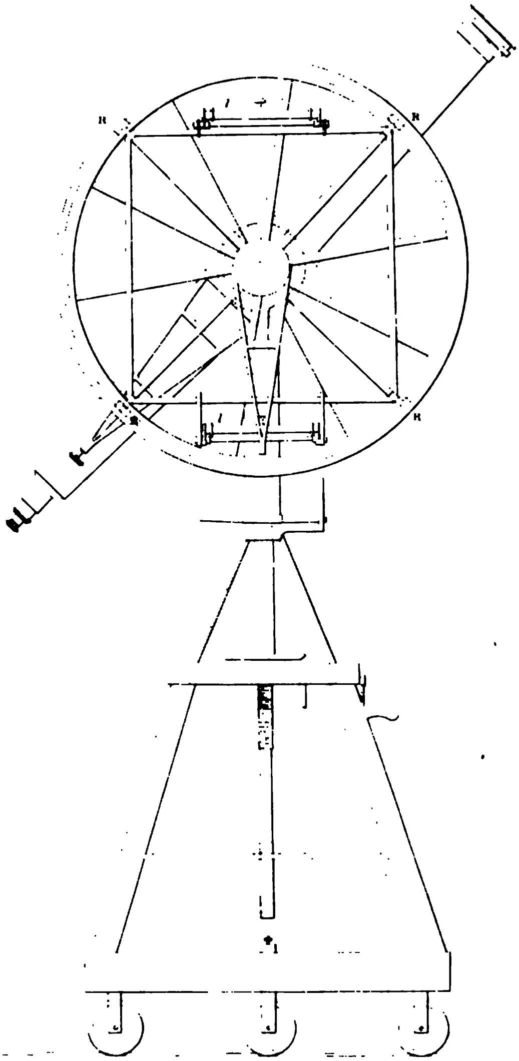

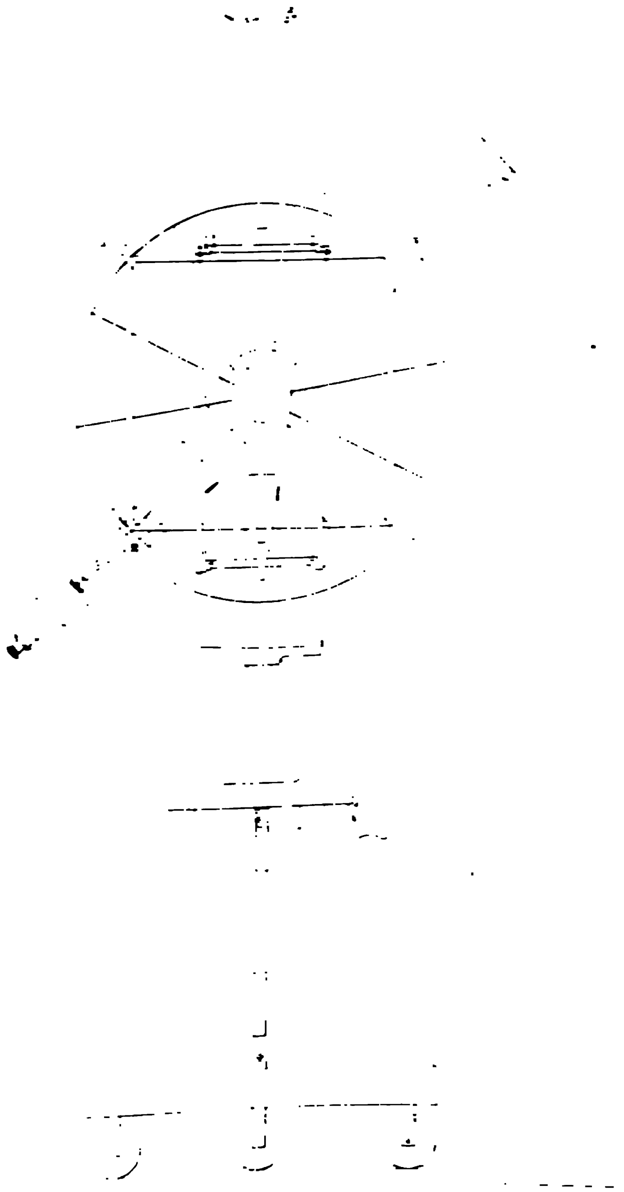

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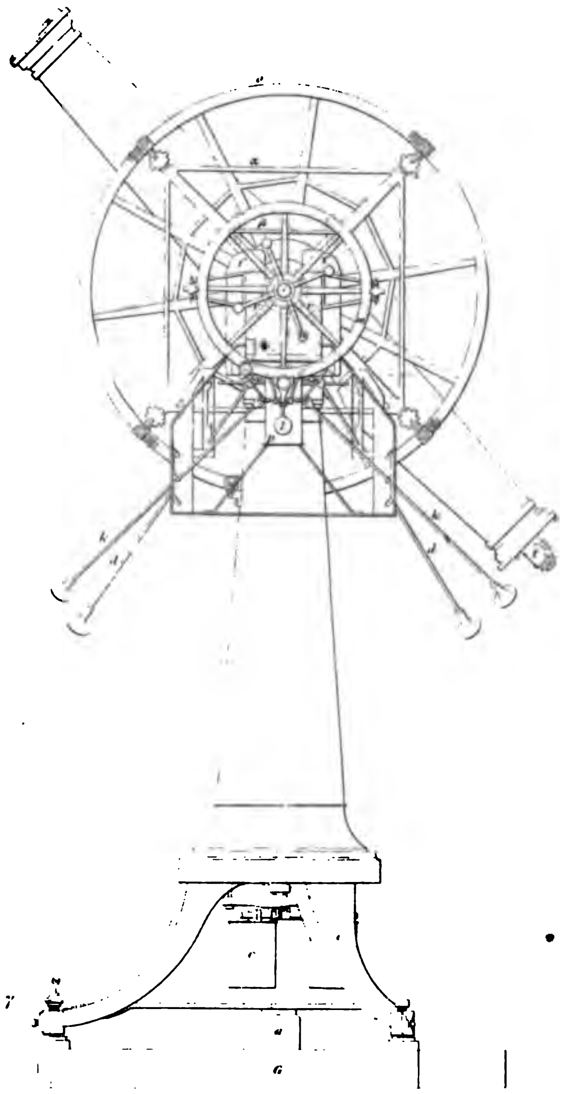

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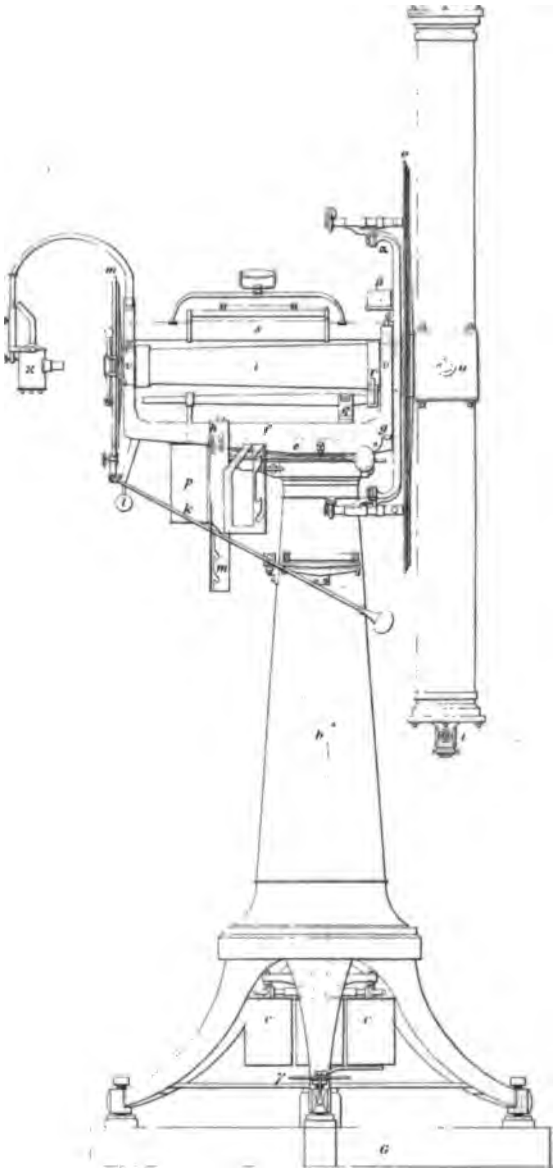

Illman Bros Engravers

PLATE XII.

Fig 1

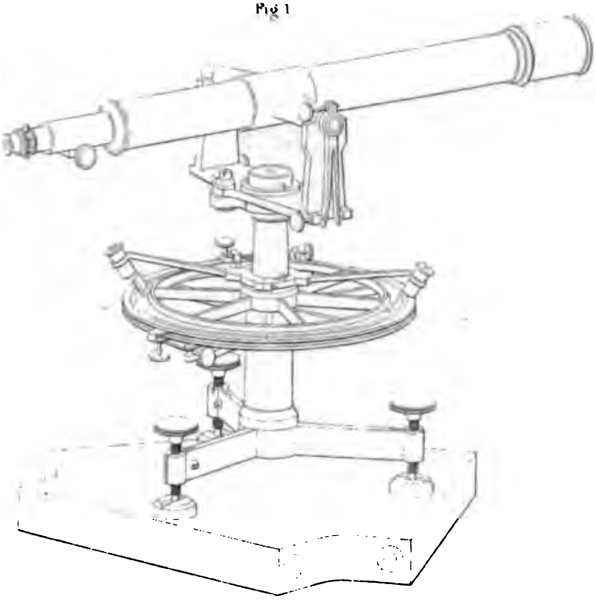

Fig 2

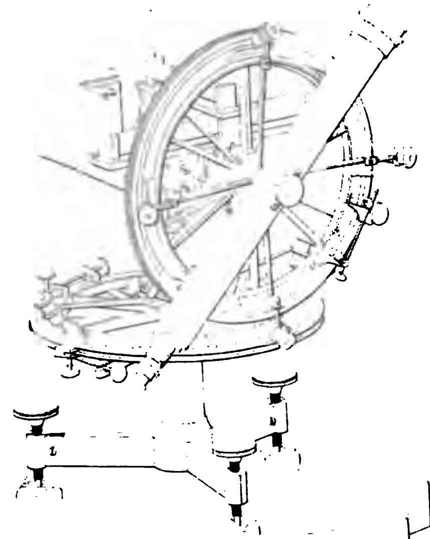

PLATE XIII